V.A. Belinski

DYNAMIC METEOROLOGY

В. А. БЕЛИНСКИЙ

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Annotation

The book should serve as a textbook of dynamic meteorology, suitable for students of hydrometeorological institutes. For the reading of the book, a physico-mathematical training on the level of higher technical educational institutions is required, as well as a knowledge of thermodynamics, hydrodynamics, and meteorology. Separate chapters are devoted to the exposition of the works of Soviet scientists in the field of dynamic meteorology, and therefore the book may also be used by post-graduate students and scientific personnel. For the reading of the chapters indicated, a more extensive mathematical training is required.

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PREFACE

This book is intended to serve as a manual of dynamic meteorology, written first and foremost for the meteorological and military faculties of hydrometeorological institutes.

The book is based on lectures on dynamic meteorology which the author has read for some years, beginning in 1938 at the Moscow Hydrometeorological Institute, later at the Leningrad Hydrometeorological Institute and Leningrad University and elsewhere.

According to the curricula of higher educational institutions for hydrometeorology, a course in dynamic meteorology is given in the third and fourth years, after the students have already learnt higher mathematics, physics, theoretical mechanics and general meteorology. The course of dynamic meteorology starts in conjunction with a course in hydrodynamics, and its last part is given parallel with a course in synoptic meteorology. Consequently it was assumed in planning the course of dynamic meteorology that the students have a knowledge of higher mathematics, physics, mechanics, hydrodynamics and general meteorology as is given in the syllabuses of the higher educational institutions for hydrometeorology.

The number of books devoted to dynamic meteorology is not great, and none of them can be used as a textbook for higher educational institutions for hydrometeorology.

The two-volume monograph written in 1935–1938 by Soviet authors and edited by N. E. Kochin and B. I. Izvekov is interesting in many respects, but the excessive size of the book and a somewhat arbitrary selection of the necessary material is an obstacle to its use as a textbook. Furthermore, this book has become partly outdated because it does not contain the most interesting results obtained by Soviet meteorologists in recent years.

Among the foreign books, that of Haurwitz, published in 1941, is to be noted. This book comes closer to the course of lectures given by the author in 1938 at the Moscow Hydrometeorological Institute, and differs from the others in the right proportions of its composition and the lucidity of its exposition. However, it suffers from two fundamental shortcomings. It is intended for readers who have no knowledge of meteorology, and therefore the subject matter is too elementary. A graver shortcoming of Haurwitz's book is the absence of the results obtained by Soviet authors. Thus, a makeshift solution of some problem as proposed by German or English authors is often given, whereas the more complete and exact solution of a similar problem obtained by Soviet authors is not touched upon.

This fault is also characteristic of the other foreign books (we could mention, for example, the book by J. Holmboe, G. E. Forsythe, W. Gustin: Dynamic Meteorology, published in 1946).

The author's aims in preparing the present course were the following:

1. To give a clear and complete idea of the problems and the possibilities of modern dynamic meteorology, with special regard to weather forecasting, which is the basic practical problem of dynamic meteorology. To form a closed system of weather equations on a basis as strict as possible, to analyze these equations and

to show to what extent they can be used for weather forecast.

2. To expound as fully as possible the theory of the elementary meteorological processes, which forms a basis for drafting a subsequent course of synoptic meteorology.

3. To examine some of the practically important applications, particularly the prognostic ones, which follow from the theory of the elementary meteorological processes.

The material of the course is divided into 21 chapters. Some of these deal with comparatively elementary problems. However, the contents of these chapters is absolutely indispensable for a clear understanding of the more complicated material, and also for being able to draw such practical conclusions as are widely used by meteorologists in operational practice and research work.

Even a superficial acquaintance with the contents of this book may convince the student of the value of the contribution to dynamic meteorology made by Soviet scientists, in particular by the Soviet school of theoretical meteorologists founded by A. A. Fridman.

A. A. Fridman's scientific interests were extremely wide, but on meteorology he left the deepest imprint, creating a school of Soviet theoretical meteorologists. The outstanding feature of this school is the turning of all contemporary achievement of hydromechanics and thermodynamics to the service of meteorology.

Without enumerating all the Soviet works which form an integral part of this book, I shall only indicate the principal ones. In Chapter VIII E. S. Kuznetsov's detailed and rigorous derivation of a closed system of equations for hydrodynamics, thermodynamics and the theory of radiation is described. At the end of the same chapter the author briefly acquaints the students with the fruitful and promising ideas of Fridman and Keller on the theory of turbulence, which find their expression in the so-called Fridman-Keller equations.

In Chapter IX, the reader is acquainted with two important results, obtained by Soviet scientists, which make it possible to simplify very complicated equations of motion. In the same chapter are given detailed tables, constructed by A. A. Fridman (in collaboration with Hesselberg) of the orders of magnitude of the meteorological elements and their derivatives. In this chapter are also stated the results obtained by N. E. Kochin, who applied the boundary layer method to atmospheric motion.

Chapter XV deals with an interesting and important result concerning the spectral structure of turbulence. This result is known in literature as "the law of two thirds" of Kolmogorov–Obukhov.

Almost all of Chapter XVI is devoted to the variation of wind with height in a turbulent atmosphere, and is based on Soviet scientific work. A considerable part of Chapter XVIII, dealing with the general circulation of the atmosphere, is likewise based on Soviet research. In this chapter special attention is paid to the work of academician V. V. Shuleikin who showed the part played by the continents and oceans in the mechanism of the general circulation, to the mathematical theory of the general circulation of academician N. E. Kochin, and finally to the hydrodynamic theory of pressure waves and of action centers of the atmosphere developed by E. N. Blinova.

In Chapter XIX, the problem of small oscillations of an interface and the loss of its stability, likewise studied by N. E. Kochin, is discussed in detail.

Finally, the last chapters (XX and XXI) are devoted entirely to an account of the results of the integration of the weather equations recently obtained by Soviet scientists. These results are particularly interesting and promising, inasmuch as they concern the basic practical problem of dynamic meteorology – forecasting the weather.

Experience over many years has shown that the main difficulty encountered by students attending the course consists in the complex task of making the applied mathematical apparatus correspond with the physical reality of the phenomena. The author therefore constantly strove to draw the attention of the students to this physical reality. In some sections, treating the work of Soviet scientists, the author was obliged to retain a more complicated mathematical apparatus transcending the mathematical syllabus of higher educational institutions for hydrometeorology without giving any detailed mathematical appendixes. The preclusion of these results would have impoverished the subject matter of the book. On the other hand, detailed digressions in the domain of higher mathematics would have unnecessarily overloaded the book with additional material. The student interested in any particular problem will always find the necessary reference in the numerous mathematical textbooks.

The question of the terminology used in the book must be noted separately. The same term frequently has a different meaning in hydrodynamics and in meteorology; we may mention gradient, convection, circulation and others. The author did not, however, consider it possible to introduce new terms in these cases, but used the terms mainly in their meteorological sense assuming that the readers of this book will mainly be meteorologists, not hydrodynamicists.

A bibliography of the basic literature used in compiling this textbook, and which is recommended to students for a more thorough study, is given at the end of the book.

In conclusion I feel bound to express my gratitude to my teachers – academician V. V. Shuleikin, corresponding member of the Academy of Science of the USSR, I. A. Kibel' and Professor E. S. Kuznetsov, to my colleagues at the Institute, Professors S. P. Khromov and A. Kh. Khrgian, as well as to my pupils L. Matveev, S. Titov, P. Smirnov and I. Slavin, who read the manuscript and gave me valuable advice and suggestions which contributed to the improvement of this book. I especially thank Sh. N. Gofshtein for his help in the preparation of this book.

V. A. Belinskii

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Chapter I

INTRODUCTION

§ 1. The Scope and Aims of Dynamic Meteorology

Dynamic meteorology is a part of general meteorology, i. e., the science of the earth's atmosphere and of the physical processes which take place in it.

The branches of science which deal with inanimate nature may be generally classified into two groups: 1) the exact physico–mathematical sciences, such as physics, mechanics, aero– and hydrodynamics and 2) the descriptive sciences such as geography, geology and mineralogy.

A distinguishing feature of the exact sciences is the possibility of a wide application of the analytical method by making use of versatile mathematical apparatus. Another distinctive feature of the exact sciences is the wide use of experimentation which provides the basic method of physical investigation.

Both possibilities are excluded from the descriptive sciences, in which the basic method consists in observations and their subsequent elaboration.

Meteorology was rightly considered a descriptive science 30–40 years ago. However, the considerable successes in the development of meteorology in recent years are due to the gradual, but more and more thorough application of analytic methods to this field. At the same time the meteorology of the twentieth century turns more often to experiments both under laboratory conditions and in nature. Modern meteorology as a whole therefore can no longer be regarded as a descriptive science.

The entire intermediate group of the geophysical sciences, including oceanography , hydrology, seismology, and others, in this sense shows the same trend as meteorology.

The essential feature of the geophysical sciences is the study of processes occurring on our planet on the basis of the general laws of physics. The part played by observation in geophysics, including meteorology, still prevails over both experimental and analytic investigation, and will retain its importance in the course of further development of meteorology. Observational results will always furnish the initial and boundary conditions, indispensable for the solution of any theoretical problem.

In this connection we should recall what the great naturalist and artist, the founder of hydrodynamics, Leonardo da Vinci said in the XVI century: "If you want to study the motion of a fluid, first observe and then apply reasoning." These words, of course, fully apply to meteorology, too. However, so many observations of atmospheric motions have been accumulated, that it is high time to "apply reasoning", by establishing observation on the basis of the results gathered by humanity in the fields of mathematics, physics and hydromechanics.

By now it is perfectly clear that meteorology as a science of atmospheric processes can only achieve decisive successes by using new methods. To this end meteorology constructs physically realistic theoretical models for the representation

of atmospheric processes, testing them afterwards experimentally. The word "experiment" is used here in its broadest sense, as a synonym for "practice".

Let us recall the genial thesis formulated by Lenin, which must be used for guidance in the development of any science including meteorology: "From an active contemplation to abstract thinking and from that to practice – such is the dialectic method of knowledge of the truth, of the knowledge of objective reality".

Meteorology began as a descriptive science studying particular phenomena occurring in the atmosphere. "One should study the objects before beginning to study the processes" (Engels). Now the necessity has arisen for studying the atmospheric processes. Such studies are facilitated by the possibility of studying these processes theoretically.

We will define the aim of dynamic meteorology in the following way. Dynamic meteorology aims to construct, with the help of hydrodynamics and thermodynamics, theoretical models of the processes occurring in the atmosphere, subjecting these models to subsequent experimental verification. Therefore it would be more exact to call dynamic meteorology, theoretical meteorology.

The primary practical task of dynamic meteorology is to forecast the weather.

However, dynamic meteorology is yet far from solving this practical problem.

Dynamic meteorology is a new science which is still not finally formed. It is far from that classical perfection which characterizes theoretical mechanics. The number of problems in dynamic meteorology waiting to be solved is appreciably greater than those which have already found their solution.

The difficulties faced by meteorology in the solution of its primary problem can be divided into three groups.

Difficulties are already encountered in working out a closed system of equations, mathematically expressing the physical laws which govern atmospheric processes; this is explained by the extreme complexity of atmospheric processes.

The most important characteristics of the atmosphere, which are responsible for the complexity of processes occurring in it, are the following:

1) Air is a baroclinic, compressible fluid, i. e., its density depends not only on pressure but also on the temperature; therefore the equation of motion of the air is connected with the equation of energy balance.

2) The energy exchange in the atmosphere is brought about by radiation, by eddy flux of heat and by latent heat liberated in the course of a change in the water phases, all three forms of energy exchange strongly affecting the atmospheric processes. The radiation and the turbulent exchange in the atmosphere are particularly complicated.

3) The processes of heat exchange are made exceedingly complicated by the presence of water vapor in the atmosphere. The turbulent heat exchange is maintained by radiation and the condensation of water vapor. These processes depend, in their turn, on the quantity of water vapor contained in the air. The amount of water vapor changes with the motion of the atmosphere. This close relationship between atmospheric motion and the content of water vapor – the main energetic factor in the atmosphere – extremely complicates the problem of working out a system of equations expressing the laws of atmospheric processes.

4) The motions of the atmosphere take place in the Coriolis field of force generated by the rotation of the earth, and this field essentially influences the large-scale motion of the atmosphere.

5) All air motion in the atmosphere is of a turbulent nature. Due to this fact considerable forces of turbulent internal friction, essentially affecting the character of the motion, are frequently evolved. Therefore it is necessary to apply averaged equations of atmospheric motion, and these equations are incompletely determined,

Thus, we are obliged to proceed from the conclusions of the hydromechanics of a compressible baroclinic fluid, derived for turbulent motions taking place in the Coriolis field of force, in order to work out a system of equations for atmospheric processes. We must hereby take into consideration the thermodynamic conclusions regarding ideal gases and water phases, and the thermodynamics of radiant energy.

The system of equations thus obtained appears not strictly closed, because the number of unknown functions entering the averaged equations of atmospheric motion is greater than the number of equations. Therefore, the system of equations is unsolvable in principle.

However, this system of equations, which we call for short "the weather equations" can apparently be put in closed form by adopting some sufficiently correct assumptions. Thus, the weather equations, i. e., a system of equations expressing the laws governing the atmospheric processes, can be worked out, and the first difficulty can be overcome in this way.

However, after the first difficulty is overcome, i. e., after having formulated a system of weather equations,and when its solution is attempted, we are faced with a new difficulty. Namely, it appears impossible to solve correctly the obtained system of equations, owing to difficulties of mathematical nature which are, for the time being, unsurmountable. Therefore, having been convinced of the impossibility of obtaining an exact solution of the weather equations, we are forced to abandon the attempt to solve them in their general form, and to use simplifications by substituting approximate equations for the exact ones, or restricting ourselves to the separate examination of particular processes and states of the atmosphere.

One must be particularly cautious in substituting approximate weather equations for exact ones, because of the risk of distorting the processes described by such equations.

The division of the atmosphere into separate layers, for which simplifications of the weather equations are possible in principle, is a very successful method for working out approximate weather equations. Thus we pass from a single-layer problem to a multi-layer one. From an exact system of weather equations, correct for any point of the terrestrial atmosphere, but which cannot be integrated, we pass to a set of systems of approximate equations, each valid within the limits of only one layer, but which can be integrated. The solutions obtained for individual layers are "cemented" together.

It is often necessary to apply the method of successive approximation in order to solve a problem within a single layer, owing to the impossibility of exactly solving even the simplified equations. The principle of this method consists in first estimating all the terms in the equations. The main and most essential terms are chosen

among them, and the remaining ones are rejected. We then solve this approximate system of equations. The obtained solution is the first (rough) approximation to the required solution of the problem. The first approximation appears sufficient, and can be regarded as definitive if the processes are of such a kind that the rejected terms are negligible. However, this is often not sufficient. A second approximation must then be found. For this purpose small terms are retained in the initial equations, after first substituting the approximation obtained for the unknown functions into these terms, and the "corrected" system of equations is again integrated. The second approximation is obtained as a result of these operations.

Theoretically it is possible to obtain a third and higher approximations but in practice we usually confine ourselves to the second approximation.

After all this, the third difficulty which arises in the solution of the fundamental problem of dynamic meteorology consists in the fact that the mathematical scheme (the algorithm) of the obtained solution must be presented in such a simple form that it can be applied in operational practice.

The immense difficulties, which have already been briefly mentioned and which we will discuss in greater detail in their proper place, resulted in the fact that weather forecasting has been considered hopeless until lately.

The first step in this field was made by Richardson. In his detailed investigations Richardson arrived at the not very comforting conclusion that the method of theoretical weather forecast is of little practical use.

However, the work of the Soviet scientist I. A. Kibel' showed that the weather forecast is by no means such a hopeless undertaking, even at the present state of our knowledge.

Moreover, Professor Kibel' proposed a method of weather forecast, based on the direct use of the equations of hydromechanics and thermodynamics; this method has been successfully developed by him and his pupils in the Central Institute for Weather Forecast.

Unlike previously proposed methods for weather forecasting, I. A. Kibel's method is an astute and original attempt of direct application of the equations of hydromechanics and thermodynamics to the weather forecast under real conditions, where the synoptic chart of the day in all its diversity must be used as initial conditions.

Our course of dynamic meteorology is devoted to the systematic exposition of the principles of the thermodynamics and the dynamics of the atmosphere, giving special attention to those problems which have been completely solved, and which have a great practical importance because they furnish a theoretical basis for modern synoptic meteorology.

Let us not forget that correct weather forecasts are possible only on the indispensable condition that the forecaster is able to explain and quantitatively evaluate the physical processes taking place in the atmosphere.

§ 2. The System of Units Used in Meteorology. Dimension

Every mechanical system is characterized by definite physical units, such as velocity, density, energy, force, etc. By virtue of their interrelations, the units characteristic of any mechanical system can be reduced to three fundamental ones. The choice of these fundamental units depends on the nature of the problem at hand.

Length, mass, and time are the fundamental units used in physics. Length, force, and time are used in technology. Other fundamental units may also be chosen.

The metric system of units is now widely accepted. The following are adopted by international convention: the unit of length – 1 meter – the length of a standard rod made of platinum – iridium and kept by the French Board of Weights and Measures; the unit of mass – 1 kilogram – the mass of a platinum–iridium standard, kept in the same place; the unit of time – 1 second – equal to $\frac{1}{24 \cdot 3600}$ of the mean solar day.

In the study of mechanical phenomena three independent fundamental units suffice: the units of length, mass and time; the units of other physical quantities can easily be expressed by these fundamental units.

In studying thermal phenomena use can also be made of these three units. For example, the quantity of heat can be expressed in the same units as mechanical energy. However, for some problems of thermodynamics it is more convenient to choose the unit of the quantity of heat independently of the unit of mechanical energy. The calorie is the accepted unit of heat; it is determined by experiment, independently of the mechanical unit. Degrees are introduced for measuring the temperature and they, too, are independent of the three fundamental mechanical units.

The expression of a certain unit by units accepted as fundamental give us the dimensions of this unit. We shall express the dimension symbolically by monomials, in which L designates length, M – mass and T – time.

Maxwell has proposed that the dimension of any quantity a should be expressed by the symbol [a]. Accordingly, the dimensions of the fundamental units are:

$$[\text{length}] = [L],$$

$$[\text{mass}] = [M],$$

$$[\text{time}] = [T].$$

For completeness we shall add a fourth quantity, the temperature, which is necessary in thermodynamics. We shall agree to express the dimension of temperature in the following way:

$$[\text{temperature}] = \theta.$$

All the derived physical units, encountered in meteorology, can be easily expressed by the fundamental ones: the length L, mass M, time T and temperature θ.

At present, dynamic meteorology makes use of two alternative systems of units. the CGS and MTS systems. The fundamental units in the CGS system are the centimeter, the gram-mass and the second. In the MTS system - the meter, ton-mass and second.

The CGS system is universal; it is most convenient in experimental physics, where mechanical quantities of the same order of magnitude as the fundamental units of this system are usually considered.

However, the dimensions of the mechanical systems investigated in meteorology are enormous as compared with the units of the CGS system, so that it is highly inconvenient for practical use in meteorology. The introduction of larger units of measure is understandable enough. This situation led the international conference of 1911 to adopt the MTS system for meteorological use. Nevertheless, even now no unanimity exists as to the use of either system in meteorology. In this book we will sometimes use CGS, sometimes the MTS units, depending on the character of the examined problem.

The dimensions of the most important physical magnitudes are given in Appendix 1 both in CGS and MTS units.

The problem of dimensions is of considerable importance in the formulation of quantitative relationships. In order that these relationships do not vary upon alteration of the fundamental units of measure, it is necessary that the dimensions of both sides of the equation should be identical. This condition is automatically fulfilled if the considered relationship is simultaneously a definition of the method of measurement of one of the magnitudes it includes. For example, Newton's second law not only relates mass, acceleration and force to each other, but also serves as a method for measuring one of these magnitudes. If this is not the case, and the method of measurement of the physical magnitudes is independent of the relationship in question, it is necessary to insert a special dimensional coefficient in this relationship, in order to ensure identical dimensions. For example, a dimensional coefficient is introduced in the equation of universal gravitation, the so-called gravitational constant. This can be explained by the fact that the dimensions of the physical magnitudes entering in the equation of universal gravitation, i. e., the masses, lengths, and forces, are determined independently of the gravitation law.

Dimensional considerations provide a good criterion for checking the correctness of an equation. Moreover, by the same method we can find an equation involving an incompletely known function of different physical magnitudes, assuming that no dimensional coefficients enter in the desired equation. We shall use this method in the study of turbulence for determining certain functions.

§ 3. Size and Shape of the Earth

The earth moves round the sun in an elliptic orbit with an average velocity of 29.8 km/sec, simultaneously revolving about its own axis from west to east with a constant angular velocity.

Let us calculate the angular velocity ω of the earth's rotation. The earth performs forms a complete rotation around its axis in a sidereal day; hence

$$\omega = \frac{2\pi}{\text{sidereal day}}. \qquad (1)$$

The sidereal day differs from the solar day, owing to the simultaneous revolution of the earth around the sun. A year equals nearly $365\frac{1}{4}$ solar days. That is to say, the earth turns $365\frac{1}{4}$ times around its axis relative to the sun. But the earth makes an additional revolution from west to east in absolute space while traveling its whole yearly orbit, so that it will turn $366\frac{1}{4}$ times relative to the fixed stars. Thus: 1 year = $365\frac{1}{4}$ solar days = $366\frac{1}{4}$ sidereal days. Consequently:

$$1 \text{ sidereal day} = \frac{365\frac{1}{4}}{366\frac{1}{4}} \text{ solar days},$$

and

$$\omega = \frac{2\pi \times 366\frac{1}{4}}{365\frac{1}{4}} = \frac{2\pi}{86164} = 7.292 \times 10^{-5}\ \text{sec}^{-1}.$$

The earth would have been spherical had it not revolved on its axis and had it been completely homogeneous, or composed of concentric homogeneous layers. But, owing to its rotation the earth was deformed into a spheroid, i. e., a slightly flattened ellipsoid of revolution, when it still was in a liquid state.

Notwithstanding, the true form of the earth is yet different to some extent from an ellipsoid of revolution owing to the heterogeneity of the earth's crust and to the nonuniform distribution of the masses of different densities in its interior. Consequently, the theoretical level surface of the earth, i. e., the surface to which gravity is perpendicular at each of its points, has a special form; this complicated form is called a geoid.

In practice the slight deviations of the geoid from the ellipsoid of revolution may be completely neglected. Furthermore, in the majority of meteorological problems it appears possible to neglect even the deviation of the earth's form from the spherical, and to consider the earth as being a sphere.

The basic proportions of the terrestrial spheroid, as accepted by the international conference of 1924 are given in Appendix 2.

§ 4. The Distribution of Gravity at the Earth's Surface and in the Atmosphere

Gravity is the most important force acting on the atmosphere. Therefore, it is necessary to know the distribution of gravity in the atmosphere and its point to point variation. We will consider the gravity in reference to a unit of mass or, in other words, the acceleration of gravity. Gravity is the resultant of two forces – the Newtonian force of gravitation and the inertial centrifugal force.

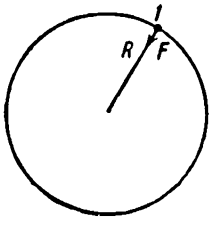

Figure 1. Gravity directed to the earth's center

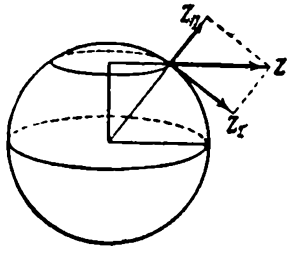

Figure 2. Resolution of the centrifugal force, arising as a result of the rotation of the earth, into normal and tangential components

Let us suppose the earth has the form of a sphere (Figure 1). We will consider a physical object of unit mass situated on the surface of this sphere. The Newtonian force of gravitation alone would have acted on this body if the earth were immobile. This force is directed to the center of mass, and its magnitude is proportional to the product of the interacting masses and inversely proportional to the square of the distance between them:

$$F = K \cdot \frac{M}{R^2}, \qquad (1)$$

where F is the force of gravity, M–the mass of the Earth, K–the gravitation constant, R–the earth's radius.

An inertial centrifugal force Z is developed by the rotation of the earth (Figure 2), directed along the radius of the circle of latitude emanating from the axis of rotation; this force is equal to

$$Z = r\omega^2,$$

where r is the radius of the circle of latitude, ω –the angular velocity of rotation of the earth. Since

$$r = R \cos \varphi,$$

where φ is the latitude of the locality, the centrifugal force is equal to

$$Z = R \cos \varphi \omega^2.$$

Let us resolve the centrifugal force Z into two components: one in the direction of gravitation, and the other normal to it. The first will evidently be the normal component Z_n, and the second – the tangential component Z_t relative to the surface of the earth.

These components have the following magnitudes:

$$Z_n = R \cos \varphi \cdot \omega^2 \cdot \cos \varphi = R \cos^2 \varphi \cdot \omega^2, \qquad (3)$$

$$Z_t = R \cos \varphi \cdot \omega^2 \cdot \sin \varphi = \frac{1}{2} R \sin 2\varphi \cdot \omega^2. \qquad (4)$$

The normal component of the centrifugal force, which is directed opposite to gravitation, causes an apparent decrease in the weight of the body. The body, situated on the surface of the spherical earth, tends to be displaced in the direction of the equator under the action of the tangential component of the centrifugal force. The form of an ellipsoid of rotation (spheroid) acquired by the earth can be explained by this fact. The surface of the spheroid is the level surface of gravity.

The resultant of the force of gravitation acting in the direction of the earth's center and of the centrifugal force, called gravity, is obviously perpendicular to the surface of the spheroid, and its tangential component (relative to the spheroid) vanishes. Gravity, however, is not directed to the center of the mass, as is the force of gravitation, but to some point situated nearer to the surface of the earth.

The value of the force of gravitation is different for different latitudes, because the earth has lost is spherical form ; this force has a maximum value at the poles, and a minimum on the equator. The centrifugal force also changes with latitude, being zero at the pole, and reaching a maximum value on the equator. Thus gravity is a function of latitude.

The following empirical equation, which describes the dependence on latitude of the gravity acceleration at sea level, was established on the basis of numerous gravity determinations at different latitudes:

$$g_\varphi = 980.616\,(1 - 0.002644 \cos 2\varphi + 0.000007 \cos 2\varphi)\ \text{cm} \cdot \text{sec}^{-2}$$

The number in front of the bracket is the "normal" gravity acceleration, i.e., the value corresponding to 45° latitude at sea level. Actually, $g_{45} = 980.616\ \text{cm/sec}^2$, by substituting $\varphi = 45°$.

For all the problems of dynamic meteorology the use of a simpler equation

$$g_\varphi = 980.6(1 - 0.0026\cos 2\varphi)\ \text{cm sec}^{-2} \qquad (5)$$

appears possible.

On the equator $\varphi = 0$, and the acceleration of gravity reaches a minimal value $g_0 = 980.6\ (1-0.0026)$; at the pole $\varphi = 90°$, and this acceleration reaches a maximal value $g_{90} = 980.6\ (1 + 0.0026)$, hence

$$\frac{g_{90} - g_0}{g_{45}} = 0.0052 = 5.2\,^0/_{00}.$$

It follows that the variation of gravity on the earth's surface at sea level is not great. This variation only amounts to 0.5% of its normal value. Therefore the variation of gravity with latitude can be neglected in the majority of cases.

Gravity also varies with altitude above sea level. Let us take two points: the point M_1, situated at the height z, and M—at sea level (Figure 3). Then the following equation can be written correct to terms of the second order:

$$\frac{g_z}{g_0} = \frac{R^2}{(R+z)^2}, \qquad (6)$$

hence

$$g_z = g_0 \frac{1}{\left(1+\frac{z}{R}\right)^2} \cong g_0\left(1 - 2\frac{z}{R}\right),$$

where we disregard the terms quadratic in the small magnitude z/R.

Replacing R by its value we obtain an equation, which determines the relation of the acceleration of gravity at any point to its height above sea level, at the same latitude:

$$g_z = g_0(1 - 3.14 \cdot 10^{-7} z), \qquad (7)$$

where z is expressed in meters.

It can be easily deduced that the acceleration of gravity varies within the limits of 1% of its value up to the height of 30 km.

Combining equations (5) and (7) we obtain an equation, which determines the acceleration of gravity at arbitrary latitudes and heights in the atmosphere:

$$g_{\varphi,z} = 980.6(1 - 0.0026\cos 2\varphi)(1 - 3.14 \cdot 10^{-7} z)\ \text{cm}\cdot\text{sec}^{-2}. \qquad (8)$$

It is to be noted that equations (7) and (8) are obtained for a free atmosphere. For points lying in mountainous territory we obtain instead of (8) another equation:

$$g_{\varphi,z} = 980.6\,(1 - 0.0026\cos 2\varphi)(1 - 1.96\cdot 10^{-7} z), \qquad (9)$$

which accounts for the gravitation of the mountains.

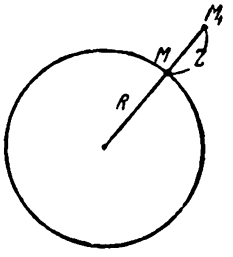

Figure 3. Gravity is inversely proportional to the square of the distance from the earth's center

In Appendix 3 are given the normal values of the acceleration of gravity, for different latitudes and heights above sea level.

§ 5. The Composition of the Atmosphere

As a rule, we will consider the atmosphere as composed of three different constituents:

1) dry and pure air
2) water vapor
3) atmospheric dust and other impurities.

Let us consider each of these atmospheric constituents separately.

1) Dry and pure air is a mechanical mixture of different gases.

Appendix 4 lists the likeliest composition of pure and dry air near the earth's surface, in accordance with contemporary knowledge. The two main constituents of this mixture – nitrogen and oxygen – form 99% of the air, both by volume and by mass, and 99.9% when including argon. The content of carbon dioxide in the atmosphere varies both with time and in space, depending on the properties of the underlying surface. The average content of carbon dioxide in the air amounts to 0.03% by volume. The concentration of other gases is negligible, as can be seen from the table.

The composition of pure and dry air does not vary with latitude and height, apart from the decrease with height of carbon dioxide and radium emanation, and the increase of ozone concentration. Samples of air taken during flights in aerostats, and by means of sounding balloons, make it possible to assert that the composition of dry and pure air in the atmosphere does not vary essentially up to heights of 20–30 km. The composition of air apparently varies above this altitude as a consequence of the increasingly important role of the lighter gases at very great heights. Anyhow, the upper layers of the atmosphere form as yet no object of investigation for dynamic meteorology.

The critical temperatures of the components of dry and pure air are so low that they are always in a gaseous state under conditions observed in the atmosphere; they can therefore be regarded as ideal gases. Xenon and carbon dioxide are exceptions. But their amount in the atmosphere is so small that they cannot be liquefied, and they too can therefore be regarded as ideal gases. Thus, the entire mechanical

mixture of gases which constitute dry and pure air, can be regarded as an ideal, i. e., a nonliquefiable gas.

2) In contrast to the permanent components of atmospheric air, the amount of water vapor in the atmosphere varies both in time and space, within very wide limits – from 0 to 4% by volume. Its critical temperature is 374°C. Therefore, water vapor is a real gas under atmospheric conditions; it can change its state of aggregation by turning into water drops or into crystals of ice. It nearly conforms to the laws of ideal gases if it is not saturated. On the other hand, saturated water vapor diverges noticeably from these laws. The processes of condensation and sublimation of water vapor, as well as the vaporization of the condensation products, largely determine the diversity of atmospheric processes. Furthermore, water vapor plays a leading part in the processes of absorption and emission of radiant energy.

3) Apart from the above-mentioned components of the atmosphere, admixtures of very fine suspended solid particles or molecules of gases may be found in it, in greater or lesser amounts. Among these admixtures both the hygroscopic gases (vapors of nitric acid HNO_3, sulfurous acid H_2SO_3, etc), and the hygroscopically active solid particles (smoke, grains of salt, etc) play a particularly important part. They initiate and promote the processes of condensation and sublimation of water vapor, forming nuclei of condensation and sublimation. The neutral dust particles, sometimes present in great amounts in the lower layers of the atmosphere, do not take an active part in the processes of vapor condensation and sublimation, but may play an important part both in the heating and cooling of the air as a result of radiant energy absorption and emission.

We will consider mainly the first two constituents of the atmosphere, disregarding the different admixtures contained in it.

§ 6. The Extent of the Atmosphere

The atmosphere surrounding the earth is composed of several gases, including water vapor (see § 5).

A fundamental property of every gas is the lack of forces of mutual attraction between the molecules. In contrast, the forces of mutual attraction in the solid and liquid states preponderate to such an extent that the body is unable to increase its volume spontaneously.

Every volume of gas tends to expand indefinitely if left to itself. This is explained by the fact that every gas is a chaotic assembly of independent molecules moving in different directions at enormous speeds and colliding with each other. Thus the molecules of oxygen at normal pressure and at a temperature of 15°C move at an average speed of 430 m/sec, whereby the number of collisions in a cubic millimeter equals $4 \cdot 10^{22}$ per second.

As a result of such a chaotic motion of the gas molecules, billions of them leave the original volume, in this way causing its expansion. The gases which compose the atmosphere also tend to leave the earth and disperse in outer space. But gravity counteracts this phenomenon. Only isolated, particularly rapid molecules a are able to get away from the field of gravity. Therefore the earth loses only a negligible part of its atmosphere.

The earth carries along its surrounding atmosphere, or rather it is the so-called "quiescent atmosphere" that participates in the continuous rotation together with the earth. The wind arises merely as a difference in the motion of the atmosphere and of the earth's surface. A systematic lag of the atmosphere in relation to the earth may be realized only at very great heights.

It is impossible to indicate the exact upper limit of the atmosphere. The earth's atmosphere gradually merges into outer space, which is not an absolute vacuum, but contains molecules of different gases. Consequently, the earth's atmosphere can be regarded as a local condensation of the interplanetary atmosphere.

As an outer limit of the atmosphere one can accept, from the theoretical point of view, the maximum height at which the atmospheric particles still participate in the rotation of the earth, and gravity still prevails over the centrifugal force, which tends to draw the atmosphere away from the field of gravitation.

Theoretical calculations show that the atmosphere extends away from the poles a distance of no more than 28,000 km, and at the equator over 42,000 km. It thus forms a spheroid which is more oblate than the earth. This calculation determines the limits past which the atmosphere can not possibly extend.

However, no phenomenon can be indicated that would prove the presence of atmosphere at such great heights. Only at heights not exceeding 2,000 km is the density of the atmosphere sufficient to reveal its presence by weak optical phenomena which are sometimes observed from the earth's surface.

At heights of 500 to 1,000 km aurorae polaris in the form of weak uniform arches are observed. Ignitions of meteors are sometimes detected up to a height of 300 km.

However, the density of the air ρ is insignificant at these heights, Thus at the height of 300 km ρ is equal to $6.7 \cdot 10^{-27}$ g/cm^3, whereas at the height of 20 km $\rho = 0.9 \cdot 10^{-4}$ g/cm^3 and at the earth's surface, $\rho = 1.3 \cdot 10^{-3}$ g/cm^3 (Appendix 5).

The concept "mass of the entire atmosphere" also has no precise meaning. However, on the basis of indirect considerations it is estimated at $5.26 \cdot 10^{21}$ g which is approximately one millionth part of the earth's mass.

On the basis of calculations it has been shown that nearly 50% of the earth's atmosphere is contained within a layer extending to 5 km above the earth's surface; 75% within the layer up to 10 km, 95% – within the layer up to 20 km, and more than 99% within the layer up to 40 km.

It is generally assumed that the atmosphere lying higher than 25–30 km has no direct dynamic influence on the air layers close to the earth's surface. Therefore, in dynamic meteorology the atmosphere can be regarded as a thin layer 20-25 km thick, which envelopes the globe.

This circumstance has a singular effect on the nature of the principal processes occurring in the lower layers of the atmosphere. Namely, all these processes appear to be protracted along the horizons, and flattened in the vertical direction. Thus, the cyclones and anticyclones occupy vast regions of many hundreds and even thousands of kilometers along the horizons, but extend merely a few kilometers in the vertical direction. This peculiarity of the meteorological processes is of essential value for their understanding and study.

Chapter II

THE THERMODYNAMICS OF IDEAL GASES

§1. The Thermodynamics of the Atmosphere and Physical Thermodynamics

The thermodynamics of the atmosphere is based on the results obtained by physical thermodynamics.

The subject matter of physical thermodynamics is the study of those physical phenomena which are statistical consequences of events taking place in the microcosm (the universe of molecules, atoms, etc). Typical examples of such physical phenomena are cooling and heating which are accompanied by a change of velocity of the elementary particles of matter, a change in the state of aggregation (evaporation, condensation, fusion, crystallization, etc); diffusion, absorption etc (i. e., an irregular penetration of the molecules of one substance into the assembly of molecules of another substance).

Unlike the mechanical system, which may consist of any number of particles (down to two [or even one]), the thermodynamic system must consist of a sufficiently large number of particles to ensure an equalization of random events occurring in the microcosm.

However, within the limits of the heights investigated by dynamic meteorology, even very small volumes of air meet this requirement. Only in the very high layers of the atmosphere does the number of elementary particles contained in a unit volume become so small that statistical laws can no longer be applied, and the conclusions of the thermodynamics cease to be accurate, because they are not applicable to individual molecules or to an insufficiently large number of molecules.

§2. The First and Second Laws of Thermodynamics

From a great number of results obtained by experiments, thermodynamics selects two or three, the most reliable, which are fundamental and of quite general application. Then, by strictly logical reasoning, a series of particular rules are derived, enabling one to predict the different properties of substances, and the characteristics of processes taking place in them. The development of thermodynamics as a separate science was initiated by the discovery of the two fundamental laws which form its empirical basis. These are therefore called the laws of thermodynamics. Many different formulations exist of the first and particularly of the second law of thermodynamics, from among which we will make use of the following:

The first law: generation and destruction of energy are impossible.

The second law: a process which results solely in the transformation of heat into work cannot take place.

Let us consider the concepts of perpetual motion machines of the first and second kind, in order to appreciate the importance of these laws. The perpetual motion machine of the first kind is one that produces more work than the energy which it consumes from without, i. e. that creates energy. The perpetual motion machine of the second kind is one which is capable of transforming into work all the energy absorbed in the form of heat from any source (heater). The perpetual motion machine of the first kind would have allowed to obtain energy from nothing, and the perpetual motion machine of the second kind—to master the practically unlimited natural sources of heat. Thus, a perpetual motion machine of the second kind would be able to set in motion all the mills and factories of the globe by transforming into work the heat of the oceans, thereby decreasing the temperature of the water in them by one hundredth of a degree after 1700 years.

A great number of attempts to construct perpetual motion machines of the first and second kinds have been made. Naturally, all these attempts have failed. The laws of thermodynamics provide the reason for these failures.

From the first law of thermodynamics follows the impossibility of constructing a perpetual motion machine of the first kind.

From the second law of thermodynamics follows the impossibility of constructing a perpetual motion machine of the second kind.

The mathematical expressions of the first and second laws of thermodynamics will be given below.

§ 3. The Parameters of State of a Thermodynamic System

We will study the following thermodynamic systems which take part in the thermodynamics of the atmosphere.

1) Dry air, which can practically be regarded as a physically homogeneous medium (single component system).

2) Pure water in one, two, or three phases. This thermodynamic system can be homogeneous if the water appears in one phase, and nonhomogeneous, if two or three phases are present (it is also a single component system in one, two, or three phases).

3) Unsaturated moist air; this, like dry air, can be regarded as a homogeneous medium (a two component system: dry air + water vapor).

4) Moist air containing water in a liquid or solid phase together with water vapor (a two component system: dry air + water in two or three phases).

The quantitative variables, which determine the physical properties of any thermodynamic system, are called parameters or characteristics of state of this system. Any element of the thermodynamic system can be characterized by four variables: the mass δM, the volume $\delta\tau$, the pressure p and the temperature T. Thus, in addition to the mechanical variables δM, $\delta\tau$, p, a specific thermodynamical variable—the temperature—is introduced to characterize the state of the thermodynamic system. The system should be in a mechanical and thermal equilibrium for the temperature to be exactly determined. Thus thermodynamics is concerned only with systems which are in equilibrium or in a state infinitesimally removed from equilibrium. Hence, in order to apply thermodynamical analysis to the atmosphere, we shall take it to be in a state of equilibrium or quasi-equilibrium.

§4. Mass, Volume and Density

We shall assume that each thermodynamic system is composed of the same* particles. Therefore the total mass of the system, remaining constant, is not considered a parameter of state, although it is taken into account during calculations. The masses of the separate components (2) and (4) of the system will change with the change of the phases; therefore these masses are parameters of state.

The mass δM is expressed in grams, kilograms, tons or moles. A mole is that quantity whose weight in grams equals the molecular weight of the substance. The volume is measured in cm^3, liters, m^3. However, it is more convenient to use the specific volume v, instead of the total volume $\delta\tau$

The specific volume is the volume occupied by a unit mass, or in other words, the ratio of the volume occupied by the system, to its mass:

$$v = \frac{\delta\tau}{\delta M}; \qquad [v] = [L^3 M^{-1}].$$

The specific volume is measured in cm^3/g, liter/g, m^3/kg, m^3/ton. Under normal conditions (p = 760 mm Hg, t = 0°C) the specific volume of dry and pure air v_d equals

$$v_d = 773.4 \frac{cm^3}{g} = 0.7734 \frac{liter}{g}.$$

Thus, one gram of dry air occupies approximately 3/4 liter under normal conditions.

Sometimes the molar volume is also used. The molar volume is the volume occupied by one mole of the substance. According to the empirical law of Avogadro, the molar volume is equal for all gases under the same conditions. The molar weight of dry and pure air is 28.97. Therefore the molar volume of dry and pure air, and hence of any other ideal gas under normal conditions, equals $28.97 \times 0.7734 = 22.40$ liter/mole.

In dynamic meteorology, instead of the specific volume, its reciprocal value —the density— is usually used as a parameter of state.

The density is the mass of substance contained in a unit volume or, in other words, the ratio of the mass of the substance to its volume:

$$\rho = \frac{\delta M}{\delta\tau}; \qquad [\rho] = [ML^{-3}].$$

The unit of mass adopted in the CGS system is the gram, i. e., the mass of one cubic centimeter of chemically pure water at 4°C; the unit of volume is 1 cm^3.

The density of water at 4°C thus serves as the unit of density.

Sometimes the density of dry and pure air, under normal conditions, is adopted as a unit of density (p = 760 mm Hg; t = 0°C). This value is equal to $0.001293 \ g \cdot cm^{-3}$.

We can estimate the mass of a substance by its weight, and the density by the specific weight. Therefore in practice the specific weight is always determined and the density is taken to be equal to it, because the numerical values of the density and the specific weight practically agree.

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* [Translator's note—apparently the meaning is that each system remains composed of the same particles throughout its evolution].

It is evident that the specific volume and the density are mutually related by

$$v \cdot \rho = 1.$$

It follows from Avogadro's law that the ratio of the densities of gases measured under the same conditions is equal to the ratio of their molecular weights.

§5. Pressure

Pressure is the normal component of forces uniformly distributed over the surface of a body; it is directed towards the interior of the body. Pressure is the force acting on a unit surface. The pressure compressing the body is considered to be positive. The pressure exerted by gases is also called gas tension.

The kinetic theory of gases explains pressure as being the result of impingements of the molecules on the surface confining the gas.

Experiment shows that the pressure of motionless air acts equally in all directions. Therefore the pressure of motionless air can be regarded as a scalar.

Designating by ds the surface element and by dF the elementary force of pressure, we obtain

$$p = \frac{dF}{ds}, \qquad [p] = [ML^{-1}T^{-2}].$$

The bar is adopted as a unit of pressure in the CGS system; it is equal to a pressure of 1 dyne on a surface of 1 cm^2. However, it is a small unit.
V. Bjerknes introduced a unit of pressure one million times greater which he also called bar:

$$1 \text{ bar} = 10^6 \text{ dyne cm}^{-2}$$

the $\frac{1}{1000}$ th part of a bar is called a millibar (mb)

$$1 \text{ mb} = 0.001 \text{ bar} = 10^3 \text{ dyne cm}^{-2}.$$

In use are also:

$$1 \text{ centibar} = 10^{-2} \text{ bar} = 10 \text{ mb} = 10^4 \text{ dyne cm}^{-2}$$
$$1 \text{ decibar} = 10^{-1} \text{ bar} = 100 \text{ mb}$$

The centibar is used in the MTS system as a unit of pressure.

The pressure of air is usually measured in mm Hg. However, the height of a column of mercury does not determine unambiguously the pressure exerted on it. It is necessary to take into consideration the temperature of the mercury and the local acceleration of gravity at the site of observation.

Therefore it is necessary to convert the measured height of the mercury column in the barometer to its height at standard conditions, i. e., standard temperature and standard gravity, in order to obtain a measure of the pressure which is independent of the particular measurement.

Moreover, the scale measuring the height of mercury in the barometer gives the exact number of millimeters only at a definite temperature (usually at 0°C).

Therefore the height b_t of the mercury in the barometer at a temperature t is each time reduced to its height at 0°C according to the equation:

$$b_0 = b_t[1 - (\beta - \gamma) \cdot t],$$

where β and γ are respectively the coefficients of expansion of the mercury and of the scale.

The mercury column of height b_t mm and cross section 1 cm^2 will then have a mass of

$$13.595 \cdot \frac{b_t}{10} \cdot [1 - (\beta - \gamma)] = 1.3595 b_0.$$

Here $b_0 = b_t + \delta_t$ where b_0 is the barometer reading reduced to 0°C and δ_t the correction of the barometer reading for the temperature. This quantity of mercury exerts a pressure of:

$$p = 1.3595 b_0 g_{\varphi, z},$$

where $g_{\varphi, z}$ is the acceleration of gravity at latitude φ and height z; it is determined by the equations (8) and (9) of § 4, Chapter I.

Restricting ourselves, for conciseness, to the first case we obtain:

$$p = 1.3595 \cdot 980.616 (1 - 0.0026 \cos 2\varphi) \cdot (1 - 3.14 \cdot 10^{-7} z) b_0,$$

or

$$p = 1333.19 \cdot b \frac{\text{dyne}}{\text{cm}^2} \tag{1}$$

where

$$b = b_0 (1 - 0.0026 \cos 2\varphi)(1 - 3.14 \cdot 10^{-7} z)$$

this is "the corrected barometer reading", expressed in millimeters. Correct to small second order terms it can be written as:

$$b = b_t + \delta_t + \delta_\varphi + \delta_z; \tag{2}$$

where b_t is the observed height of mercury in the barometer.

δ_t –the correction of the barometer reading for the temperature

$$\delta_t = -(\beta - \gamma) b_t \cdot t, \tag{3}$$

δ_φ –the correction for the latitude of the locality

$$\delta_\varphi = -0.0026 \cos 2\varphi b_t, \tag{4}$$

δ_z –the correction for the height above sea level

$$\delta_z = -3.14 \cdot 10^{-7} \cdot z \cdot b_t. \tag{5}$$

Apart from the corrections considered above, each barometer has its own "instrumental" correction, which is obtained by comparing its readings with the readings of a standard barometer.

On account of the great number of corrections which must be applied to the readings of a barometer, it seems useless to divide the barometer scale into

millibars, as has been proposed. The division of the scale into millibars would not eliminate the necessity for introducing the above-mentioned corrections, and would only introduce additional confusion. In a barometer one merely observes the height of the column of mercury, balancing the pressure. Therefore it is natural enough to express the pressure in units of length, i. e., in millimeters. The value of the pressure corresponding to this reading is determined by a subsequent calculation.

It is not difficult to see that normal pressure, exerted by a column of mercury 760 mm high, having a temperature of t = 0°C, and placed at sea level and at a latitude $\varphi = 45°$ is equal to

$$p = 1333.19 \cdot 760 = 1\,013\,226 \frac{\text{dyne}}{\text{cm}^2}$$

Conversely, the pressure of 1 bar $= 10^6 \frac{\text{dyne}}{\text{cm}^2}$ corresponds to a height of the barometric mercury column $b = \frac{10^6}{1333.19} = 750.08$ mm Hg.

Thus, between the two examined units of pressure exists the following relationship:

$$1 \text{ bar} = 750.08 \text{ mm Hg} \quad 760 \text{ mm Hg} = 1.013226 \text{ bar},$$
$$1 \text{ mb} = 0.75 \text{ mm Hg} \quad 1 \text{ mm Hg} = \tfrac{4}{3} \text{ mb}.$$

In order to convert the pressure from millibars to mm Hg, use can be made of tables and graphs, the plotting of which is obvious.

§6. Temperature

The temperature of a body can be determined by the readings of a thermometer, i. e., an instrument which is in thermodynamic equilibrium with the body in question. In meteorological stations the air temperature is usually measured by liquid (mercury or alcohol) thermometers, in which the volume of the liquid indicates its heat content. The readings of the thermometer, however, depend on the substance whose expansion is used to measure the temperature. Therefore the readings of mercury and alcohol thermometers differ slightly. This is due to the variations in the coefficient of expansion from substance to substance.

Experiment shows, however, that gases, in particular those not easily liquefiable (hydrogen, helium, oxygen, nitrogen), expand in a similar manner when heated: their volume increases at 0°C (at constant pressure) by 1/273rd part of this volume, if the temperature is raised by 1°C. If heating is carried out at a constant volume, the gas pressure varies in the same manner. Therefore the gas thermometer, and in particular the hydrogen thermometer, is the best.

The unit of temperature is the degree-a special unit which cannot be expressed in terms of the other fundamental units.

Three scales for temperature measurement exist at the present time: 1) the Celsius scale-°C; 2) the Fahrenheit scale-°F; 3) the Kelvin scale-°K. The following are adopted as the fundamental points of the Celsius scale: the melting point of pure ice at normal pressure, marked at 0°C, and the boiling point of pure water at normal pressure, marked at 100°C. The interval between these fundamental points is divided into 100 equal parts, called Celsius degrees.

The melting point of ice corresponds to 32°F, and the boiling point to 212°F.

The interval between the fundamental points of the Fahrenheit scale is thus divided into 180 parts, called Fahrenheit degrees.

In Kelvin's scale, also called the absolute scale, 0°K is fixed as that temperature at which an ideal gas would contract to zero volume. This temperature equals −273°C (exactly, −273.16°C) and is called the absolute zero. The melting point of ice corresponds to 273°K, and the boiling point of water to 373°K.

It is obvious that a Kelvin degree is equal to a Celsius degree. It is to be noted that the temperature on Kelvin's scale is essentially positive; it is denoted by T.

The exact definition of temperature is one of the most important achievements of thermodynamics. Kelvin showed that on the basis of the second law of thermodynamics one can give a precise thermodynamic definition of the temperature, independent of the properties of the substance; the thermodynamic scale thus obtained coincides with the scale of the hydrogen thermometer.

In meteorology and in everyday life the Celsius scale is mostly used. The Fahrenheit scale is widely used in the Anglo-Saxon countries (England, the United States, Canada, Australia, India). The Kelvin scale is particularly suitable for theoretical work.

The following obvious relationships give the conversion factors of degrees of the different scales

$$1^\circ\,\mathrm{C} = 1^\circ\,\mathrm{K} = \frac{9}{5}\,^\circ\,\mathrm{F},$$

$$1^\circ\,\mathrm{F} = \frac{5}{9}\,^\circ\,\mathrm{C} = \frac{5}{9}\,^\circ\,\mathrm{K}.$$

Finally, Table 1 gives equalities used to convert the temperature from any scale into another.

Table 1

The conversion of temperature from one scale into another

To determine / Given	t° C	x° F	T° K
t° C	t° C	$(9/5\,t + 32)^\circ$ F	$(273 + t)^\circ$ K
x° F	$5/9\,(x - 32)^\circ$ C	x° F	$[273 + 5/9\,(x - 32)]^\circ$ K
T° K	$(T - 273)^\circ$ C	$[9/5\,(T - 273) + 32]^\circ$ K	T° K

§7. The Equation of State

The three parameters of state which we have examined are dependent on each other.

Above we have made use of the relationship between volume and temperature in order to measure the temperature. Experiment shows that only two of the parameters can be arbitrarily assigned, the third being uniquely determined from the relationship called the equation of state.

The general form of the equation of state is the following:

$$f(p, v, T) = 0, \tag{1}$$

so that the equations

$$p = p(v, T), \quad v = v(T, p), \quad T = T(p, v) \tag{2}$$

can be used to determine one of the parameters, if the other two are given. Equation (1) can be represented by a surface of state in a system of coordinates, p, v, T (see Figure 7). Any point situated on this surface characterizes a possible state of the system. Any point not situated on this surface represents a physically unattainable state. The transition of the system from one state to another corresponds to a change in the position of the point on the surface.

We will examine here the equation of state only for ideal gases, because the majority of the gases which form dry air are practically ideal gases under atmospheric conditions.

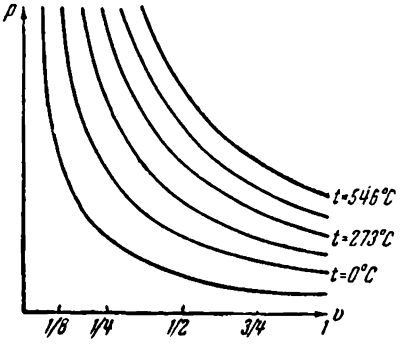

Figure 4. The isotherms in the p–v plane, according to the law of Boyle–Mariotte

The gases follow the Boyle–Mariotte law, when the process is isothermic (T = const).

$$p \cdot v = p_0 \cdot v_0 = \text{const}. \tag{3}$$

Graphically, the Boyle–Mariotte law is represented in the p, v–plane by a set of equilateral hyperbolas corresponding to the isotherms (Figure 4).

An isobaric process (p = const) follows the Gay–Lussac Law:

$$v = v_0(1 + \alpha t) \quad \text{or} \quad \frac{v}{T} = \frac{v_0}{T_0}, \tag{4}$$

where $\alpha = \frac{1}{273} = 0.00366 \text{ degree}^{-1}$ is the coefficient of volume expansion for all gases.

Finally, the isochoric process (v=const) also follows the Gay–Lussac law in the following form:

$$p = p_0(1 + \alpha t) \quad \text{or} \quad \frac{p}{T} = \frac{p_0}{T_0}, \tag{5}$$

where α has the same numerical value.

The graphs of equations (4) and (5) appear as parallel sets of straight lines, which pass respectively through the origins of the (v, T) and (p, T) planes (Figures 5 and 6).

During an arbitrary process, when the temperature, the pressure and the specific volume vary simultaneously, an equation of state of a more general form, Clapeyron's equation:

$$p \cdot v = RT, \tag{6}$$

is applied, where R is the specific gas constant, depending on the properties of the gas:

$$[R] = [L^2 T^{-2} \theta^{-1}].$$

Let us assume that the initial state of the gas is p_o, v_o, T_o and its final state p, v, T. We choose any intermediate state in such a way that the temperature in this state equals the initial one T_o, and the pressure equals the final pressure, p. The volume in the intermediate state will be denoted by v'.

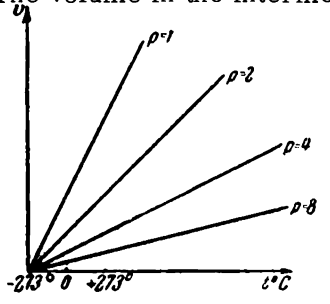

Figure 5. The isobars in the v–T plane, representing the Gay–Lussac law

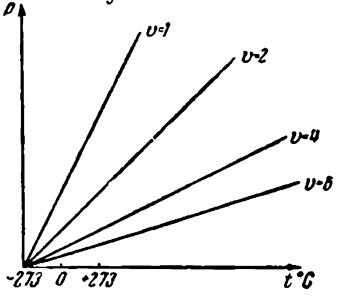

Figure 6. The isochores* in the p–T plane, representing the Gay–Lussac law

Applying the Boyle–Mariotte law to the initial and intermediate states, and then the Gay–Lussac law to the intermediate and final ones, we obtain:

$$\left.\begin{array}{lll} p_0 & v_0 & T_0 \\ p & v' & T_0 \\ p & v & T \end{array}\right\} \quad \left.\begin{array}{l} p_0 v_0 = p v' \\ \dfrac{v}{T} = \dfrac{v'}{T} \end{array}\right\}$$

or by eliminating v',

$$\frac{p \cdot v}{T} = \frac{p_0 v_0}{T_0} = R \tag{7}$$

which is equivalent to (6).

Now we can determine the gas constant for dry air. Assuming p_o = 760 mmHg = = 1,013,226 dyne/cm^2; T_o = 273.16° K, v_o = 773.4 cm^3/g, we obtain:

$$R = \frac{1\,013\,226 \cdot 773.4}{273.16} = 2.87 \cdot 10^6 \frac{\text{cm}^2}{\text{sec}^2\ \text{deg}} = 287 \frac{\text{m}^2}{\text{sec}^2\ \text{deg}}$$

(see Appendix 6).

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* [Translator's note–the Russian text reads isotherms, an obvious error].

§8. Mixtures of Ideal Gases

It is not necessary to determine the specific volume of the gas in order to calculate the specific gas constant. The knowledge of the molecular weight μ of the gas is sufficient. Let us write Clapeyron's equation for one mole of gas:

$$\mu \cdot p \cdot v = \mu R T. \tag{1}$$

According to Avogadro's law, the molar volume $\mu \cdot v$ is equal for all under the same conditions. The product

$$\mu R = R_0, \tag{2}$$

which is a constant, equal for all the ideal gases, is called the universal gas constant. Since

$$p_0 = 1\,013\,226 \frac{\text{dyne}}{\text{cm}^2}\ ;\ T_0 = 273.16^\circ \text{K},\ \mu v = 22\,400,$$

we have

$$R_0 = \frac{p_0 \cdot \mu v_0}{T_0} = \frac{1\,013\,226 \cdot 22\,400}{273.16} = 8.314 \cdot 10^7 \frac{\text{cm}^2}{\text{sec}^2 \text{deg}} = 8\,314 \frac{\text{m}^2}{\text{sec}^2 \text{deg}}.$$

Now we can write Clapeyron's equation in the form:

$$p \cdot v = \frac{R_0}{\mu} \cdot T \tag{3}$$

or, by introducing the density,

$$p = \frac{R_0}{\mu} \cdot \rho \cdot T. \tag{4}$$

Because dry or moist air is a mixture of different gases we shall explain the method of calculating the gas constant of gas mixtures.

Let us suppose that M grams of a gas mixture occupy a volume V, the components of the mixture being characterized by the following values:

Gas	1,	2, ... , N
Mass	M_1,	M_2, ... , M_N
Molecular weight	μ_1,	μ_2, ... , μ_N
Specific gas constant	R_1,	R_2, ... , R_N
Partial pressure	p_1,	p_2, ... , p_N

Let $\sum_{i=1}^{N} M_i = M$. According to Dalton's empirical law, each ideal gas expands among the other gases as though in a vacuum, providing there is no chemical interaction; we can therefore write:

$$p_i V = M_i R_i T. \tag{5}$$

The total pressure of the mixture, again according to Dalton's law, is equal to the sum of the partial pressures

$$p = \sum_{i=1}^{N} p_i.$$

By summing the equations (5) for all the components of the mixture, we obtain

$$p \cdot V = \left(\sum_{i=1}^{N} M_i R_i \right) \cdot T \tag{6}$$

or by introducing the specific gas constant of the mixture as the weighted mean

$$M \cdot R = \sum_{i=1}^{N} M_i R_i, \tag{7}$$

we obtain

$$p \cdot V = MRT; \tag{8}$$

finally by introducing the specific volume $v = \frac{V}{M}$, we find

$$p \cdot v = RT. \tag{9}$$

In this way we obtain the following rule: the equation of state for a mixture of ideal gases has the same form as Clapeyron's equation for a single ideal gas, the specific gas constant of the mixture being defined as a mean of the gas constants of the components. Let us calculate the value of the specific gas constant for pure and dry air. Using the data given in Appendix 4, we obtain according to equation (7):

$$R = \frac{R_0}{M}\left[\left(\frac{M}{\mu}\right)_{N_2} + \left(\frac{M}{\mu}\right)_{O_2} + \left(\frac{M}{\mu}\right)_{Ar} + \left(\frac{M}{\mu}\right)_{CO_2}\right] =$$

$$= 8.314 \cdot 10^5 \left[\frac{75.52}{28.016} + \frac{23.15}{32.000} + \frac{1.28}{39.944} + \frac{0.05}{44.000}\right] = 2.87 \cdot 10^6 \ \frac{\text{cm}^2}{\text{sec}^2\text{deg}}.$$

Knowing R and R_0 it is easy to obtain the molecular weight of the gaseous mixture. For dry and pure air

$$\mu = \frac{R_0}{R} = \frac{8.314 \cdot 10^7}{2.87 \cdot 10^6} = 28.97.$$

§9. The Equations of Change of the Gas State

Clapeyron's equation connects the three parameters of state and by means of the coordinates p, v, T represents the thermodynamic surface of state of the ideal gas (Figure 7).

This surface intersects the planes T = const along hyperbolas, corresponding to the Boyle–Mariotte law (3) § 7. On the other hand, the intersections of this surface with the planes p = const are straight lines, corresponding to Gay–Lussac's law (4) § 7. Finally the intersections of this surface with the planes v = const are straight lines, again corresponding to Gay–Lussac's law (5) § 7. Any state of the ideal gas will be represented by a point situated on this surface, denoted by M. The change of state of the gas will be represented by a curve M . . . N, lying entirely on the surface. The final state of the gas can be attained as a result of an infinite number of individual processes.

The iso–processes, taking place when one of the parameters of state remains constant, are of special interest. Thus, a gas can be brought from state M to state N as a result of the following pairs of iso–processes:

1) the isothermic (MK) and the isochoric (KN);

2) the isothermic (MQ) and the isobaric (QN);

3) the isochoric (MP) and the isothermic (PN);

4) the isobaric (ML) and the isothermic (LN).

In the derivation of Clapeyron's equation we assumed that the gas first changes its state isothermically (MQ), and then isobarically (QN), the intermediate state being represented on Clapeyron's surface by the point Q. Clapeyron's equation is not concerned with the process giving rise to a change of state, i. e., a "thermodynamic transition". This equation only shows that whatever the process, it must take place in such a way that both the point representing the state of the gas and the curve representing the variation of this state should lie on the thermodynamic surface.

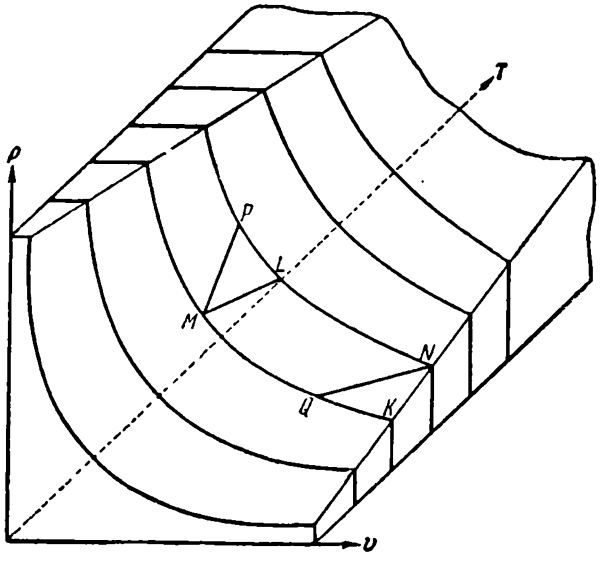

Figure 7. The thermodynamic surface of state in the p, v, T, space corresponding to Clapeyron' s equation

We shall encounter the following problem further on.

Any gaseous body is characterized in its initial state by the three parameters (p_o, v_o, T_o), connected by Clapeyron's equation. Under an imposed subsidiary condition only one of these parameters, (not two, as before) may vary arbitrarily. The condition can be, for example, that the process should evolve without an exchange of heat with the surrounding medium, or, on the contrary, that a supply or loss of heat should take place. It must be shown what values would the variables (p, v, T) assume for every future state. This subsidiary condition will determine the particular thermodynamic transition which the air mass will undergo, when passing from the initial state to the new arbitrary one. The analytic expression of this subsidiary condition is called the equation of the thermodynamic path or briefly the equation of the path .

Thus, we have two equations: the equation of state

$$f(p, v, T) = 0 \tag{1}$$

and the equation of the thermodynamic path

$$\Phi(p, v, T) = 0, \tag{2}$$

containing three parameters of state. One of the parameters can obviously be eliminated. One equation then remains, containing only two variables. This equation is called the equation of change of state of the gas. The curve obtained as the intersection of the surface of state with the surface determined by the equation of the thermodynamic path is called the curve of change of state.

In the above-mentioned examples of iso-processes, the equations $T = \text{const}$, $p = \text{const}$, $v = \text{const}$ serve correspondingly as equations of the thermodynamic path, whereas the Boyle-Mariotte and Gay-Lussac laws give the equations of the change of state. The first law of thermodynamics, in a more general case, assumes the form of the equation of the thermodynamic path.

§10. The Quality of Heat. Specific Heat

Any gaseous body possesses, as does any other physical body, some store of energy. The energy of the gas increases, if it absorbs energy from without. On the other hand, its store of energy decreases if it transfers a part of its energy to another body. The transfer of energy from one body to another can occur when work is done, or heat is carried off.

In doing work, one of the bodies develops forces, which act on the other body; in this case the energy is transferred by means of the direct macroscopic action of one body on another.

During heat exchange, one body transfers energy to the other body by radiation, or by thermal conduction, or by change of the state of aggregation, i. e., by means of microscopic action. Three magnitudes—energy, work and heat—can be expressed in the same units. However these expressions are not equivalent. When work is discussed, a process comes first to mind. A reserve of potential, as yet unaccomplished work is envisaged when referring to energy. Finally, any sort of stored energy is sometimes meant, when talking about heat. This erroneous notion survives from the time when the caloric theory was predominant in physics. In fact, heat is a form of energy transferable by means of microscopic physical interactions.

Thus, heat transfer and work are not stores of energy contained in the body, but different modes of energy transference from one body to another.

The calorie is the unit of quantity of heat, i. e., the quantity of energy supplied during the heat-transfer. One calorie is the quantity of heat required to raise the temperature of 1 gram of water from 14.5°C to 15.5°C.

If a body receives energy by heat transfer, and work is produced at the expense of this energy, then we may briefly say that the heat is converted into work. The quantity of work, obtained from the conversion of a unit of heat, i. e. of 1 calorie, is called the mechanical equivalent of heat and has been designated by the letter J in honor of Joule. It has been determined by careful experiments as:

$$J = 4.186 \times 10^7 \ \text{erg/cal} .$$

The reciprocal of the mechanical equivalent of heat is called the thermal equivalent of work, denoted by the letter A:

$$A = \frac{1}{J} = 2.389 \times 10^{-8} \frac{\text{cal}}{\text{erg}} .$$

If a quantity of energy δQ is supplied to a body in the form of heat (by heat

transfer), the temperature of this body will then change by δT; the ratio $\frac{\delta Q}{\delta T}$ is called the thermal capacity C of the body:

$$C=\frac{\delta Q}{\delta T}, \quad [C]=[ML^2T^{-2}\theta^{-1}]. \tag{1}$$

The thermal capacity per unit mass is called the specific thermal capacity [or specific heat] c of the body:

$$c=\frac{\delta q}{\delta T}, \quad [c]=[L^2T^{-2}\theta^{-1}], \tag{2}$$

where δq is the quantity of energy supplied by heat transfer to a unit of mass of the body.

However, the quantity of heat δq, necessary to heat a unit mass of a body, and consequently the thermal capacity of the body, depends on the conditions under which the heating takes place. The mere indication of the increase of the body's temperature, e. g., from t°C to (t + 1)°C still does not define the change of state which the body has undergone. The state of a body is determined by two parameters, of which one can be the temperature, and the second any other magnitude x which is free to vary independently of t, such as, in particular, x = p or x = v. The specific heat at p = const or v = const, i. e., the specific heats c_p at constant pressure and c_v at constant volume are defined by:

$$c_p=\left(\frac{\delta q}{\delta T}\right)_{p=\text{const.}}; \quad c_v=\left(\frac{\delta q}{\delta T}\right)_{v=\text{const.}} \tag{3}$$

The specific heat c_p is almost always meant when referring to the specific heat of solid or liquid substances. Both sorts of specific heat are encountered in the thermodynamics of the atmosphere.

§11. The Equation of the First Law of Thermodynamics

The experimental determination of the mechanical equivalent of heat has corroborated the first law of thermodynamics; the essence of this law can be expressed by the fact that the energy of any process of nature is not destroyed, and cannot arise from nothing, but is only converted from one form into another.

Let us express the first law of thermodynamics in mathematical form Energy absorbed by a body can then be converted into another form, under appropriate conditions. We shall restrict ourselves to the thermal and mechanical changes in a body. Their action may 1) change the body's volume, 2) change the body's temperature, 3) change the molecular structure of the body (for example, its state of aggregation).

The volume of a body generally increases by heating. The change of volume δV can be expressed as a product of the body's surface area S and the distance δn, taken along the normal from the initial to the expanded surface of the body (Figure 8). If an external pressure p acts on the surface of the body, the body will perform during its expansion a work δW, against the external pressure:

$$\delta W=p\cdot S\cdot\delta n=p\delta V; \tag{1}$$

a part of the body's energy will be spent in the work of expansion (1). The work per unit of mass will be denoted by δw. Then

$$\delta w=p\delta v. \tag{2}$$

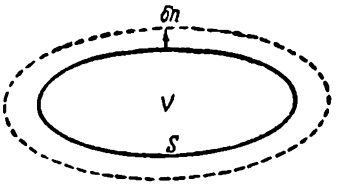

Figure 8. The work performed by heat against external pressure during expansion

From molecular physics we know that the temperature of a body characterizes the velocity of its molecules. The molecules in motion have a certain kinetic energy. Besides, every body also has potential energy, determined by the form of arrangement of its molecules.

The sum of these two forms of molecular energy is called the internal energy U. The kinetic and the potential energies of the molecules both vary when additional energy is supplied to the body, and consequently the internal energy of the body varies, too. Disregarding the other changes in the body's state which may arise (for example, electric and magnetic changes) we obtain the following equation, expressing the mathematical content of the first law of thermodynamics

$$\delta Q = \delta U + p\delta V \tag{3}$$

or, by referring all the magnitudes to a unit of mass

$$\begin{gathered} \delta q = \delta u + p\delta v, \\ [\delta Q] = [\delta U] = [\delta W] = [ML^2T^{-2}], \\ [\delta q] = [\delta u] = [\delta w] = [L^2T^{-2}]. \end{gathered} \tag{4}$$

§12. The Internal Energy of an Ideal Gas

We shall now establish a definite expression for the internal energy of an ideal gas. Generally speaking, the internal energy is a function of p, v, T. One of the variables, p, for example, can be eliminated by means of the equation of state, hence, the [specific] internal energy u can be regarded as a function of the two variables T and v:

$$u = u(T, v), \tag{1}$$

implying

$$\delta u = \frac{\partial u}{\partial T}\delta T + \frac{\partial u}{\partial v}\delta v \tag{2}$$

or, by substituting in equation (4) § 11, we obtain

$$\delta q = \frac{\partial u}{\partial T}\cdot\delta T + \left(\frac{\partial u}{\partial v} + p\right)\delta v. \tag{3}$$

If the process is isochoric (v = const), then, according to (3) § 10,

$$c_v = \left(\frac{\delta q}{\delta T}\right)_{v=\text{const.}} = \frac{\partial u}{\partial T}; \tag{4}$$

hence

$$\delta u = c_v\delta T + \frac{\partial u}{\partial v}\delta v. \tag{5}$$

The dependence of the internal energy of an ideal gas on its volume must be determined by experiment. Joule showed experimentally that the temperature of a gas does not change during its adiabatic expansion into a vacuum. We will prove that the internal energy of an ideal gas is independent of its volume.

When gas expands into a vacuum it does no work against an external pressure, i. e., $\delta w = 0$, and $\delta q = 0$ because the process is adiabatic. Then

$$\delta u = c_v \delta T + \frac{\partial u}{\partial v} \delta v = 0,$$

hence

$$\frac{\partial u}{\partial v} = - c_v \frac{\delta T}{\delta v}. \tag{6}$$

Owing to $\delta T = 0$ according to Joule's experiments,

$$\frac{\partial u}{\partial v} = 0. \tag{7}$$

Following Joule, similar but more accurate experiments did show some temperature variation. But according to these experiments $\frac{\partial u}{\partial v}$ is very small, and differs less from zero the closer does the gas under consideration follow the laws of ideal gases; it is therefore expedient to agree, in defining an ideal gas, that its internal energy should not depend on the volume, i. e., that

$$\delta u = c_v \delta T. \tag{8}$$

Since

$$\frac{\partial c_v}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial T}\right) = \frac{\partial}{\partial T}\left(\frac{\partial u}{\partial v}\right) = 0, \tag{9}$$

we see that c_v is a function of the temperature only (see Appendix 7).

The results of experiments show that the temperature variation of c_v is smaller, the less the gas differs from an ideal one.

The constancy of c_v is the third and last condition which an ideal gas must satisfy. Thus, δu is a total differential:

$$du = c_v dT. \tag{10}$$

The integration of (10) shows that the internal energy of an ideal gas

$$u = c_v T + \text{const}. \tag{11}$$

is a function of the gas temperature only or, in other words, only of the state of the gas, and does not depend on the character of the process or on the thermodynamic path by which the gas has reached this state.

§13. The Relationship Between the Work Performed During Expansion and the Thermodynamic Path. p–v Diagram

For many thermodynamic purposes the p–v diagram is useful, on the coordinate axes of which the parameters of state p, v are plotted in a linear scale. Because the state of an ideal gas is determined without ambiguity by two parameters p and v, any state of such a gas will be represented on the diagram by some point. An arbitrary change of state from (v, p) to $(v + \delta v, \ p + \delta p)$, called an elementary

physical process, will be represented on the p–v diagram by an infinitesimal segment CD (Figure 9).

The continuous curve AB on the diagram represents a finite physical process, made up of a succession of elementary processes, i. e., the thermodynamic path.

The elementary work $\delta w = p\delta v$ is represented by an infinitely narrow strip CDPN having a height p and a width δv.

The whole work performed by the body during expansion from the volume at the initial state to the volume at the final state, and expressed by the line integral

$$w = \int_A^B p\delta v,$$

is represented on the p–v diagram by the area of the domain ABQM, bounded at the top by the curve AB, at the sides by the two coordinates p_A, p_B and below by a segment of the abscissa. This area can be regarded as an algebraic quantity, if we agree to consider it as being positive in the case where the change of state is accompanied by an increase of the volume (i. e., the gas itself produces work), and negative when the change of state is accompanied by a decrease of volume (the work is used up to compress the gas).

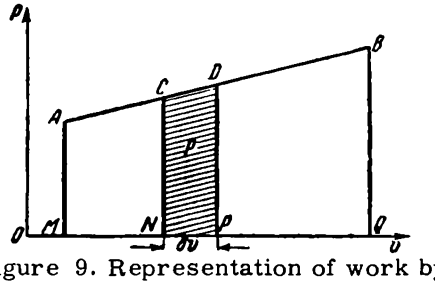

Figure 9. Representation of work by area in the p–v diagram

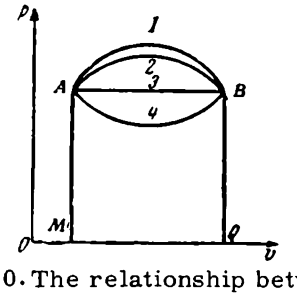

Figure 10. The relationship between the work performed during expansion and the thermodynamic path

Figure 10 shows to what extent the work, performed during expansion (represented by the area A1BQM, A2BQM, etc), and used to bring the gas from the initial state A to the final state B, depends on the thermodynamic path.

It follows from the above that the elementary work δw is not a total differential; but

$$\delta q = \delta u + \delta w,$$

where δu is a total differential and δw is not; therefore the elementary increment of energy δq, transferred to the gas in the form of heat in its transition from the state A to the state B, is also not a total differential.

§14. The Different Forms of the Equation of Heat Balance for Ideal Gases

The equation of the first law of thermodynamics is often called the equation of heat influx*. The equation of heat balance for an ideal gas, according to (4) § 11 and (8) § 12 has the following form:

$$\delta q = c_v \delta T + p \delta v. \tag{1}$$

Let us convert equation (1) into a form often used in meteorology. By differentiating Clapeyron's equation (6) § 7 we get

$$p \delta v = R \delta T - v \delta p$$

and by substituting the result of this differentiation into equation (1), we obtain

$$\delta q = (c_v + R) \delta T - v \delta p. \tag{2}$$

According to equation (3) § 10 we have for an isobaric process (p = const)

$$c_p = \left(\frac{\delta q}{\delta T}\right)_{p = \text{const.}} = c_v + R.$$

Thus we have obtained a very important relationship, linking the specific heats c_p and c_v of a gas, known as Mayer's equation

$$c_p = c_v + R; \quad c_p - c_v = R. \tag{3}$$

Mayer's equation together with the conclusions of § 12 show that the specific heat at constant pressure is constant for an ideal gas. The real values of the specific heat c_{pd} for dry air vary somewhat with the change of the temperature. These changes can be disregarded, and the quantity c_{pd} can be regarded as constant.

The following combinations of the specific heats c_p and c_v are often introduced into the equations of thermodynamics:

$$\varkappa = \frac{c_p}{c_v}; \quad \lambda = \frac{R}{c_p} = \frac{c_p - c_v}{c_p} = \frac{\varkappa - 1}{\varkappa} \tag{4}$$

$$[\varkappa] = [\lambda] = [0].$$

For dry and pure air:

$$\varkappa = 1.4; \quad \lambda = 0.285; \quad R = 0.07 \ \frac{\text{cal}}{\text{g.deg}}$$

(see Appendix 7).

Thus, the equation of heat balance has the form

$$\delta q = c_p \delta T - v \delta p \tag{5}$$

or, by eliminating v by means of Clapeyron's equation:

$$\delta q = c_p \delta T - RT \frac{\delta p}{p}. \tag{6}$$

The increment δq may be transformed into a total differential by dividing

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*[Translator's note–Russian terminology. Hereafter translated as "equation of heat balance"].

both sides of the equation of heat balance by T and applying Clapeyron's equation:

$$\frac{\delta q}{T}=c_v\frac{\delta T}{T}+R\frac{\delta v}{v}, \tag{7}$$

$$\frac{\delta q}{T}=c_p\frac{\delta T}{T}-R\frac{\delta p}{p}. \tag{8}$$

Eliminating $\delta T/T$ from the equation of heat balance by a logarithmic differentiation of Clapeyron's equation

$$\frac{\delta T}{T}=\frac{\delta v}{v}+\frac{\delta p}{p}, \tag{9}$$

we obtain

$$\frac{\delta q}{T}=c_p\frac{\delta v}{v}+c_v\frac{\delta p}{p}. \tag{10}$$

Collecting (7), (8), (10) we obtain the following forms of the equation of heat balance:

$$\frac{\delta q}{T}=c_v\frac{\delta p}{p}+c_p\frac{\delta v}{v}=R\frac{\delta v}{v}+c_v\frac{\delta T}{T}=c_p\frac{\delta T}{T}-R\frac{\delta p}{p}. \tag{11}$$

In this group of equations, each equation is obtained from the preceding one by subjecting the variables (p, v, T), and the constant coefficients (c_v, c_p, R) to the same cyclic permutation, where R becomes negative in the second permutation.

It is easy to see that the right sides of equations (11) are total differentials of functions of the parameters of state:

$$\frac{\delta q}{T}=\delta\left[\ln\left(p^{c_v}\cdot v^{c_p}\right)\right]=\delta\left[\ln\left(v^{R}\cdot T^{c_v}\right)\right]=\delta\left[\ln\left(T^{c_p}\cdot p^{-R}\right)\right]; \tag{12}$$

consequently, the left side is also a total differential of a certain function of state:

$$d\varphi=\frac{\delta q}{T};\quad [\varphi]=[L^2T^{-2}\theta^{-1}]; \tag{13}$$

the quantity φ, which is a function of state, is called specific entropy.

We know that the line integral of a total differential taken around a closed contour is equal to zero; therefore, considering the p–v diagram and integrating (13) around a closed contour we obtain

$$\oint d\varphi=0 \quad \text{or} \quad \oint\frac{\delta q}{T}=0.$$

This is one of the most prevalent expressions of the second law of thermodynamics. It is apparent from the foregoing argument that the mathematical expression for the entropy, and the equation of the second law of thermodynamics for an ideal gas, can be derived from the first law of thermodynamics. It is quite impossible to derive the mathematical expression for the entropy of other substances, such as water, from the first law of thermodynamics alone. To this end it is also necessary to apply the second law of thermodynamics. We will return to this question in the following.

If the process is isobaric, then according to equation (5)

$$\delta q=c_p\delta T \tag{14}$$

is a total differential of a certain quantity which is a function of state, called the enthalpy; we denote this quantity by ϑ. It is obvious that

$$\vartheta=c_p\cdot T+\text{const}. \tag{15}$$

or, according to (3)

$$\vartheta = c_v T + RT = u + pv. \tag{16}$$

Thus the enthalpy is equal to the sum of the internal energy and the potential energy of expansion, i. e., the work performed by the gas during an isobaric expansion until reaching the volume it presently occupies. It is easy to see that the enthalpy varies in the same manner as the internal energy, a constant relationship existing between the two functions:

$$\frac{\vartheta}{u} = \frac{c_p T}{c_v T} = \frac{c_p}{c_v} = \varkappa, \tag{17}$$

hence

$$\vartheta = \varkappa \cdot u, \quad [\vartheta] = [L^2 T^{-2}].$$

§ 15. Polytropic Processes

A process is called polytropic when taking place at a constant value of a specific heat (c_π = const). All the iso-processes (the isobaric, isochoric, isothermic, isentropic) are examples of polytropic processes.

Let us consider the equation of heat balance for a polytropic process in an ideal gas:

$$\delta q = c_\pi \cdot \delta T = c_p \delta T - RT \frac{\delta p}{p}; \tag{1}$$

By separating the variables we obtain:

$$(c_p - c_\pi) \frac{\delta T}{T} = R \frac{\delta p}{p} = (c_p - c_v) \frac{\delta p}{p},$$

Integrating, we obtain:

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{c_p - c_v}{c_p - c_\pi}}. \tag{2}$$

This is known as the polytropic equation.

Introducing the density ρ, instead of the pressure p we obtain:

$$\frac{T}{T_0} = \left(\frac{\rho}{\rho_0}\right)^{\frac{c_p - c_v}{c_v - c_\pi}}; \tag{3}$$

dividing equation (2) by equation (3) we find:

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\frac{c_p - c_\pi}{c_v - c_\pi}}; \tag{4}$$

and by introducing the specific volume $v = 1/\rho$, we obtain the polytropic equation in its better known form:

$$p \cdot v^k = p_0 v_0^k, \tag{5}$$

where

$$k = \frac{c_p - c_\pi}{c_v - c_\pi} \tag{6}$$

is the polytropic index.

Thus, in the case of a given polytropic process (the index k of the polytrope

is given) the state of an ideal gas is completely determined by the initial state and any parameter of state. Therefore the polytropic equation is an equation of the change of state of the gas. In Table II are assembled all the equations which determine for a polytropic process any parameter of state when any other parameter is given.

Table 2

Polytropic equations

	$f(T)$	$f(p)$	$f(\rho)$	$f(v)$
$T=$	T	$T_0\left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}$	$T_0\left(\frac{\rho}{\rho_0}\right)^{k-1}$	$T_0\left(\frac{v}{v_0}\right)^{1-k}$
$p=$	$p_0\left(\frac{T}{T_0}\right)^{\frac{k}{k-1}}$	p	$p_0\left(\frac{\rho}{\rho_0}\right)^{k}$	$p_0\left(\frac{v}{v_0}\right)^{-k}$
$\rho=$	$\rho_0\left(\frac{T}{T_0}\right)^{\frac{1}{k-1}}$	$\rho_0\left(\frac{p}{p_0}\right)^{\frac{1}{k}}$	ρ	$\frac{1}{v}$
$v=$	$v_0\left(\frac{T}{T_0}\right)^{\frac{1}{1-k}}$	$v_0\left(\frac{p}{p_0}\right)^{-\frac{1}{k}}$	$\frac{1}{\rho}$	v

Turning to the diagrammatic representation of the polytropic curves, we shall notice that the polytropes lie on the thermodynamic surface representing Clapeyron's equation in the p, v, T space (Figure 11). It is obvious that each polytrope AC is the curve of intersection of the surface of state and the surface representing the equation of heat balance for the polytropic process

$$c_n \delta T = c_v \delta T + p \delta v. \tag{7}$$

The projections of the polytropes on the coordinate planes will be the curves whose equations are given in Table 2. The equations of polytropes include, in particular, all the equations of change of state for iso-processes.

During an adiabatic (isentropic) process, the specific heat of the gas identically vanishes, $c_n = 0$. This means that while the temperature changes during the adiabatic process, the gas does not gain energy by heat transfer, but only by conversion of the work done in compression (both the internal energy of the gas and its temperature increase).

The specific heat of the gas equals c_v during the isochoric process. This means that all the energy obtained in the form of heat during the isochoric process is spent to increase the internal energy of the gas.

The specific heat of the gas is c_p during the isobaric process. All the energy obtained in the form of heat during this process is spent to increase the enthalpy of the gas.

The specific heat of the gas is infinitely great during the isothermic process. In this case all the energy received by heat transfer is spent in the work of expansion, whereas the internal energy, the enthalpy and the temperature remain unchanged.

Let us now determine the work of expansion done by the gas during a polytropic process. According to equation (2) § 11

$$w = \int_{v_0}^{v} p\,dv. \tag{8}$$

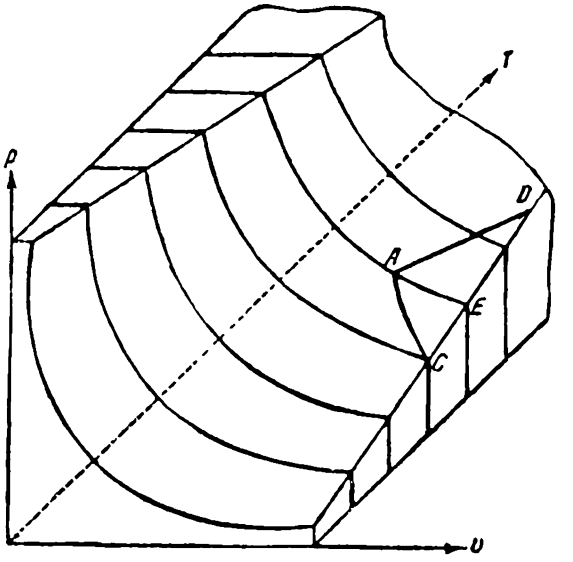

Figure 11. All the polytropes lie on the thermodynamic surface of state

Expressing p as a function of v, according to the polytropic equation (5), we obtain:

$$w = \int_{v_0}^{v} p\,dv = p_0 v_0^k \int_{v_0}^{v} \frac{dv}{v^k} = \frac{1}{1-k} p_0 v_0^k (v^{1-k} - v_0^{1-k}) \tag{9}$$

or

$$w = \frac{1}{k-1}(p_0 v_0 - pv), \tag{10}$$

or, by introducing T according to Clapeyron's equation,

$$w = (c_v - c_n)(T_0 - T). \tag{11}$$

For the isothermic process we obtain:

$$w = p_0 \cdot v_0 \int_{v_0}^{v} \frac{dv}{v} = p_0 v_0 \ln \frac{v}{v_0}$$

or finally

$$w = RT \ln \frac{v}{v_0}, \tag{12}$$

as well as

$$w = RT \ln \frac{p_0}{p}. \tag{13}$$

The equations (10), (11), and (12) show how important is the influence of the thermodynamic path on the magnitude of the work done during expansion.

Considering the p–v diagram (Figure 12) we can see that among iso-processes the greatest amount of work is done by an isobaric process (area ADPM), less by an isothermic process (area AEPM), still less by an adiabatic process (area ACPM) and least of all (equal to zero) by the isochoric process. The variations of the internal energy and of the enthalpy do not depend on the index of the polytrope nor

in general, on the thermodynamic path. These variations respectively equal:

$$\Delta u = c_v (T - T_0); \; \Delta \vartheta = c_p (T - T_0).$$

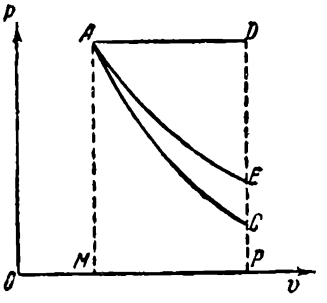

Figure 12. The maximum work is done by the isobaric process, less by the isothermic process, and still less by the adiabatic process

The amount of energy transferred to a unit mass of the gas in the form of heat is determined by the character of the process, and equals:

$$\Delta q = c_n (T - T_0);$$

Table 3 contains a summary of the fundamental characteristics of the most important polytropic processes.

§16. The Dry-Adiabatic Process. Potential Temperature

One of the most important processes of atmospheric thermodynamics is the adiabatic process, taking place in dry air. The real processes occurring in the atmosphere are evidently not adiabatic. Any atmospheric air body receives from the surroundings and gives off some amount of energy by heat exchange. The heat exchange is accomplished in dry air by absorption and emission of radiation, by thermal conductivity, turbulent mixing, and by the transformation of mechanical energy into heat.

All these processes continually affect each air body, their combined action being too complicated to permit detailed thermodynamic analysis. However, in the majority of cases their influence on the change of state of the air body is very small as compared with that of other important types of processes which do admit detailed analysis.

The processes of this latter type are connected with the motion of air, principally in the vertical direction. Moreover, the vertical motions are generally so rapid that the heat exchange between the moving air body and its environment can be neglected. Such processes are known as adiabatic. An adiabatic process taking place in dry air is called dry-adiabatic. We have already mentioned the adiabatic processes in the previous paragraphs. The fundamental equations for an adiabatic process can be obtained directly from the equations of heat balance by putting $dq = 0$.

$$du + dw = 0; \qquad (1)$$

Table 3

Process	Polytropic	Adiabatic (Isentropic)	Isochoric	Isobaric	Isothermic
Specific heat	c_n	0	c_v	c_p	∞
Polytropic index	$k=\frac{c_p-c_n}{c_v-c_n}$	$\varkappa=\frac{c_p}{c_v}$	∞	0	1
Equation of heat balance	$dq=c_n dT=c_v dT+p dv$	$c_v dT+p\,dv=0$	$dq=c_v dT$	$dq=c_p dT$	$dq=p\,dv$
Equation of the specific work of expansion	$(c_n-c_v)\,dT=p\,dv$	$c_v dT+p dv=0$	$v=\text{const.}$	$p=\text{const.}$	$T=\text{const.}$
Equation of change of state as a function of the variables p, v	$p\cdot v^k=\text{const.}$	$p\cdot v^{\varkappa}=\text{const.}$	$v=\text{const.}$	$p=\text{const.}$	$p\cdot v=\text{const.}$
Equation of change of state as a function of the variables v, T	$T\cdot v^{k-1}=\text{const.}$	$T\cdot v^{\varkappa-1}=\text{const.}$	$v=\text{const.}$	$v=v_0(1+\alpha t)$	$T=\text{const.}$
Equation of change of state as a function of the variables T, p	$T\cdot p^{\frac{1-k}{k}}=\text{const.}$	$T\cdot p^{\frac{1-\varkappa}{\varkappa}}=\text{const.}$	$p=p_0(1+\alpha t)$	$p=\text{const.}$	$T=\text{const.}$
Equation of the surface of heat flow	$w=(c_v-c_n)(T_0-T)$	$w=c_v(T_0-T)$	0	$w=R\quad(T-T_0)$	$w=RT\ln\frac{p_0}{p}$
Variation in internal energy	$\Delta u=c_v(T-T_0)$	$\Delta u=c_v(T-T_0)$	$\Delta u=c_v(T-T_0)$	$\Delta u=c_v(T-T_0)$	0
Heat transferred to the gas	$\Delta\vartheta=c_p(T-T_0)$	$\Delta\vartheta=c_p(T-T_0)$	$\Delta\vartheta=c_p(T-T_0)$	$\Delta\vartheta=c_p(T-T_0)$	0
The amount of heat transferred to the gas	$\Delta q=c_n(T-T_0)$	0	$\Delta q=c_v(T-T_0)$	$\Delta q=c_p(T-T_0)$	$\Delta q=RT\ln\frac{p_0}{p}$

hence

$$c_v dT + p\,dv = 0, \tag{2}$$

$$c_p dT - v\,dp = 0, \tag{3}$$

$$c_v \frac{dp}{p} + c_p \frac{dv}{v} = R\frac{dv}{v} + c_v \frac{dT}{T} = c_p \frac{dT}{T} - R\frac{dp}{p}, \tag{4}$$

$$d(\ln v^R \cdot T^{c_v}) = d(\ln T^{c_p} \cdot p^{-R}) = d(\ln p^{c_v} \cdot v^{c_p}), \tag{5}$$

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\lambda}; \quad \frac{T}{T_0} = \left(\frac{v}{v_0}\right)^{1-\varkappa}; \quad p \cdot v^{\varkappa} = p_0 v_0^{\varkappa};$$

$$\frac{T}{T_0} = \left(\frac{\rho}{\rho_0}\right)^{\varkappa-1}; \quad \frac{p}{\rho^{\varkappa}} = \frac{p_0}{\rho_0^{\varkappa}}. \tag{6}$$

Equation (6) was derived by Poisson in 1823. Poisson's equation is most frequently used in the thermodynamics of the atmosphere in the following form:

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}. \tag{7}$$

This equation is known as the dry-adiabatic equation.

During the adiabatic descent of an air body its temperature increases, owing to the increase in external pressure, and may exceed the temperature of underlying air bodies even though they were initially warmer than the descending body. The concept of potential temperature is used in order to enable to make a comparison of the states of different air bodies.

The potential temperature is the temperature acquired by dry air, if its pressure is adiabatically reduced to standard pressure. By standard pressure, a pressure of 1,000 mb is usually meant. From Poisson's equation (7) we obtain:

$$\theta = T\left(\frac{1000}{p}\right)^{\frac{R}{c_p}}. \tag{8}$$

Tables and diagrams computed according to equation (8) can be used for calculating the potential temperature.

The most important property of the potential temperature is its constancy during the dry-adiabatic processes. We will introduce the potential temperature into the equation of heat balance in order to prove the existence of this property. By differentiating equation (8) logarithmically, and by multiplying the results by c_p, we obtain:

$$c_p \frac{d\theta}{\theta} = c_p \frac{dT}{T} - R\frac{dp}{p}. \tag{9}$$

By comparing equation (9) with equation (8) § 14 we obtain

$$\frac{dq}{T} = c_p \frac{d\theta}{\theta} = d(c_p \ln \theta). \tag{10}$$

Thus the potential temperature does not vary, if the process is adiabatic ($dq = 0$):

$$\theta = \text{const}. \tag{11}$$

The potential temperature increases [$d\theta > 0$] if heat is supplied [$dq > 0$], whereas it decreases [$d\theta < 0$] when heat is emitted [$dq < 0$]. Use can be made of the constancy of the potential temperature in order to identify air bodies during a short interval of time (24–36 hours), if neither evaporation nor condensation occur. On the other hand, by observing the same body of air over a long period of time, its transformation (warming up or cooling) can be detected by means of the potential temperature.

We will determine the relationship between the specific entropy φ and the potential temperature of dry air by combining equation (13) § 14 and equation (10):

$$\varphi = c_p \ln \theta + \text{const.} \tag{12}$$

Entropy becomes a very important energetic characteristic due to the fact that $d\varphi$ is a total differential. Besides, according to equation (12) one can substitute the expression in the potential temperature for the entropy of dry air.

§17. General Theory of Thermodynamic Diagrams

During the last 15–20 years a great number of different diagrams have been proposed in order to represent the processes of aerological ascents. Among such diagrams, those on which the potential energy of the inconstant atmospheric stratification can be plotted and measured easily and graphically, are of special interest.

Emagrams, aerograms, tephigrams, sondograms and several other diagrams belong to these. In this section we will describe the general theory of plotting diagrams used for the energy calculations.

We made repeated use of the p–v diagrams in order to study the work done in expansion. We have seen that the work of expansion is represented on the p–v diagram by an area. Consequently, p–v diagrams are used in heat technology for the calculation of energy. However the use of the p–v diagram is not practicable, because air volumes are not measured and generally not even calculated in meteorology.

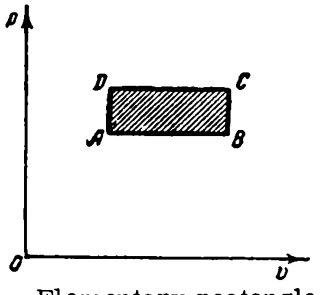

Figure 13. Elementary rectangle on the p-v diagram represents the elementary work of expansion

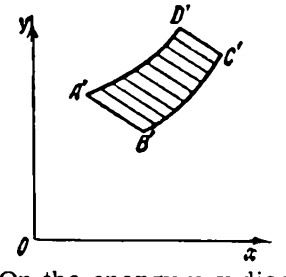

Figure 14. On the energy x-y diagrams the elementary work of expansion is represented by the elementary curvilinear quadrilateral

When considering the values which can be measured and calculated in meteorology, it is essential to explain which combinations of these values can be used for the plotting of the type of diagram on which the area is to represent the work of expansion, in the same manner as on the p–v diagram.

Let us consider the p–v diagram in Figure 13. Every point on this diagram corresponds to some state of the gas.

An infinitesimal rectangle on this diagram corresponds to that infinitesimal work of expansion, which is performed by an infinitesimal pressure dp during a change of the gas volume by dv. Let us imagine another thermodynamic diagram (Figure 14) on whose abscissa is plotted some function of state of the gas

$$x = \varphi(v, p). \tag{1}$$

and on the ordinate, any other function of state

$$y = \psi(v, p). \tag{2}$$

Then every point on this new diagram will also correspond to some state of the gas. The isobars and the isochores represented on the p–v diagram by straight lines will be represented on the new x–y diagram by curves. On the x–y diagram any infinitesimal curvilinear rectangle A'B'C'D', formed as a result of the intersection of two infinitely close isobars and isochores, will correspond to the infinitely small rectangle ABCD which on the p–v diagram represents the infinitesimal work of expansion. If the area of the rectangle A'B'C'D' is equal to the area of ABCD, or is a constant multiple of it, then the possibility exists of measuring the work due to expansion by a corresponding area on the new diagram, in the same way as was done on the p–v diagram.

If the ratio between these areas is not constant, but depends on p and v in such a manner that it has different values at different points of the x–y plane, the use of the x–y diagram is unfeasible for the calculation of energy, even though the infinitesimal areas on the x–y diagram, corresponding to the infinitesimal areas on the p–v diagram, might represent the elementary work of expansion.

Let us determine the condition which the functions φ, ψ must fulfill in order to make the x–y diagram suitable for calculations of energy, and above all in order to make it possible to represent the work of expansion by a corresponding area on this diagram.

The elementary area A'B'C'D' can be taken, correct to infinitesimals of the second order, as twice the area of the curvilinear triangle A'B'C' which can be regarded, to the same accuracy, as the area of a rectilinear triangle A'B'C'.

Let us write down the co-ordinates of the vortexes of the quadrilaterals

$$\begin{array}{ll} A'(x, y), & A(v, p), \\ B'(x_1, y_1), & B(v+dv, p), \\ C'(x_2, y_2), & C(v+dv, p+dp), \\ D'(x_3, y_3), & D(v, p+dp); \end{array}$$

then one can, obviously, write:

$$x = x(v, p), \quad y = y(v, p),$$

$$x_1 = x(v+dv, p) = x + \frac{\partial x}{\partial v} \cdot dv + \varepsilon_1; \quad y_1 = y(v+dv, p) = y + \frac{\partial y}{\partial v} \cdot dv + \varepsilon_1';$$

$$x_2 = x(v+dv, p+dp) = x + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial p} dp + \varepsilon_2; \quad y_2 = y(v+dv, p+dp) = y + \frac{\partial y}{\partial v} \cdot dv + \frac{\partial y}{\partial p} dp + \varepsilon_2';$$

$$x_3 = x(v, p+dp) = x + \frac{\partial x}{\partial p} dp + \varepsilon_3; \quad y_3 = y(v, p+dp) = y + \frac{\partial y}{\partial p} dp + \varepsilon_3',$$

where $\varepsilon_1 \varepsilon_1' \ldots \varepsilon_3'$ are sums of infinitesimals of the second and higher orders.

It is known that twice the area of a triangle, whose vertexes are situated at the points A' (x, y), B' (x_1, y_1), C' (x_2, y_2), is determined by the equation

$$2 \triangle A'B'C' = \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix}.$$

Denoting the area A'B'C'D' by $d\sigma'$, and the area ABCD by $d\sigma$, and passing to the limit we obtain:

$$d\sigma' = \begin{vmatrix} x, & y, & 1 \\ x + \frac{\partial x}{\partial v} dv, & y + \frac{\partial y}{\partial v} dv, & 1 \\ x + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial p} dp, & y + \frac{\partial y}{\partial v} dv + \frac{\partial y}{\partial p} dp, & 1 \end{vmatrix}.$$

By subtracting the second row from the third, and the first from the second, we obtain:

$$d\sigma' = \begin{vmatrix} x, & y, & 1 \\ \frac{\partial x}{\partial v} dv, & \frac{\partial y}{\partial v} dv, & 0 \\ \frac{\partial x}{\partial p} dp, & \frac{\partial y}{\partial p} dp, & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial v}, & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial p}, & \frac{\partial y}{\partial p} \end{vmatrix} \cdot d\sigma. \tag{3}$$

Let us set

$$D = \begin{vmatrix} \frac{\partial x}{\partial v}, & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial p}, & \frac{\partial y}{\partial p} \end{vmatrix}. \tag{4}$$

This functional determinant is known as the Jacobian, and its value is called the coefficient of distortion. Then

$$d\sigma' = D\, d\sigma. \tag{5}$$

Thus, it is necessary and sufficient that the absolute value of the Jacobian D should be equal to unity:

$$|D| = 1.$$

Under this condition any domain chosen in the p, v plane, and bounded by two isobars and two isochores, will preserve its area on being mapped onto the x, y plane. If the Jacobian is equal to any constant number D, differing from one, then the areas of all surface elements in the p, v plane will be multiplied by D on being mapped onto the x, y plane. The work of expansion in this case will be represented on the x-y diagram by an area D times greater than the corresponding area on the p-v diagram. If the Jacobian is not constant but some function D (v, p), then the areas of surface elements in the x, y plane vary differently at different points. Such a diagram is not suitable for calculating the work of expansion or for other calculations of energy.

Let us explain which of the above-mentioned diagrams are suitable for calculation of energy. The conclusions of this explanation are given in Table 4.

From this table it follows that emagrams, tephigrams and aerograms are suitable for calculations of energy if R = const, i. e., if we study the change of state of dry air, or if it is assumed that R = const for the change of state of moist air and the influence of humidity is accounted for by the introduction of a virtual temperature (see § 4, Chapter IV).

It is evident that the Stüve diagram is unsuitable for exact calculations of energy, because the coefficient of distortion D is a function of y, and therefore the same work of expansion will be represented by a greater area for small p and by a smaller area for large p.

Let us now consider the inverse problem. We will plot diagrams of energy, i. e., suitable for calculations of energy.

Equation (4) is the starting point for plotting energy diagrams; it can be written in the form:

$$\frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial p} - \frac{\partial x}{\partial p} \cdot \frac{\partial y}{\partial v} = D. \tag{6}$$

Table 4

The energy properties of aerological diagrams

	Emagram	Tephigram	Stuve diagram	Aerogram	Leichtmann's sondogram
x	$T=\frac{p\cdot v}{R}$	$-T=-\frac{pv}{R}$	$T=\frac{pv}{R}$	$\ln T=\ln p+$ $+\ln v+\text{const.}$	$\ln RT=\ln p+\ln v$
y	$\ln\frac{p_0}{p}$	$\varphi_d=c_p\ln\theta+\text{const.}=$ $=c_p\ln T-R\ln p+$ $+\text{const.}=c_v\ln p+$ $+c_p\ln v+\text{const.}$	$-\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}$	$T\ln\frac{p_0}{p}=\frac{pv}{R}\ln\frac{p_0}{p}$	$RT\ln\frac{p_0}{p}+c_p\ln\frac{T}{T_0}=$ $=pv\ln\frac{p_0}{p}+c_p\ln\frac{pv}{p_0v_0}$
$\frac{\partial x}{\partial v}$	$\frac{p}{R}$	$-\frac{p}{R}$	$\frac{p}{R}$	$\frac{1}{v}$	$\frac{1}{v}$
$\frac{\partial y}{\partial v}$	0	$c_p\frac{1}{v}$	0	$\frac{p}{R}\ln\frac{p_0}{p}$	$p\ln\frac{p_0}{p}+c_p\frac{1}{v}$
$\frac{\partial x}{\partial p}$	$\frac{v}{R}$	$-\frac{v}{R}$	$\frac{v}{R}$	$\frac{1}{p}$	$\frac{1}{p}$
$\frac{\partial y}{\partial p}$	$-\frac{1}{p}$	$c_v\cdot\frac{1}{p}$	$-\frac{R}{c_p}\left(\frac{p_0}{p}\right)^{\frac{R}{c_{p_0}}}\frac{1}{p}$	$\frac{v}{R}\left(\ln\frac{p_0}{p}-1\right)$	$v\left(\ln\frac{p_0}{p}-1\right)+c_p\frac{1}{p}$
$D=\frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial p}-$ $-\frac{\partial x}{\partial p}\cdot\frac{\partial y}{\partial v}$	$-\frac{1}{R}$	1	$\frac{1}{c_p}y$	$-\frac{1}{R}$	-1

One of the coordinates of the new diagram is assumed to be a given function of (v, p); it is, for example, assumed that

$$x = \varphi(v, p).$$

Hence the partial derivatives

$$P(v, p) = \frac{\partial x}{\partial v}; \quad Q(v, p) = \frac{\partial x}{\partial p} \tag{7}$$

will also be given functions v, p.

We will determine the function

$$y = \psi(v, p),$$

plotted on the other axis of the diagram, by selecting it in such a way that the diagram would represent energy relations. To realize this aim it is necessary and sufficient that the Jacobian D should have a constant value. It follows from equations (6) and (7) that

$$P\frac{\partial y}{\partial p} - Q\frac{\partial y}{\partial v} = D. \tag{8}$$

It is known that the solution of first order partial differential equations of the form (8) reduces to the solution of a system of ordinary differential equations:

$$\frac{dp}{P} = -\frac{dv}{Q} = \frac{dy}{D}. \tag{9}$$

The first of the equations (9) can be written in the following form:

$$\frac{\frac{dp}{\partial x}}{\frac{\partial v}{}} = -\frac{\frac{dv}{\partial x}}{\frac{\partial p}{}} \quad \text{or} \quad \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial p}dp = 0.$$

This equation satisfies the relationship

$$x(p, v) = x_0 = \text{const.}; \tag{10}$$

through equation (10) we can express the variable v in terms of p:

$$v = v(p, x_0)$$

and substitute in equation (7)

$$P(v, p) = P'(p, x_0).$$

The y can be determined by integrating the second of equations (9)

$$\frac{dp}{P'(p, x_0)} = \frac{dy}{D},$$

hence

$$y = D\int \frac{dp}{P'(p, x_0)} + \Phi(x), \tag{11}$$

where $\Phi(x)$ is an arbitrary function of x.

We shall look at some examples writing certain calculations for conciseness in the form of Table 5.

From the explanations given in this chapter it follows that it is possible to plot a great number of various diagrams, expressing the energy properties of the atmosphere. Those diagrams should then be selected for which:

Table 5

x	$v^{\alpha}p^{\beta}$	$(vp)^{a}$	$\gamma \ln v + \delta \ln p$	$b \cdot \ln (pv)$
$P = \frac{\partial x}{\partial v}$	$\alpha v^{\alpha-1} \cdot p^{\beta}$	$a\,(vp)^{a-1} \cdot p$	$\gamma \cdot \frac{1}{v}$	$b \cdot \frac{1}{v}$
v	$x_0^{\frac{1}{\alpha}} \cdot p^{-\frac{\beta}{\alpha}}$	$x_0^{\frac{1}{a}} \cdot p^{-1}$	$e^{\frac{x_0}{\gamma}} \cdot p^{-\frac{\delta}{\gamma}}$	$e^{\frac{x_0}{b}} \cdot p^{-1}$
$P'(p, x_0)$	$\alpha \cdot x_0^{\frac{\alpha-1}{\alpha}} \cdot p^{\frac{\beta}{\alpha}}$	$a \cdot x_0^{\frac{a-1}{a}} \cdot p$	$\gamma \cdot e^{-\frac{x_0}{\gamma}} \cdot p^{\frac{\delta}{\gamma}}$	$b \cdot e^{-\frac{x_0}{b_0} p}$
y	$\frac{D}{\alpha-\beta} \cdot v^{1-\alpha} \cdot p^{1-\beta} + \Phi(x)$	$\frac{D}{a} \cdot (p \cdot v)^{1-a} \ln p + \Phi(x)$	$D \cdot \frac{pv}{\gamma - \delta} + \Phi(x)$	$D \cdot \frac{pv}{b} \cdot \ln p + \Phi(x)$

Setting	We find for a suitable zero-point
$a = 1$ $D = -1$ $\Phi(x) = \ln p_0$	Emagram $x = RT;\quad y = \ln \frac{p_0}{p}$
$a = 1$ $D = -R$ $\Phi(x) = R \ln p_0 + c_p \ln \frac{T}{T_0}$	Tephigram $x = RT;\quad y = R \ln \frac{p_0}{p} + c_p \ln \frac{T}{T_0} = \varphi_d - \varphi_{d_0}$
$\gamma = c_p$ $\delta = c_v;\ D = R$ $\Phi(x) = \theta$	Tephigram $x = c_p \ln \frac{v}{v_0} + c_v \ln \frac{p}{p_0} = c_p \ln \frac{\theta}{\theta_0} = \varphi_d - \varphi_{d_0};\quad y = RT$
$b = 1$ $D = -1$ $\Phi(x) = RT \ln p_0$	Aerogram $x = \ln RT;\quad y = RT \ln \frac{p_0}{p}$
$b = 1$ $D = -1$ $\Phi(x) = c_p \ln \frac{T}{T_0} + RT \ln p_0$	Sondogram $x = \ln RT;\ y = RT \ln \frac{p_0}{p} + c_p \ln \frac{T}{T_0}$

1) The most important iso-lines (isobars, isotherms, adiabates, etc) are straight lines, because the smaller the curvature of the lines, the easier it is to use such a diagram.

2) The various diagrammatic calculations, in particular, the determination of the heights of aerologic ascents, may be carried out without undue complication.

We shall resume this question later on.

§18. The Mathematical Expression of the Second Law

Consider an ideal heat engine, the working substance of which is an ideal gas. We first choose an ideal gas because its equation of state pv = RT, and its internal energy u are known. Let us assume that the ideal gas is subjected to a reversible cyclic process (i. e. a process made up of a continuous succession of equilibrium states), as a result of which the substance returns to the initial state. This general requirement comprises the following specific requirements:

1) Heat losses to the surroundings are precluded.

2) Nowhere can a finite difference appear, i. e., a discontinuity of temperature.

3) The external pressure can only differ by an infinitely small value from the pressure of the gas.

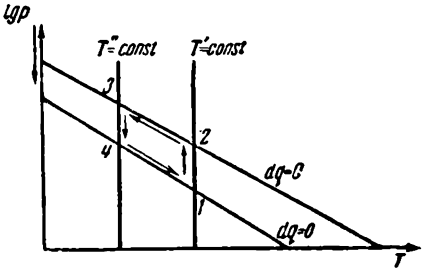

Figure 15. The Carnot cycle represented on an emagram

The efficiency, i. e., the ratio of the useful work accomplished to the work expended, depends on the form of the cyclic process, but above all on the temperature limits within which the gas performs the cycle. The Carnot cycle is the most favorable as regards efficiency because during this process the gas first expands isothermally, then adiabatically, as a result of which its temperature decreases; afterwards the gas is compressed until its temperature increases again.

The gas will return to its initial state at the end of the process. In Figure 15 the cyclic process is represented on an emagram. On the basis of the arguments of §15,, we can explain this process by Table 6.

Thus, the following work is carried out during the cyclic process:

$$w = RT' \ln \frac{p_1}{p_2} + RT'' \ln \frac{p_3}{p_4}. \qquad (1)^*$$

We will simplify equation (1). Dividing the equation of the adiabate (2→3) by the equation of the adiabate (4 → 1), we obtain

$$\frac{p_1}{p_2} = \frac{p_4}{p_3}, \qquad (2)$$

* [Translator's note -- w should be W].

and by substituting (2) in (1), we obtain:

$$W = R \ln \frac{p_1}{p_2} (T' - T''). \tag{3}$$

Table 6

Calculation on the emagram of the work performed during a cyclic process

Thermodynamic	$1 \to 2$	$2 \to 3$	$3 \to 4$	$4 \to 1$
The equation of the thermodynamic path	$dT = 0$	$dq = 0$	$dT = 0$	$dq = 0$
Equation of change of state	$T' = \text{const.}$	$T' \cdot p_2^{-\lambda} = T \cdot p^{-\lambda} = T'' \cdot p_3^{-\lambda}$	$T'' = \text{const.} < T'$	$T' \cdot p_1^{-\lambda} = T \cdot p^{-\lambda} = T'' \cdot p_4^{-\lambda}$
Work performed by the gas	$RT' \ln \frac{p_1}{p_2} > 0$	$c_v (T' - T'') > 0$	$RT'' \ln \frac{p_3}{p_4} < 0$	$c_v (T'' - T') < 0$
Energy transferred to the gas	$Q' = RT' \ln \frac{p_1}{p_2} > 0$	0	$Q'' = RT'' \ln \frac{p_3}{p_4} < 0$	0

On the other hand, the amount of energy transferred to the gas during a cycle is equal to:

$$Q' = RT' \ln \frac{p_1}{p_2}. \tag{4}$$

Dividing equation (3) by equation (4) we obtain the efficiency of the Carnot cycle

$$\eta = \frac{W}{Q'} = \frac{T' - T''}{T'}. \tag{5}$$

Thus the efficiency of the Carnot cyclic process is the same for all ideal gases. It is independent of the values characteristic of particular gases (R, c_p, c_v etc) and depends only on the temperature limits T' and T'' within which the cyclic process takes place.

It is obvious that a gas can accomplish work only in case the influx of energy Q' during the isothermic expansion is greater than the loss $|Q''|$ during the isothermic compression, i. e., when

$$Q' > |Q''|, \quad \text{where } Q'' < 0.$$

The amount of heat transformed into work equals

$$W = Q' + Q'',$$

i. e., expression (5) can be rewritten as follows:

the same amount of heat being removed from the colder sink. This fact is not in contradiction with the first law of thermodynamics but it does not agree with experiments nor with the second law.

Because the described process is realizable in its essentials, it is obviously necessary to admit that the accomplished work equals zero in order to make this process conform with the second law of thermodynamics. Hence

$$\eta_m = \eta_n = \frac{T' - T''}{T'}, \tag{7}$$

which was to be proved

Table 7

Evaluation of the direct and reverse processes

	In the direct process	In the reverse process	Net result
Energy transferred from the "higher" source, at temperature T'	$Q'_m > 0$	$Q'_n < 0$	$Q'_m + Q'_n = 0$ $Q'_m = -Q'_n = Q'$
Energy transferred from the "lower" source, at the temperature T"	$Q''_m < 0$ $Q'_m + Q''_m = \eta Q'_m$	$Q''_n > 0$ $Q'_n + Q''_n = \eta_n Q'_n$	$Q''_m + Q''_n = Q'(\eta_m - \eta_n) \geqslant 0$ $\eta_m Q'_m + \eta_n Q'_n = Q'(\eta_m - \eta_n) \geqslant 0$

Thus the second law of thermodynamics leads to the following proposition.

The efficiency of the Carnot reversible cyclic process, taking place in any thermodynamic system, is independent of the properties of this system, but is determined only by the temperatures limits between which the change of state of this system occurs. Thus the relationship (6) is correct for any substance.

Now this relationship should be generalized for any reversible cyclic process.

Let us consider an arbitrary cyclic process, represented on the diagram by a closed contour (Figure 16). The area bounded by this contour can be divided by the adiabates into a great number of strips, in such a way that the adiabates and the isotherms form a zig-zag curve whose vertexes lie on the contour representing the cyclic process. Each separate strip can then be regarded as a Carnot diagram.

Enumerating the strips we can write:

for strip 1 $$\frac{\Delta Q'_1}{T'} + \frac{\Delta Q''}{T''} = 0,$$

for strip 2 $$\frac{\Delta Q'_2}{T'} + \frac{\Delta Q''_1}{T''} = 0$$

and so on.

Summing the results for all strips we obtain:

$$\sum_{i=1}^{n} \frac{\Delta Q_i}{T'} + \sum_{i=1}^{n} \frac{\Delta Q''_i}{T''} = 0.$$

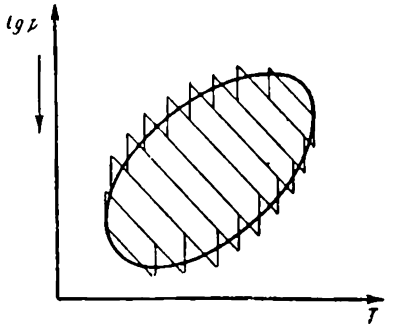

Figure 16. An arbitrary cyclic process on the emagram

The first sum refers to the "upper" isotherms, whereas the second one to the "lower" ones. By adding together all these sums, we obtain:

$$\sum \frac{\Delta Q}{T} = 0, \tag{8}$$

where the summation extends over the whole contour. We are obliged to add (or subtract) a certain quantity of heat ΔQ in all the parts of the the zig-zag curve formed by the isotherms in order to realize this process in practice. Each of these quantities of heat must be added or subtracted at the temperature of the isotherm.

Passing to the limit we obtain

$$\oint \frac{\delta Q}{T} = 0, \tag{9}$$

i. e., the sum of the "transferred" quantities of heat $\delta Q/T$ over the whole thermodynamic path is equal to zero for any reversible process.

Thus the value of the function $\int \delta Q/T$ is independent of the contour of integration, but is determined, up to an arbitrary additive constant, by the parameters of state only:

$$\Phi = \int \frac{\delta Q}{T} + \text{const.};$$

this function is called the entropy. Therefore a quite definite value of entropy corresponds to each state of the thermodynamic system, determined by the variables of state (p, v), or (v, T), or (T, p). For the value of entropy it is immaterial by what thermodynamic path has the system reached the p, v state.

The entropy φ, referring to a unit mass of the substance, is called the specific entropy; it is obvious that

$$\varphi = \int \frac{\delta q}{T} + \text{const.}$$

Chapter III

THE THERMODYNAMICS OF WATER PHASES

§ 1. The Phases of Water and Their Equilibrium

In the previous chapter we studied the change of state of a thermodynamic system consisting of a single monophasic component, i.e., an ideal gas, for dry air can be practically considered to be one. The state of such a system was determined by three parameters, related to each other by Clapeyron's equation. They could vary indefinitely without giving rise to the appearance of new phases in the system, i.e., the changes of state of such a system were due to the change of the parameters (p, T, v) only.

Experiment shows that a second component of atmospheric air, water vapor, can be found in the atmosphere in three states of aggregation, or phases; it is therefore impossible to apply the laws of ideal gases to it without reservation. We can change the parameters of state between their extreme limits without changing the number of phases in the system.

Therefore, we can use the laws of ideal gases to a sufficiently accurate approximation only until a state of saturation is reached; but then the picture sharply changes.

In the present chapter we shall also consider the monocomponent system of water vapor, which can be monophasic (water, vapor, or ice), or diphasic (water vapor + water, water vapor + ice, or ice + water) or, finally, triphasic (water vapor + water + ice).

The change of state of such a sytem will not be limited to the change of the parameters of state, p, v, T but will also be accompanied, generally speaking, by a phase change.

These changes are of a great importance for atmospheric processes, because during the change of phases, great quantities of latent heat are liberated.

The monocomponent system is determined, as experiment shows, by two independent parameters, and it is therefore convenient to use a (p, T) graph in order to represent its properties diagrammatically.

From experimental data it is known that two phases can be in equilibrium only if a certain relationship exists between the pressure p and temperature T. The expression "the phases are in equilibrium" means that their relative quantities do not change.

Water vapor and water are in equilibrium along the curve TK (Figure 17). Its equation is:

$$p = E(T), \qquad (1)$$

where E is the saturation pressure. This curve is called the curve of evaporation; it passes through the points E = 6.10mb, t = 0°C and E = 1013.2mb (=760 mm Hg), t=100°C and ends in the critical point K, representing the critical

state of water. This state is characterized by the disappearance of the difference between (liquid) water and water vapor. The critical temperature equals T_k = 647°K for water, the critical pressure p_k = 221000 mb = 218 atmospheres, the critical specific volume v_k = 3.1 cm^3/g. It can be seen, by the way, that vapor cannot be transformed into water at $T>T_k$ or at $p>p_k$, because, by following the curve of evaporation up to the point K, we shall enter the region of vapor.

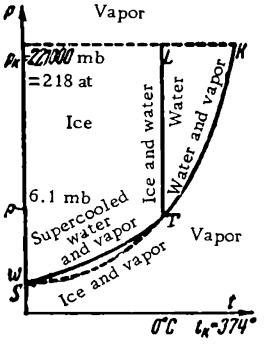

Figure 17. Schematic diagram of the water phases
T - the triple point; K - the critical point.

Water, vapor and ice are in equilibrium along the curve TS, called the curve of sublimation. Its equation is

$$p = E_i(T), \tag{2}$$

where $E_i(T)$ is the saturation pressure over ice.

Finally, water and ice are in equilibrium along the curve TL; this curve is called the melting curve; its equation is

$$T = T_L(p), \tag{3}$$

where T_L is the melting temperature.

The three curves intersect at one point T, which is called the triple point. This point is the only one at which all the three water phases are in equilibrium. Its exact co-ordinates are the following:

$$p_T = 6.1\,\text{mb}, \quad t_T = 0.0075^\circ\,\text{C}.$$

The region KTL, bounded by the curves of evaporation and of melting corresponds to water in liquid form, the region LTS - to ice, and the remaining area - to vapor. Thus, the temperature of water can be lowered beneath the melting point, i. e., supercooled water can be obtained, by carefully cooling liquid water. But such supercooled water will be in an unstable state and will be immediately transformed into the solid phase – ice – as soon as it comes into contact with even the finest crystal of ice. The curve of evaporation TK must be extended into the region of negative temperatures, because of the existence of supercooled water in the atmosphere. This curve is determined by the saturation pressure of the water vapor relative to the supercooled water (at negative temperatures). The presence of supercooled water is of great importance for meteorological processes. Supersaturated water vapor can be obtained in exactly the same way under certain conditions. Thus, the region WTS,

bounded by the curve of sublimation ST and the evaporation curve WT at negative temperatures, will correspond to supersaturated water vapor – relative to the ice, although saturation would not have been reached relative to the water.

§ 2. Water Vapor

We will hypothetically conduct the following experiment. Let us introduce drops of water with a pipette into the mercury column of the barometer B (Figure 18). The drops of water that emerge on the surface of the mercury will spread and evaporate into the space called Torricelli's vacuum. By determining the pressures on the barometers A and B, and by comparing them, we shall find that the pressure of the water vapor in the barometric tube will increase as the evaporation of water continues; the vapor pressure will stop growing after reaching a certain maximum value E. The evaporation of water will then cease. It is said in such a case that the space is saturated by water vapor. The value E is called the saturation pressure. It follows from the above that water vapor is saturated when in equilibrium with water (or ice) with which it is in contact.

Experiment shows that the saturation pressure E is a function of temperature only:

$$E = E(T) \tag{1}$$

An equation in the form (1) is characteristic of a diphasic system "vapor + water" or "vapor + ice".

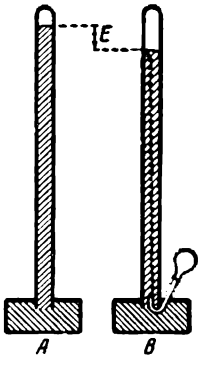

Figure 18. Water vapor above the mercury in the barometer B is saturated. It reached a pressure E equal to the difference in the pressures exerted by the mercury columns in the barometers A and B

Attempts to solve this equation have led to several empirical and semi-empirical equations. The Magnus equation is the one most prevalent in meteorology:

$$E = E_0 \cdot 10^{\frac{at}{b+t}}, \tag{2}$$

where t is the temperature in Celsius degrees, $E_0 = 4.58$ mmHg $= 6.10$ mb is the saturation pressure at 0°C, a and b are constants having the following values:

over ice $a = 9.5$, $b = 265.5°$ C,
over water $a = 7.5$, $b = 237.3°$ C.

It is evident that the saturation pressure is smaller over ice than over water (see Appendix 8).

The graph for the Magnus equation is shown in Figure 19.

The steep rise of E with increasing t should be noted.

It follows from the experiment described, that the phenomenon of saturation is independent of the presence in the space of other gases, such as air, if these gases do not react with the water vapor. This follows from the more general proposition, Dalton's law, which can be formulated for the case under consideration in the following way: "saturated vapor expands in another chemically inert gas as in a vacuum". The presence of air only retards the attainment of equilibrium.

Therefore it is not quite correct to say: "the air is saturated with vapor". It is not the air, but the water vapor which is saturated. Thus the phenomenon of "saturation" of a certain space merely means that the water vapor is in a dynamic equilibrium with water (or with ice), and that the volume of air under consideration can, at a given temperature, contain a well defined maximum amount of water vapor. Some justification for this incorrect expression can be given by the fact that the water vapor contained in the atmosphere has the same temperature as the air. Therefore the use of the expression "air saturated with water vapor" will be continued, for brevity.

Water vapor is called unsaturated, if its pressure is less than the pressure of saturated vapor at the same temperature.

$$e < E. \tag{3}$$

Water or ice, when in contact with unsaturated water vapor, are not in equilibrium with it, and will evaporate.

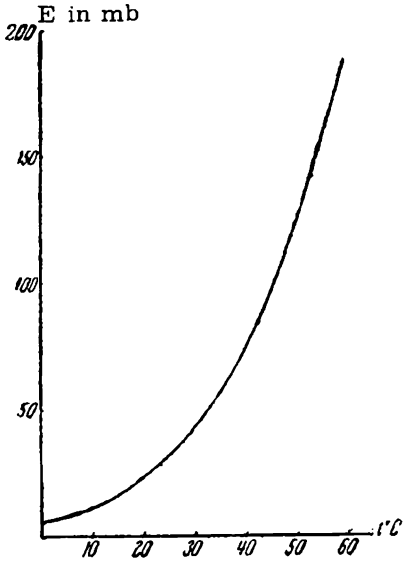

Figure 19. Graph of saturation pressure as a function of the temperature

Saturated vapor and ideal gas are, according to their properties, two extreme cases of the gaseous phase . Unsaturated vapor has intermediate properties.

The density ρ_w of the water vapor (or its reciprocal - the specific volume v_w), its pressure e, and its temperature T, are parameters of state of the water vapor.

Unsaturated water vapor follows the laws of ideal gases with reasonable accuracy, namely:

Process	Law	Equation
Isothermic	Boyle-Mariotte .	$e \cdot v_w = e_0 v_{w_0}$, $\frac{e}{e_0} = \frac{\rho_w}{\rho_{w_0}}$, (4)
Isobaric	Gay-Lussac ..	$v_w = v_{w_0}(1 + \alpha t)$; $\rho_w \frac{\rho_{w_0}}{1 + \alpha t}$; (5)
Isochoric	Gay-Lussac ..	$e = e_0(1 + \alpha t)$. (6)
Arbitrary	Clapeyron ...	$e \cdot v_w = R_w T$; $e = \rho_w R_w T$ (7)

The gas constant for unsaturated water vapor can be determined theoretically from the relationship

$$R_w = \frac{R_0}{\mu_w} , \tag{8}$$

where μ_w = 18.02 – molecular weight of water vapor; this gives

$$R_w = \frac{8.314 \cdot 10^7 \ \frac{cm^2}{sec^2 \ deg}}{18.02} = 4.60 \cdot 10^6 \ \frac{cm^2}{sec^2 \ deg} = 460 \ \frac{m^2}{sec^2 \ deg}$$

Table 8

The parameters of state for saturated water vapor

T...	273.2	283.2	293.2	303.2	313.2	373.2
E...	6.108	12.27	23.37	42.41	73.75	1013.226
v_w...	206 300	106 400	57 840	32 930	19 550	1673
Experimental R_w	461.3	461.1	461.0	460.6	460.3	451.3

The saturation pressure and the specific volume in Table 8 conform with the international definitions, established by the third international conference which took place in Washington in 1934, and which dealt with the formulation of vapor tables. The last line gives the value R_w, calculated according to the Clapeyron equation.

It can be seen that the mean value of the gas constant: $R_w = 460 \frac{m^2}{sec^2 \ deg}$ can be accepted for temperatures between 0^o and 40^oC.

This value agrees perfectly with the value of R_w, obtained above from the universal gas constant. Thus, Clapeyron's equation works out quite accurately for saturated water vapor, and even better for unsaturated vapor within the allowed temperature limits, which is very important for meteorology.

It will be convenient in the following to express R_w by the gas constant R_d for dry air.

Since

$$R_w : R_d = 460 : 287 = 1.604,$$

we have

$$R_w = 1.604\, R_d \tag{9}$$

It is also useful to compare the density ρ_w of water vapor with the density ρ_d of dry air under the same conditions.

Since

$$\rho_w = \frac{e}{R_w \cdot T}; \qquad \rho_d = \frac{p}{R_d T},$$

we have for e=p, and the same temperature T

$$\rho_w : \rho_d = R_d : R_w, \tag{10}$$

so that

$$\rho_w = 0.622\, \rho_d \approx \frac{5}{8}\, \rho_d \tag{11}$$

(see Appendix 9).

§ 3. The Specific Heat of Water Vapor

The specific heat of saturated water vapor is negative, as will be shown below. This means that heat does not have to be supplied but taken away in order to raise the temperature of saturated vapor.

For unsaturated water vapor, as is the case with an ideal gas, the specific heat can assume different values, depending on the character of the process which accompanies the heating. The specific heat c_{pw} of water vapor at constant pressure and the specific heat c_{vw} at constant pressure are taken into account.

We have seen above that the specific heats of an ideal gas depend neither on the temperature nor on the pressure. The specific heats of water vapor do depend slightly on pressure and temperature, but this dependence can be neglected for meteorological purposes. In fact, the specific heat of water vapor differs only by a correction factor from the specific heat of moist air.

Therefore we shall consider the specific heats c_{pw} and c_{vw} as constants, independent of both temperature and pressure; setting

$$c_{pw} = 0.445 \frac{\text{cal}}{\text{g deg}} = 1862 \frac{\text{m}^2}{\text{sec}^2 \text{ deg}},$$

$$c_{vw} = 0.335 \frac{\text{cal}}{\text{g deg}} = 1402 \frac{\text{m}^2}{\text{sec}^2 \text{ deg}},$$

we obtain from the adopted values:

$$c_{pw} - c_{vw} = 460 \frac{\text{m}^2}{\text{sec}^2 \text{ deg}} = R_w,$$

which agrees with equations (8) § 2. As water vapor differs from an ideal gas, $c_{pw} - c_{vw}$ also differs, of course, from R_w but has the same order of magnitude (see Appendix 9).

§ 4. The Thermic Properties of Liquid Water and of Ice

The specific volume of liquid water has a minimum value $v_f = 1.00 \frac{cm^3}{g}$ at 4°C. This value is adopted as the unit of specific volume; v increases both with the rise and fall of the temperature; it reaches the value of $1.043 \frac{cm^3}{g}$ at t = 100°.

Water has the greatest specific heat of all substances[sic]. The specific heat of water at 15°C is one calorie, according to the definition of the calorie: expressed in mechanical units

$$c_f = 4185 \frac{m^2}{sec^2\ deg}$$

Ice is the solid phase of water and often has a crystalline structure. It forms crystals of the hexagonal system. Several different forms of ice exist, corresponding to the different conditions of temperature and pressure to which it is subjected. Thus, at very high pressures ice can be preserved in the solid phase even at a temperature of +75°. But in nature ice exists in only one mode and therefore the other modes of ice will not be considered here.

The specific volume of ice is $v_i = 1.091 \frac{cm^3}{g}$, hence the volume of water increases in the course of freezing by 10%. Ice contracts so insignificantly on cooling down from 0°C that v_i can be regarded, for meteorological purposes, as constant for all temperatures.

The specific heat c_i also changes very little with the temperature so that its value at 0°C

$$c_i = 0.505 \frac{cal}{g\ deg} = 2114 \frac{m^2}{sec^2\ deg}$$

can be regarded as constant, equal for all temperatures (see Appendix 10).

§ 5. The Fundamental Equations of Thermodynamics

In the present section we shall derive all the fundamental equations of thermodynamics from a transformation of the first and second laws; with their help we shall obtain a very important relationship which characterizes the phase changes.

The derivation of the thermodynamic equations is based on the examination of the equation

$$dz = X\,dx + Y\,dy. \tag{1}$$

This equation would be exact, if X and Y, which are functions of x and y, satisfy the condition:

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}; \tag{2}$$

otherwise equation (1) would not be exact. This may be easily verified. Let:

$$z = f(x, y).$$

Then the total differential dz will be:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Assuming

$$X = \frac{\partial f}{\partial x}; \quad Y = \frac{\partial f}{\partial y},$$

we obtain:

$$\frac{\partial X}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}; \quad \frac{\partial Y}{\partial x} = \frac{\partial^2 f}{\partial x \partial y},$$

hence

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}.$$

The equation of the first law of thermodynamics

$$\delta q = du + p\,dv \tag{3}$$

can be written in the following form

$$\delta q = M\,dx + N\,dy, \tag{4}$$

where x, y are parameters of state of the system, so that

$$u = u(x, y), v = v(x, y), p = p(x, y), \text{ a } M(x, y) \text{ and } N(x, y)$$

are functions of state.

The equations (3) and (4) can be made identical by substituting in equation (3)

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy,$$
$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy,$$

then we obtain the similar expressions:

$$M = \frac{\partial u}{\partial x} + p \frac{\partial v}{\partial x};$$
$$N = \frac{\partial u}{\partial y} + p \frac{\partial v}{\partial y}.$$

Differentiating M with respect to y and N with respect to x, and subtracting the second derivative from the first, we obtain

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{\partial v}{\partial x} \cdot \frac{\partial p}{\partial y} - \frac{\partial v}{\partial y} \cdot \frac{\partial p}{\partial x} \neq 0, \tag{5}$$

because δq is not a total differential. This is the first fundamental equation of thermodynamics.

The equation of the second law can be written as follows:

$$d\varphi = \frac{\delta q}{T} = \frac{M}{T} dx + \frac{N}{T} dy,$$

$d\varphi$ being a total differential.

According to equations (1) and (2) we can write:

$$\frac{\partial}{\partial y}\left(\frac{M}{T}\right)=\frac{\partial}{\partial x}\left(\frac{N}{T}\right),$$

hence

$$T\frac{\partial M}{\partial y}-M\frac{\partial T}{\partial y}=T\frac{\partial N}{\partial x}-N\frac{\partial T}{\partial x}$$

or

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=\frac{1}{T}\left(M\frac{\partial T}{\partial y}-N\frac{\partial T}{\partial x}\right). \tag{6}$$

This is the second fundamental equation of thermodynamics. Finally, by combining equations (5) and (6) we obtain one of the more general equations, resulting from the application of both laws:

$$M\frac{\partial T}{\partial y}-N\frac{\partial T}{\partial x}=T\left(\frac{\partial v}{\partial x}\cdot\frac{\partial p}{\partial y}-\frac{\partial v}{\partial y}\cdot\frac{\partial p}{\partial x}\right). \tag{7}$$

§ 6. The Latent Heat of Vaporization

The transition of a liquid into a gaseous state takes place in the following way. Some molecules, situated on the surface of the liquid, break away from it as a result of the thermal motion; they pass from the range of the molecular forces into space, where they become instantly subject to different laws of motion: they soon behave as gas molecules. Clearly, the fast molecules break away more readily than the slow ones, so that the molecules remaining inside the liquid have a lesser average velocity and thus the liquid cools as a result of evaporation. If the temperature of the evaporating liquid is to remain constant, i.e., if the store of internal energy of each gram of the liquid must not change, it is necessary to supply continuously from without some quantity of heat, which is proportional to the quantity of liquid evaporated. The supplied heat is equivalent to the work which is performed in vaporization against the forces of cohesion and against the external pressure. If the liquid evaporates into a space devoid of other gases, the work of expansion is carried out exclusively against the pressure of the water vapor; if other gases are also present, it is carried out against the total pressure. In the present chapter, we will consider vaporization in the absence of air.

The latent heat L of vaporization of a liquid is composed of two parts:

$$L=L_i+L_e, \tag{1}$$

of which the most important - L_i - is spent to overcome the intermolecular forces of cohesion, i.e., to increase the internal (potential) energy; L_i is called the internal latent heat of evaporation. The second part of the latent heat of evaporation - L_e - is spent to increase the volume occupied by a unit mass of the substance, i.e., as work against the external pressure. Let us denote by u_f and u_w respectively the internal energy of liquid water and water vapor, and v_f and v_w - their specific volumes. We should note that $v_f \ll v_w$. It is evident

that

$$L_i = u_w - u_f,$$
$$L_e = e(v_w - v_f),$$
$$L = L_i + L_e = u_w - u_f + e(v_w - v_f) \approx u_w - u_f + ev_w \tag{2}$$

(because evaporation takes place at a constant pressure).

The sum of the internal energy and the isobaric work of expansion is the enthalpy ϑ; therefore,

$$L = \vartheta_w - \vartheta_f, \tag{3}$$

i.e., the latent heat of evaporation of water is equal to the difference between the enthalpy of the water vapor, $\vartheta_w = u_w + ev_w$ and the enthalpy of water $\vartheta_f = u_f + ev_f$.

The latent heat of melting L_{if}, is also necessary for melting ice; it is expressed by the following equation:

$$L_{if} = \vartheta_f - \vartheta_i = u_f - u_i + p(v_f - v_i) \approx u_f - u_i. \tag{4}$$

Finally, a latent heat of sublimation L_{iw} is necessary for the sublimation of ice:

$$L_{iw} = \vartheta_w - \vartheta_i = u_w - u_i + e(v_w - v_i) \approx u_w - u_i + ev_w. \tag{5}$$

The latent heat of vaporization decreases with the increase of temperature. The dependence of the latent heat on the temperature is expressed by different empirical formulas. We will choose the following:

$$L = 597 - 0.57t \ \frac{\text{cal}}{\text{g}}, \tag{6}$$

whence

$$L_e = 30.1 + 0.11t \ \frac{\text{cal}}{\text{g}}.$$

The values of latent heat at 0°C can be considered to equal

$$L = 597 \frac{\text{cal}}{\text{g}},$$
$$L_{if} = 80 \frac{\text{cal}}{\text{g}}, \tag{7}$$
$$L_{iw} = L + L_{if} = 677 \frac{\text{cal}}{\text{g}}.$$

The change of the latent heat of vaporization with the temperature can be calculated theoretically by certain approximations. Differentiating equation (3) with respect to T, we obtain:

$$\frac{dL}{dT} = \frac{d\vartheta_w}{dT} - \frac{d\vartheta_f}{dT}.$$

Recalling that by equation (15), § 14, Chapter II, water vapor follows the laws of ideal gases, we obtain:

$$\frac{d\vartheta_w}{dT} = c_{pw},$$

and according to the definition

$$c_f = \frac{dq}{dT} = \frac{d\vartheta_f}{dT},$$

Hence

$$\frac{dL}{dT} \approx c_{pw} - c_f = 0.45 - 1.00 = -0.55 \frac{\text{cal}}{\text{g deg}}. \tag{8}$$

Thus, the change of the latent heat of vaporization with temperature is equal to the difference of the specific heats of water vapor and of water. The specific heat is chosen for the water vapor, which one assumes will become unsaturated during heating. The integration of equation (8) gives a result which agrees with the empirical equation (6).
In the same way we obtain the following equation for the latent heat of sublimation L_{iw}:

$$\frac{dL_{iw}}{dT} = c_{pw} - c_i = 0.45 - 0.50 = -0.05 \frac{\text{cal}}{\text{g deg}}.$$

Thus, the latent heat of sublimation can be practically regarded as constant, equal to its value at 0°C.

$$L_{iw} \approx 677 \frac{\text{cal}}{\text{g}}.$$

The derivation must be carried out more exactly for the latent heat of melting; but even in this case we obtain an accurate expression, and an accurate order of magnitude:

$$\frac{dL_{if}}{dT} = c_f - c_i = 1.00 - 0.50 = 0.50 \frac{\text{cal}}{\text{g deg}}.$$

Thus, the latent heat of melting rises and falls with the temperature. The results of the calculation agree closely enough with observation.

§ 7. The Fundamental Equations Characterizing Phase Changes

When deriving the equations of change of phases, it is primarily necessary to choose the parameters rationally, in order to form the equations of heat balance. Let us consider a unit mass of water, consisting of two phases which are in equilibrium. We will choose the temperature T as one of the variables, and as the second – the mass μ_2 of the second phase. Denoting by 1 and 2 any two phases which are in equilibrium, we shall call μ_1 and μ_2 the mass of water in each of the phases, so that

$$\mu_2 = 1 - \mu_1,$$

because the total mass of water is assumed to be equal to 1g. We obtain, then, for the specific volume of the mixture:

$$v = \mu_2 v_2 + (1 - \mu_2) v_1 = v_1 + \mu_2 (v_2 - v_1), \tag{1}$$

and for the specific heat c:

$$c = \mu_2 c_2 + (1 - \mu_2) c_1 = c_1 + \mu_2 (c_2 - c_1). \tag{2}$$

Furthermore, let us denote the latent heat of transformation from phase 1 into phase 2 by L_{12}.

We can then write the equation of heat balance for the mixture under consideration as follows:

$$dq = L_{12}\, d\mu_2 + c\, dT. \tag{3}$$

Thus

$$M = L_{12}; \quad N = c = c_1 + \mu_2 (c_2 - c_1);$$
$$x = \mu_2; \quad y = T.$$

Let us apply the fundamental equations of thermodynamics (5), (6), (7) § 5 to equation (3). To this end we shall first determine all the partial derivatives:

$$\frac{\partial M}{\partial y} = \frac{dL_{12}}{dT}; \qquad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial \mu_2} = v_2 - v_1;$$
$$\frac{\partial N}{\partial x} = \frac{\partial c}{\partial \mu_2} = c_2 - c_1; \qquad \frac{\partial v}{\partial y} = \frac{dv}{dT};$$
$$\frac{\partial T}{\partial y} = \frac{dT}{dT} = 1; \qquad \frac{\partial p}{\partial x} = \frac{\partial p}{\partial \mu_2} = 0;$$
$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \mu_2} = 0; \qquad \frac{\partial p}{\partial y} = \frac{dp}{dT}.$$

$\frac{\partial T}{\partial \mu_2} = \frac{\partial p}{\partial \mu_2} = 0$; and $\frac{\partial p}{\partial y} = \frac{dp}{dT}$ are chosen so that neither the temperature, nor the pressure of transition are dependent on either phase, but each is completely determined by the other. Thus we obtain:

$$\frac{dL_{12}}{dT} + (c_1 - c_2) = (v_2 - v_1)\frac{dp}{dT}, \tag{4}$$

$$\frac{dL_{12}}{dT} - \frac{L_{12}}{T} = c_2 - c_1, \tag{5}$$

$$\frac{L_{12}}{T} = (v_2 - v_1)\frac{dp}{dT}. \tag{6}$$

We will give some examples of the application of the obtained equations. Equation (6), called the Clausius-Clapeyron equation, can be used for calculating L_{12} from the empirical values of $v_1, v_2, \frac{dp}{dT}$ or from the empirical L_{12}, v_1, v_2. Finally we can insert the empirical values in both parts of equation (6) in order to check the conclusions of the second law of thermodynamics, expressed by this equation.

We will calculate the slope of the curves of equilibrium near the triple point.

The slope of the evaporation curve:

$$\frac{dE}{dT} = \frac{L}{T(v_\partial - v_f)} = \frac{597 \times 4.185 \times 10^7}{273\,(206\,300 - 1)} \frac{\text{dyne}}{\text{cm}^2\ \text{deg}} = 440 \frac{\text{dyne}}{\text{cm}^2\ \text{deg}} = 0.44 \frac{\text{mb}}{\text{deg}}.$$

The slope of the sublimation curve:

$$\frac{dE_i}{dT}=\frac{L_{iw}}{T(v_w-v_i)}=\frac{677\times 4.185\times 10^7}{273\,(206\,300-1)}\ \frac{\text{dyne}}{\text{cm}^2\ \text{deg}}=510\ \frac{\text{dyne}}{\text{cm}^2\ \text{deg}}=0.51\ \frac{\text{mb}}{\text{deg}}\ .$$

The slope of the melting curve;

$$\frac{dp}{dT}=\frac{L_{if}}{T(v_f-v_i)}=\frac{80\times 4.185\times 10^7}{273\,(1.00-1.09)}\ \frac{\text{dyne}}{\text{cm}^2\ \text{deg}}=-137\times 10^6\ \frac{\text{dyne}}{\text{cm}^2\ \text{deg}}=$$

$$=-1.37\times 10^5\ \frac{\text{mb}}{\text{deg}}\ .$$

The value obtained for the slope of the curves of equilibrium shows that the sublimation curve is situated beneath the evaporation curve, because the saturation pressure over water decreases by 0.44 mb , whereas the saturation pressure over ice decreases by 0.51 mb , when the temperature is decreased from 0° to -1°. At the same time the melting pressure increases by $1.37\cdot 10^5$ mb = 137 bars $\approx$135 atmospheres, when the temperature is decreased from 0° to -1°.

Thus, the melting curve rises rapidly with the decrease of the temperature. The results obtained agree well with the experimental data. We can already determine the slope of the saturation curve at t = 100°C:

$$\frac{dE}{dT}=\frac{540\times 4.185\times 10^7}{373\times(1673-1)}=36\ \frac{\text{mb}}{\text{deg}}\ .$$

Let us note that the evaporation curve is parallel to the pressure axis in the neighborhood of the critical point.

We can use equation (5) to determine the specific heat of saturated water vapor only on the condition that the quantity of vapor remains constant:

$$d\mu=0,$$

then

$$c_w=c_f+\frac{dL}{dT}-\frac{L}{T};$$

at t = 0°C

$$c_w=1-0.57-\frac{597}{273}=-1.75\ \frac{\text{cal}}{\text{g deg}}$$

at t = 100°C

$$c_w=1-0.57-\frac{540}{373}=-1.0\ \frac{\text{cal}}{\text{g deg}}\ .$$

Thus, the specific heat of saturated water vapor is negative. This can be explained as follows. Some quantity of heat must be supplied to the saturated water vapor in order to increase its temperature by 1°C but the vapor ceases to be saturated after being heated; in order to restore the state of saturation without increasing the quantity of water vapor, the latter must be compressed. But compression produces some heat, and this quantity of heat must be dissipated, if the temperature $(t+1)^\circ$ of the vapor is to be maintained.

The quantity of dissipated heat of compression is found to be greater than the heat supplied during heating; the specific heat is therefore negative.

However, the specific heat of saturated water vapor plays a negligible part in atmospheric processes with a constant specific amount of water vapor. We have raised this question here, because equation (5) will make it possible to integrate further the adiabatic equation of condensation.

The Clausius-Clapeyron equation (6) makes it possible to derive theoretically the equations both of the sublimation curve and of the evaporation curve, within the temperature limits which are relevant in meteorology.

First, we will work out the equation of the sublimation curve. Since $v_w \gg v_i$, we can assume that

$$v_w - v_i \approx v_w = \frac{RT}{E},$$

and the Clausius equation can be written in the form

$$\frac{dE}{dT} = \frac{L_{iw}}{R_w} \cdot \frac{E}{T^2}. \tag{7}$$

Hence L_{iw}, according to the above, can be regarded as a constant; and then separating the variables and integrating, we obtain:

$$\ln \frac{E}{E_0} = \frac{L_{iw}}{R_w}\left(\frac{1}{T_0} - \frac{1}{T}\right); \tag{8}$$

inserting the constant values $L_{iw} = 677 \frac{\text{cal}}{\text{g}}$, $R_w = 0.10 \frac{\text{cal}}{\text{g deg}}$, $T_0 = 273^\circ$

and converting into decimal logarithms, we obtain:

$$\lg \frac{E}{E_0} = 2940\left(\frac{1}{T_0} - \frac{1}{T}\right)$$

transforming to exponential form we obtain,

$$E = E_0 \cdot 10^{\frac{10.8t}{273+t}}. \tag{9}$$

The equation of the sublimation curve thus obtained agrees closely with the following empirical equation of Magnus:

$$E = E_0 \cdot 10^{\frac{9.5t}{265.5+t}}. \tag{10}$$

In the same way we obtain the approximate equation of the evaporation curve. If L is regarded as a constant and equal to its value at t = 0°C, i. e., if L = 597, we have:

$$\lg \frac{E}{E_0} = 2593\left(\frac{1}{T_0} - \frac{1}{T}\right) \tag{11}$$

or

$$E = E_0 \cdot 10^{\frac{9.5t}{273+t}}. \tag{12}$$

The result thus obtained is of the same form as Magnus' equation for the saturation pressure

$$E = E_0 \cdot 10^{\frac{7.5t}{237.3+t}}. \tag{13}$$

However, the constants in the exponents of equations (12) and (13) differ from each other and therefore the results obtained by using the approximate theoretical equation do not agree completely with the experimental data, on the basis of which the empirical extrapolated equation (13) was derived.

All these equations are not used in practice, because it is more convenient to use tables or graphs. But the results obtained are of great interest and instructive in one respect, namely, they show how some theoretical relationships can be derived from the second law of thermodynamics, which cannot be obtained by any other method.

§ 8. Supercooled Water and Supersaturation in Relation to Ice

We have indicated above that by careful cooling of (liquid) water it is possible to reach a temperature which is lower than the freezing point, whilst the phase remains liquid. Such supercooled water will be in a stable thermodynamic equilibrium relative to water vapor, but not relative to ice; the water will freeze immediately after the introduction of even the finest crystals of ice. Observations show that supercooled water is constantly present in the atmosphere. Clouds and fog, which are formed at $t < 0^{\circ}C$, as a rule consist of supercooled water. The water still remains liquid even at a temperature of $-20^{\circ}C$. Water droplets can even be detected at temperatures lower then $-30^{\circ}C$.

Drops of supercooled water and water vapor can be in equilibrium; but if into such a system, including drops of supercooled water, crystals of ice are introduced at a given temperature, the system ceases to be in equilibrium. The water vapor is in equilibrium with the drops of water, but it will be supersaturated relative to the ice, i.e., the pressure of the water vapor e is greater than the saturation pressure relative to ice E_i

$$e > E_i.$$

As a result of this, part of the water vapor sublimates; but the decrease of the quantity of water vapor will cause the decrease of the saturation pressure relative to water, i.e., $e < E$; then the water will begin to evaporate.

As a result of these two processes, the ice crystals grow at the expense of the evaporating water drops. This process will continue until all the drops of water evaporate.

The table in Appendix 8 shows that this process takes place with particular vigor at a temperature which is close to $-12^{\circ}C$, when the difference of the saturation pressures over water and ice reaches its maximum value of 0.198 mm Hg.

Two important meteorological phenomena relating to supercooled water may be cited:

1) when a fog composed of drops of supercooled water passes over snow, it is dispersed;

2) when a small number of ice crystals get into a cloud composed of drops of supercooled water, they start to grow rapidly. These crystals cannot remain suspended in the air, and begin to fall; they melt while falling, and reach the earth's surface in the form of rain drops. Such fine crystals can exist even at a relative humidity of 55 - 60 % as regards water*. If the cloud grows upwards, intensive precipitation can only take place when its summit reaches the level where ice crystals are present.

§ 9. The Saturation Pressure over a Curved Surface

The above-mentioned arguments concerning the saturation pressure and curves of equilibrium refer to those cases, where the surface separating water and ice from water vapor is a plane. However, water condenses in the atmosphere mainly in the form of drops; condensation can occur on the surfaces of

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* [Nonfrozen water. Translator's note].

terrestrial objects (plants, stones, buildings etc) only in the immediate proximity of the earth.

We shall now derive the equation, which determines the saturation pressure over a curved surface.

Let us imagine a vessel filled with water (Figure 20) placed under a bell-jar evacuated of air and containing a vertical capillary tube in which water rises the height h up to a horizontal plane AB.

Let the vapor pressure at the surface CD be E. The pressure E' at the level AB will be smaller than E by the water head h, i.e.,

$$E' = E - h \cdot \rho_w, \tag{1}$$

where is the mean density of the water vapor.

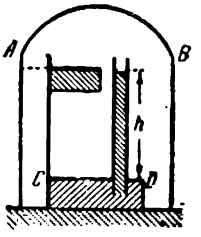

Figure 20. The saturation pressure is smaller relative to a concave surface than relative to a plane surface

The saturation pressure in the capillary tube at the level AB must be equal to E'; otherwise a perpetual motion machine would be possible. In fact, if the saturation pressure were greater in the capillary tube than under the bell-jar at the level AB, the water in the capillary tube would evaporate and descending along the tube would condense on the surface CD, thus performing work as a result of the uninterrupted upward flow of the liquid in the capillary tube. The pressure in the capillary tube can also not be less than E' (for the same reason).

The total pressure at the level CD must be the same inside and outside the tube, because only in this case can equilibrium be maintained.

The surface tension of the liquid is expressed by the Laplace equation:

$$P = K + \alpha \left(\frac{1}{R_1} + \frac{1}{R_2} \right), \tag{2}$$

where K is the pressure on a plane surface, α - the coefficient of surface tension, i.e., the force per unit length of the liquid surface, R_1 and R_2 are the radii of curvature of the principal normal sections of the liquid surface, these radii being considered positive, when pointed at the interior of the liquid.

The pressure on the plane surface CD outside the tube will then be E, and inside the capillary tube $E' + h \cdot \rho_f + \alpha \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$ where ρ_f is the liquid density.

Equating these expressions, we obtain

$$E = E' + h \cdot \rho_f + \alpha \left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

hence

$$E' = E - h \cdot \rho_f - \alpha \left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{3}$$

h being determined from equation (1)

$$h = \frac{E - E'}{\rho_w}$$

and substituted in equation (3), we obtain after transformation:

$$E' = E + \frac{\alpha \rho_w}{\rho_f - \rho_w} \left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

In the case of a round capillary tube we have: $\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{R}$; therefore,

$$E' = E + \frac{2\alpha \rho_w}{\rho_f - \rho_w} \cdot \frac{1}{R}. \tag{4}$$

Because the meniscus of the liquid is concave $1/R < 0$, and consequently $E' < E$ or a convex meniscus (in the case of a liquid which does not wet the tube) $1/R > 0$ and $E' > E$

In case of complete wetting, i.e., when the liquid forms an angle of 0° with the wall of the capillary the radius of curvature of the liquid surface will be equal to the radius r of the tube; the liquid surface will be hemispherical. Thus, equation (4) can be extended to apply to a drop with radius r. Ignoring the term ρ_w, in the denominator which is small in comparison with $\rho_f \approx 1$, and substituting for ρ_w in the numerator its value obtained according to Clapeyron's equation $\rho_w = \frac{E}{R_w T}$, we obtain the following approximate equation which determines the saturation pressure over the surface of a water drop:

$$E' = E + \frac{2\alpha}{R_w T} \cdot \frac{E}{\rho_f} \cdot \frac{1}{r}. \tag{5}$$

Calculating the relative difference of the pressures over the surface of a drop and over a plane surface, we obtain:

$$\frac{E' - E}{E} = \frac{2\alpha}{\rho_f R_w T} \cdot \frac{1}{r}. \tag{6}$$

Equation (6) is also correct for a drop surrounded by air.

Assuming $T = 273^\circ$, $\alpha = 73 \frac{\text{dyne}}{\text{cm}}$, $\rho_f = 1 \text{g/cm}^3$, $R_w = 4.60 \cdot 10^6$, we obtain:

$$\frac{E' - E}{E} = 0.12 \cdot 10^{-6} \cdot \frac{1}{r}. \tag{7}$$

where r is expressed in centimeters.

E' differs little from E as long as the radius of the drop is sufficiently great. However E' - E increases considerably with the decrease of r (see Appendix 11).

§ 10. The Role of Condensation Nuclei

Condensation on the finest drops, whose dimensions hardly exceed the dimensions of molecules, would be possible only at an exceedingly great supersaturation if the surface tension were not to be brought into equilibrium by other forces.

In fact, the very finest particles of hygroscopic substances, so-called condensation nuclei, are always present in the atmosphere.

As a result of careful chemical analyses it has been found that fog droplets always contain salt of marine origin which probably got into the atmosphere as a result of the evaporation of water sprayed from the ocean's surface. Industry is a more important source of nuclei of condensation (in particular, nuclei containing sulfurous compounds).

Measurement has shown that the usual dimensions of the nuclei vary from $7 \cdot 10^{-7}$ cm to 10^{-5} cm. The number of nuclei per 1 cm^3 varies from 1 to 10^6, but only $10^2 - 10^4$ nuclei per 1 cm^3 can be active, because the supersaturation in the atmosphere is usually so small that only the largest and most hygroscopic particles may serve as condensation nuclei.

The particle formed as a result of the condensation of water vapor on a hygroscopic nucleus, is still a highly concentrated water solution of the hygroscopic substance. The saturation pressure E'' over the water solution is less than the saturation pressure E over pure water. For solutions of low concentration, containing n molecules of the substance for every N molecules of water the following equation is used:

$$E'' - E = -E \cdot \frac{n}{N + n}. \qquad (1)$$

Equation (1) shows that the saturation pressure decreases with the increase of the solution concentration. Equation (1) is already not correct, although the saturation pressure continues to decrease with the increase of the concentration.

The nuclei of condensation can exist in the form of drops of highly concentrated solution, because water vapor is always present in the atmosphere. Thus, the saturation pressure over the drops will be less than that given in Appendix 11, and can even fall below the saturation pressure relative to a plane surface of pure water, if the size of the drops is large enough. Therefore, persistent fog and clouds are often observed at a relative humidity under 100%.

However, formation of ice crystals on the condensation nuclei cannot take place. It is assumed that special sublimation nuclei must be present in the atmosphere in order to initiate sublimation. The nature of these nuclei is still unknown. However, it appears obvious that some solid particles are centers of such sublimation nuclei. The order of magnitude of their linear dimensions is 10^{-6} cm. Their number in a volume unit is considerably less than the number of condensation nuclei.

§ 11. The Entropy of Water

We now derive the equation of entropy of water in the case of diphasic equilibrium.

Retaining the symbols of § 7, we write the equation of heat balance for a diphasic system in equilibrium,

$$\delta q = L_{12}\, d\mu_2 + c\, dT \tag{1}$$

or, since $c = c_1 + (c_2 - c_1)\,\mu_2$, then

$$\delta q = L_{12}\, d\mu_2 + c_1 dT + (c_2 - c_1)\,\mu_2 dT. \tag{2}$$

Substituting $(c_2 - c_1)$, according to the Clausius-Clapeyron equation (5), § 7, we obtain:

$$\delta q = L_{12}\, d\mu_2 + c_1 dT + \left(\frac{dL_{12}}{dT} - \frac{L_{12}}{T}\right)\mu_2 dT$$

or

$$\delta q = T d\left(\frac{L_{12}\cdot\mu_2}{T}\right) + c_1 dT. \tag{3}$$

Dividing by T, we obtain:

$$d\varphi = \frac{\delta q}{T} = d\left(\frac{L_{12}\mu_2}{T}\right) + c_1 d(\ln T). \tag{4}$$

Thus, the entropy of a diphasic system $(1 \leftrightarrow 2)$ in equilibrium is equal to:

$$\varphi = \frac{L_{12}\mu_2}{T} + c_1 \ln T + \text{const.} \tag{5}$$

The specific entropy of a diphasic system in equilibrium therefore equals:

Diphase equilibrium " system	Specific entropy
Water $\leftrightarrow$ saturated water vapor	$\varphi = \frac{L\mu_w}{J} + c_f \ln T + \text{const.}$
Ice $\leftrightarrow$ water	$\varphi = \frac{L_{if}\mu_f}{T} + c_f \ln T + \text{const.}$
Ice $\leftrightarrow$ saturated water vapor	$\varphi = \frac{L_{iw}\mu_w}{T} + c_f \ln T + \text{const.}$

(6)

Let us deduce the equation for the entropy of the triphasic system ice $\leftrightarrow$ water $\leftrightarrow$ water vapor.

In order to write down the equation of heat balance we choose the quantity μ_3 of water vapor and the quantity μ_1 of ice as independent variables. The quantity of water will then be $\mu_2 = 1 - (\mu_1 + \mu_3)$. The heat transferred to the system will manifest itself in the change of the relative quantities of the phases, whereas the usual parameters of state will remain constant (T = const; E = const; v_w = const). $L_{23}\,d\mu_3$ calories will be necessary for the production of $d\mu_3$ grams of water vapor and of $d\mu_1$ grams of ice.

$$L_{21}\, d\mu_1 = -L_{12}\, d\mu_1;$$

thus

$$\delta q = L_{23}\, d\mu_3 - L_{12}\, d\mu_1, \tag{7}$$

whence, dividing equation (7) by T we obtain:

$$d\varphi = \frac{L_{23}}{T}\, d\mu_3 - \frac{L_{12}}{T}\, d\mu_1. \tag{8}$$

Because L_{23} and L_{12} are constant for a given T, the entropy of the triphasic system will be expressed as follows:

$$\varphi = \frac{L_{23}\mu_3}{T} - \frac{L_{12}\mu_1}{T} + \text{const.} \tag{9}$$

§ 12. The Thermodynamic Surface of State for Water

The equation of state

$$f(p, v, T) = 0 \tag{1}$$

cannot be expressed in the case of water by any elementary function. But the character of the equation of state is such that all the points at which equation (1) holds are situated on a continuous surface in the p, v, T, space, which is called the thermodynamic surface of state for water (Figure 21) and represents all the states of water, which are important for meteorology.

On this surface lie five characteristic curves, which divide the surface into six different regions.

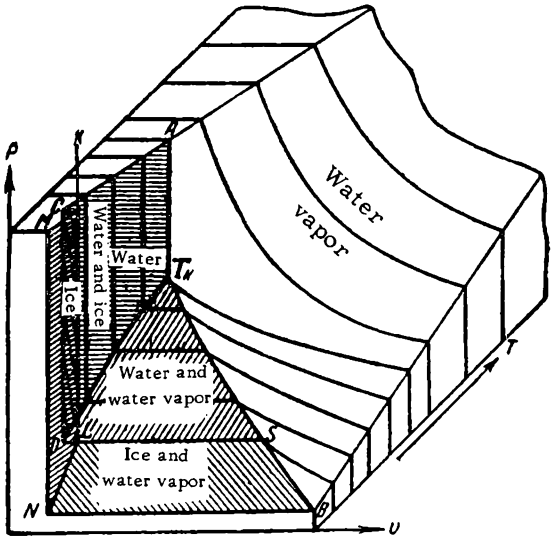

Figure 21. The thermodynamic system of state of water

1) The monophasic region of unsaturated water vapor, bounded both by the isotherm AT_k, corresponding to the critical temperature, and by the curve T_kB, corresponding to the transition of the water vapor into a state of saturation. This part of the surface recalls the thermodynamic surface of Clapeyron's equation of state for ideal gases. It is determined by a family of isotherms, which resemble more the hyperbolic isotherms of ideal gases, the higher the temperature, the greater the specific volume, and the smaller the pressure.

2) The monophasic region AT_kDC for liquid water is bounded by the isotherm of the critical temperature AT_k, by the curve T_kD corresponding to the onset of evaporation of water, and by the curve CD corresponding to the onset of freezing of water; this region of the surface is determined by a family of almost vertical, straight isotherms.

Chapter IV

THE THERMODYNAMICS OF MOIST AIR

§ 1. The Parameters of State of Moist Air

In the present chapter we will examine a two-component system -- moist air -- composed of dry air and water vapor. The quantities referring to dry air will be distinguished by the subscript d; those which refer to water vapor, by the subscript w, and those referring to the mixture, i.e. to the moist air, will bear no subscript at all.

We assume that the temperatures of both components are identical, and equal the temperature of the mixture. The state of moist air and of its components is characterized by the following parameters:

	Pressure	Density	Specific volume	Temperature
Dry air	p_d	ρ_d	v_d	T
Water vapor	e	ρ_w	v_w	T
Moist air	p	ρ	v	T

It has been empirically shown that the state of a two-component system is determined by two independent parameters, and that all the thermodynamic functions of state of such a system can be expressed by means of these parameters.

The pressure of moist air for each separate air mass is equal, according to Dalton's law, to the sum of the partial pressure of dry air p_d and of water vapor e:

$$p = p_d + e. \tag{1}$$

One can easily check for oneself the correctness of this law for an enclosed volume of air; the manometer will indicate a decrease of pressure e, equal to the previously determined vapor pressure, if the water vapor is removed by adsorption on any hygroscopically active substance ($CaCl_2$, H_2SO_4 etc).

We shall show in the following that water vapor does not form an independent atmosphere; its partial pressure e is therefore not expressed by the static pressure of the water vapor contained in the atmosphere. We might imagine that all the water vapor contained in the atmosphere should condense and be precipitated; we would then find the atmospheric pressure to have decreased only by a value equal to the weight of the condensed water, but not at all by the pressure of the water vapor.

Observations have shown that the pressure of water vapor changes very erratically with height, and does not conform to any law. An increase of the pressure of water vapor with height can be observed in separate layers. Nevertheless, the total pressure p decreases with height in a regular way; in the case of a static equilibrium, it is equal at at any height to the weight of a

column of moist air of unit cross section rising above the considered level. Consequently, the partial pressure of the dry air decreases with the same irregularity as the water vapor pressure. Thus dry air also does not form a separate atmosphere.

It follows from the above that the water vapor entering into each volume element of the atmosphere, displaces that amount of dry air, which has a partial pressure exactly equal to the pressure of the water vapor which replaces it. The water vapor pressure e increases while water is evaporating in an open air volume, and the total pressure p does not change (providing the weight of the evaporated water is not taken into consideration), because the partial pressure p_d of the dry air decreases so as to counterbalance the increase of e.

It is similarly necessary to consider the density of moist air as a sum of the partial densities of dry air and of water vapor:

$$\rho = \rho_d + \rho_w. \tag{2}$$

But in this case we must keep in mind two different circumstances in which moist air is obtained, and which differ in principle. If the moist air is produced in an enclosed volume, its density will increase together with the pressure, in proportion to the evaporation of the enclosed water, because a fresh quantity of water vapor will be added to the dry air.

In a free atmosphere, as we have just seen, the water vapor expels and replaces a part of the dry air, the total pressure of the resultant moist air remaining constant, while the partial pressure changes.

Therefore for the density of moist air we obtain:

$$\rho = \rho_d + \rho_w = \frac{p-e}{R_d T} + \frac{e}{R_w T}$$

or

$$\rho = \frac{p - 0.378e}{R_d T}; \tag{3}$$

thus moist air is lighter than dry air at the same pressure and temperature.

§ 2. The Characteristics of Humidity

We have pointed out in the previous section that the state of moist air and of its components is characterized by three parameters, which may be arbitrarily chosen. It is convenient to choose these variables in such a way that two of them should directly characterize the state of the air and vapor mixture, and the third - the state of the water vapor. We will call this third parameter - a humidity characteristic.

The following variables can be chosen as characteristics of humidity:

a) the water vapor pressure e,

b) the water vapor density ρ_w,

c) the absolute humidity a,

d) the specific humidity s,

e) the mixing ratio w,

f) the relative humidity r,

g) the dew point τ.

a) As we have already said, the partial pressure which is exerted by the water vapor is called the water vapor pressure e. It is measured, like air pressure, in dynes per square centimeter, in millibars, or millimeters Hg:

$$[e] = [ML^{-1}T^{-2}].$$

b) The mass of the water vapor, contained in a unit volume of air is called the water vapor density ρ_w. The density is measured in grams per cubic centimeter

$$[\rho_w] = [ML^{-3}].$$

According to the foregoing section, the following relationship exists between the water vapor pressure and its density:

$$\rho_w = \frac{e}{R_w T} = 0{,}622 \frac{e}{R_d T} = 0{,}22 \cdot 10^{-6} \cdot \frac{e}{T} \frac{\text{g}}{\text{cm}^3},$$

if e is measured in dynes per square centimeter;

$$\rho_w = 0{,}22 \cdot 10^{-3} \frac{e}{T} \frac{\text{g}}{\text{cm}^3}, \tag{1}$$

if e is measured in millibars, and finally

$$\rho_w = 0{,}29 \cdot 10^{-3} \frac{e}{T} \frac{\text{g}}{\text{cm}^3}, \tag{2}$$

if e is measured in millimeters Hg.

c) The mass of water vapor measured in grams, contained in 1 cubic meter of air is called the absolute humidity a. The absolute humidity is, therefore measured in g/m^3:

$$[a] = [ML^{-3}].$$

It is obvious that the absolute humidity numerically equals the density, but is expressed in other units:

$$a = \rho_w \cdot 10^6 \frac{\text{g}}{\text{m}^3}. \tag{3}$$

The following relationship exists between the absolute humidity and the water vapor pressure:

$$a = 0{,}22 \frac{e}{T} \frac{\text{g}}{\text{m}^3},$$

if e is measured in dynes per cm^2;

$$a = 220 \frac{e}{T} \frac{\text{g}}{\text{m}^3}, \tag{4}$$

if e is measured in millibars. Finally

$$a = 290 \frac{e}{T} \frac{\text{g}}{\text{m}^3} = \frac{1.06}{1 + \alpha t} \cdot e, \tag{5}$$

if e is measured in mm Hg.

The following approximate relationship between a and e may be established in the latter case. Since the coefficient $\frac{1.06}{1+\alpha t}$ differs little from unity, and at $t = 16{,}4^\circ$ C exactly equals unity, we have

$$a \approx e, \tag{6}$$

i.e., the absolute humidity numerically approximates the water vapor pressure, measured in mm Hg. The water vapor pressure is therefore sometimes called absolute humidity. One should keep in mind, however, that such usage is possible only in case the water vapor pressure is measured in mm Hg.

d) The ratio of the mass of water vapor to the mass of moist air contained in the same volume, is called the specific humidity

$$s = \frac{M_w}{M} \qquad [s] = [0]$$

In other words, the specific humidity is the ratio between the density of water vapor and the density of moist air, under the same conditions. According to equation (3) §1

$$s = \frac{\rho_w}{\rho} = 0{,}622\,\frac{e}{p - 0{,}378e}. \tag{7}$$

e) The ratio of the water vapor's mass to the mass of dry air contained in the same volume, is called the mixing ratio w.

$$w = \frac{M_w}{M_d} \qquad [w] = [0]$$

or, in other words, w is the ratio of the water vapor density to the density of dry air under the same conditions, i.e.,

$$w = \frac{\rho_w}{\rho_d} = 0{,}622\,\frac{e}{p - e}. \tag{8}$$

In equations (7) and (8) e can be expressed in any units, but certainly in the same units, as p.

Since e is numerically very small in comparison with p, s and w are fractions of the order of magnitude of thousandths. The units in which w and s are measured are 1,000 times smaller, in order not to have to deal with such small values; the definition of these characteristics will change somewhat, accordingly.

The quantity of water vapor in grams per one kilogram of dry air will be called specific humidity, i.e.,

$$s = 622\,\frac{e}{p - 0{,}378e}\,\frac{\text{g}}{\text{kg}}; \qquad w = 622\,\frac{e}{p - e}\,\frac{\text{g}}{\text{kg}}, \tag{9}$$

it follows from the definition of s and w that

$$s = \frac{M_w}{M} = \frac{M_w}{M_w + M_d} = \frac{w}{1 + w}; \qquad w = \frac{M_w}{M_d} = \frac{M_w}{M - M_w} = \frac{s}{1 - s}, \tag{10}$$

$$w - s = sw > 0; \tag{11}$$

since s and w are essentially positive, w is always greater than s. This is also a direct result of equation (9). Since w in the atmosphere is usually smaller than 0.04, $w - s < 0.0016$, i.e. this difference is so small that it can practically be disregarded. Since usually $e \ll p$, e can be omitted from the denominator, and thus we obtain

$$s \approx w \approx 0{,}622\,\frac{e}{p} \approx \frac{5}{8}\,\frac{e}{p}. \tag{12}$$

It will be proved in the following that s and w remain constant during dry-adiabatic processes, and generally when there is no moisture exchange with the environment. s and w therefore are invariant characteristics of humidity, and are used for indentification of air masses.

f) The ratio of the mass of the moisture actually present in the air to the mass of the maximal content of moisture, with which the air becomes saturated at the

same temperature, is called the relative humidity. The relative humidity is a dimensionless variable and is generally measured in percentage. If we respectively denote by E, A, s, W the water vapor pressure, absolute humidity, specific humidity and mixing ratio corresponding to saturation, we obtain:

$$r=\frac{e}{E}\cdot 100^0/_0=\frac{a}{A}\cdot 100^0/_0=\frac{s}{S}\cdot 100^0/_0=\frac{w}{W}\cdot 100^0/_0. \tag{13}$$

The relative humidity can exceed 100% in the case of supersaturation. At negative temperatures, we should make a distinction between relative humidities over ice and over water .

Since the saturation pressure E over water is greater than the saturation pressure E_1 over ice, we have at the same temperature

$$r_i>r. \tag{14}$$

Air is saturated, if r=100%; air is not saturated with water vapor if $r < 100\%$.

g) The dew point is the temperature at which isobarically cooled air becomes completely saturated. The humidity characteristics are so numerous because water vapor plays a very important role in the atmosphere.

Each of these characteristics has its advantages and disadvantages. The water vapor pressure, which is not constant, can be easily determined by the difference between the readings of the dry and the moist thermometers in a psychrometer.

The water vapor density, which bears a simple relationship to the absolute humidity and the vapor pressure is a very important index of humidity, of use in the study of the problems of emission and absorption of radiation in the atmosphere. It can be measured by means of an absolute hygrometer.

The relative humidity and the mixing ratio are invariant characteristics of humidity, and hence widely used in dynamic and synoptic meteorology, despite their being obtained not by a direct measurement but by computation from other characteristics.

The relative humidity is the most important characteristic. It is measured by means of a hair hygrometer. In meteorological stations and in aerologic ascents the relative humidity is generally measured and more rarely the water vapor pressure. The remaining characteristics are computed.

The dew point occupies a rather special place, being a thermo-hygrometric characteristic.

The dew point can be measured directly by means of a condensation hygrometer.

§ 3. Graphic Representation of Humidity Characteristics

Three parameters of moist air: p, T and r, are usually determined by aerologic sounding. In this section we will examine the graphic methods of obtaining the most important characteristics of humidity.

For this purpose it is expedient to use the thermodynamic diagrams mentioned in Chapter II, in order to avoid plotting special diagrams. On each diagram isobars and isotherms are plotted, whose points of intersection well define the state of dry air, or of saturated water vapor, insofar as the saturation pressure depends on the temperature only. The state of unsaturated moist air can

not be made to correspond to a point on the diagram. Therefore we divide the calculation of the humidity characteristics into two stages. We first determine the functions of state of the saturated water vapor according to the given p, T. From this the characteristics of humidity for the actual state will be determined by a subsequent calculation. It is not necessary to plot any additional lines in order to determine the saturation pressure. Since the saturation pressure is a function of the temperature only, any isotherm is at the same time an isoline of the saturation pressure, E = const. A supplementary scale of saturation pressure could be superimposed on the temperature scale, but such an operation is not necessary. It is necessary, however, to plot supplementary lines (lines of equal specific humidity), called isograms, in order to determine S and W. In this case we will assume, for the sake of simplicity (according to equation (12) § 2)that

$$S \approx W \approx 622 \frac{E}{p}. \tag{1}$$

The form of the isograms depends on the coordinates of the thermodynamic diagram. Since the equation of the isogram is S = const., we have, according to equation (1),

$$d \lg p = d \lg E. \tag{2}$$

along the isogram.

Let us now express the saturation pressure in terms of the temperature, restricting ourselves to the approximate equation (8) § 7 Chapter III:

$$d \lg E = -kd\left(\frac{1}{T}\right).$$

We shall then obtain the equation of the isogram in the following form

$$d \lg p + kd\left(\frac{1}{T}\right) = 0 \tag{3}$$

or, on integration

$$\lg \frac{p_0}{p} = k\left(\frac{1}{T} - \frac{1}{T_0}\right). \tag{4}$$

$x = T$, $y = \lg \frac{p_0}{p}$, on the emagram, and hence the equation of the isogram plotted with respect to the coordinates of the emagram will be:

$$xy = k\left(1 - \frac{x}{x_0}\right), \tag{5}$$

i. e., each isogram will be represented by a segment of a hyperbola on the emagram. However, the isograms are practically straight lines in the ranges of temperature and pressure occurring in the atmosphere (Figure 23). Their slope is such, that they are placed between the dry-adiabates and the isotherms. The slope increases somewhat with increasing temperature.

$x = \lg \frac{T}{T_0}$, $y = T \lg \frac{p_0}{p}$; on the aerogram; therefore the equation of the isogram plotted in the coordinate system of the aerogram will be :

$$\lg\left(1 - \frac{y}{k}\right) = x - x_0; \tag{6}$$

expanding in a power-series, and retaining the first term only, we shall obtain after simple transformations

$$y = k'(x_0 - x),$$

or

$$k' = \frac{k}{0.4343},$$

consequently, the isograms on the aerogram are also straight lines. The same can be shown for the tephigram.

The value of the specific humidity (mixing ratio) for the state of saturation, can be determined by means of an isogram: the points (T, p) are plotted on the diagram, according to the results of sounding, and the value $S (\approx W)$ is determined by interpolation between the isograms.

Let us now show how this value can be determined without marking an additional scale for the saturation pressure. Since

$$W \approx S \approx 622 \frac{E}{p},$$

the saturation pressure E, measured in millibars, is numerically equal to the specific humidity S at p = 622 mb.

Thus we arrive at the following rule : it is sufficient to determine the mixing ratio corresponding to saturation at the same temperature, and at the same pressure of 622 mb , in order to compute the saturation pressure at the temperature T . Consequently, it is sufficient to ascend along the isotherm to its intersection at the point B with the isobar p=622 mb, in order to determine the saturation pressure at the point A. The mixing ratio W_B is numerically equal to the unknown saturation pressure E. The obtained saturation pressure is measured in millibars (Figure 24). The error in the determination of E thus rarely exceeds 4% of the real value, obtained from the tables.

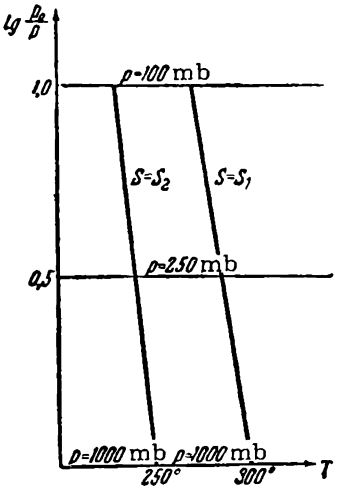

Figure 23. The isograms on the emagram are practically straight lines

We use the equations

$$w = W \cdot r; \quad e = E \cdot r.$$

in order to determine the actual humidity, on the basis both of the given relative humidity and the determined values, corresponding to saturation.

These computations are carried out graphically. Let us examine (Figure 25) a mass of saturated moist air at the point $A (p = 100$ cb, T, $W_A)$. Let us transform it, isothermically, into the state $B (p = r$ cb, T, $W_B)$. It is obvious that the saturation pressure will then remain unchanged, but the mixing ratio will again be different:

$$W_A = 622 \frac{E}{100},$$

$$W_B = 622 \frac{E}{r},$$

hence

$$\left.\begin{aligned} \frac{W_A}{W_B} &= \frac{r}{100} = r^0/_0 \\ W_A &= W_B \cdot r = w_B. \end{aligned}\right\} \qquad (7)$$

or

Thus the mixing ratio for the state of saturation at the point A is numerically equal to the mixing ratio at the point B where the moisture content is $r^0/_0$.

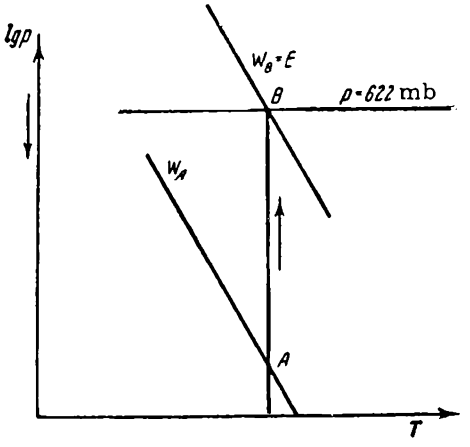

Figure 24. Graphic determination of the saturation pressure on the emagram

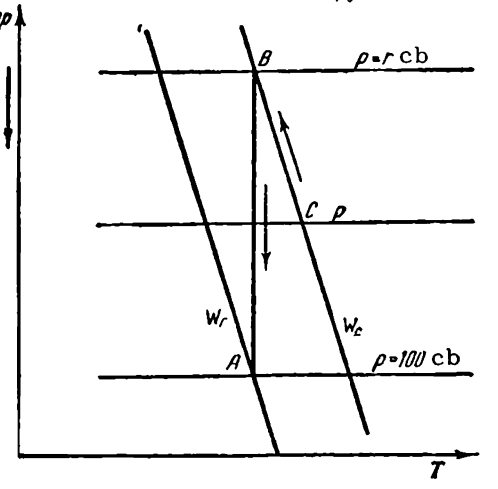

Figure 25. Graphic determination of the actual mixing ratio on the emagram

We have then the following method for determining the real specific humidity, according to the given (p, T, r). We mark the point C with the help of the fixed (p, T) and we trace the isogram through the point C to the isobar $p = r$ cb. An isotherm is traced through the point of intersection of the isogram with the isobar $p = r$ cb till it intersects the isobar $p = 100$ cb.

The mixing ratio corresponding to the isogram passing through the point A, will then equal the unknown mixing ratio for the state (p, T, r). Indeed, according to equation (7),

$$W_A = W_B \cdot r = W_C \cdot r = w. \qquad (8)$$

The calculation of the dew point τ can be effected immediately, if p is known and w is determined. It is obvious that the temperature at the point E, obtained as the intersection of the isobar p and the isogram will be equal to the dew point τ. If w is not determined beforehand, and it is required to determine τ for a fixed state (p, T, r) on the diagram (Figure 26) then it is necessary to proceed in the following way:

$$C \rightarrow B \rightarrow A \rightarrow E; \quad T_E = \tau.$$

The actual pressure e of the water vapor is easily found, once the dew point is determined. Since w, τ, and e bear to each other the same relationships as W, T, E, the method of determination of e is similar to that of E (Figure 27). It is sufficient, for this purpose, to ascend along the isotherm from the dew point E to the isobar 622, and to read there the value w. This value, as stated previously, is numerically equal to e. If the intermediate results do not interest us, and e is to be determined for the fixed values (p, T, r), then the following route must be followed:

$$C \rightarrow B \rightarrow A \rightarrow E \rightarrow F.$$

$$w_F = e.$$

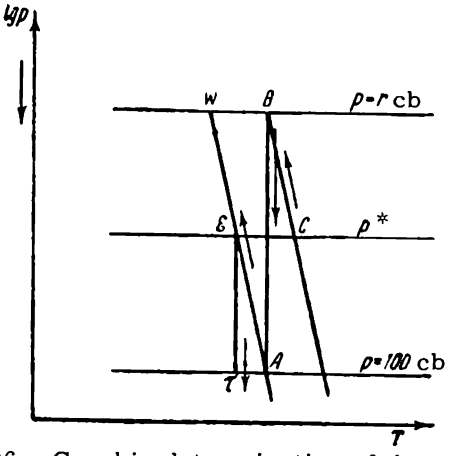

Figure 26. Graphic determination of the dew point on the emagram

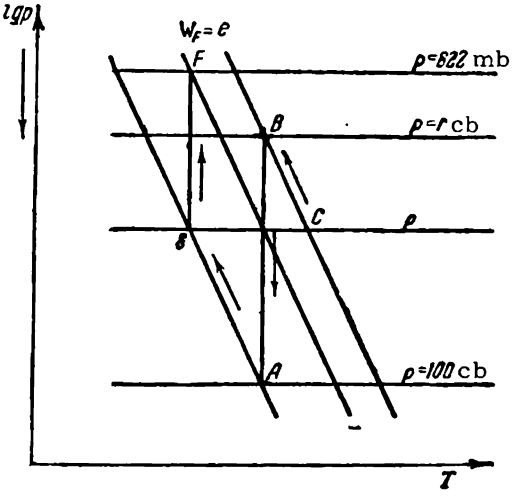

Figure 27. Graphic determination of the actual pressure of water vapor on the emagram

§ 4. The Equation of State for Moist Air

Moist air can be considered as a monocomponent system-i.e., as a homogeneous mixture of gases. This system will also be monophasic and the change of state will only cause a change of the parameters of state of this system, if a state of saturation is not reached. The system will be diphasic, and even triphasic, if saturation is reached. The change of state of such a di- or triphasic system will be accompanied by a phase change.

The parameters of state of moist air are linked together by the equation of state. We have already obtained one of the forms of this equation in the study of the problem of moist air density:

$$\rho = \frac{p - 0.378\, e}{R_d \cdot T}. \qquad (1)$$

However, since this form of the equation of state is somewhat unwieldy, it is useful to transform it into a simpler one. Let us first work out an equation for the specific gas constant. Let us consider 1 g of moist air, which contains s g of water vapor (s — the specific moisture), and $(1-s)$ g of dry air.

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* [The Russian text has p=r, an obvious misprint -- Translator's note].

The specific gas constant will then be determined for moist air, as the mean between the gas constant of dry air R_d, and of water vapor R_w:

$$R=(1-s)\cdot R_d+sR_w \tag{2}$$

or expressing R_w by R_d,

$$R_w=1{,}605\ R_d,$$

hence

$$R=(1-s)R_d+1{,}605\ s\cdot R_d \tag{3}$$

or

$$R=R_d(1+0{,}605\,s);\ R\approx R_d(1+0{,}605\,w).$$

Thus the gas constant for moist air changes with the change of humidity. $s\approx w$ is always smaller than 0.04 under atmospheric conditions; therefore, R can change within the limits $R=R_d=287$ to $R=1.024\cdot R_d=294$, i.e., at most by 2% of its minimum value. The equation of state can now be written:

$$p\cdot v=RT=R_d(1+0.605\,s)\cdot T. \tag{4}$$

Equation (1) can be reduced to the same form

$$p\cdot v=R_d\left(1+0.378\,\frac{e}{p}\right)\cdot T. \tag{5}$$

It is sometimes inconvenient to use a variable magnitude as a gas constant. In problems of this sort in meteorology, in which the influence of the humidity on the density or on the specific volume of the air is taken into consideration, it is convenient to replace the real moist air by a fictitious dry air having a higher temperature T_v, chosen in such a way that the densities of the real moist air, and of the fictitious air coincide. T_v is called the virtual temperature.

This concept makes possible the use in calculation of a fictitious air with a temperature of:

$$T_v=T(1+0.605\,s)=T\left(1+0.378\,\frac{e}{p}\right), \tag{6}$$

instead of the real moist air, which effectively simplifies the solution of many problems of atmospheric statics.

Tables have been compiled and nomograms constructed in order to compute the virtual temperature. The so-called virtual difference is usually computed

$$\Delta T_v=T_v-T=0.605\cdot s\cdot T=0.378\,\frac{e}{p}\cdot T, \tag{7}$$

which is added to the molecular temperature in order to obtain the virtual temperature.

Some aerologic diagrams have special devices for calculating ΔT_v. For instance on the emagram and the aerogram of the main isobars (1000, 900, ... mb) short dashes are marked, the distance between which corresponds to the virtual difference at complete saturation. It is sufficient to multiply the difference read off the emagram by the relative humidity in order to obtain the actual virtual difference. The virtual difference can reach 5 - 8° C at high temperatures and high humidities. At temperatures lower than -10°C, the virtual difference does not exceed 0.1° - 0.2° and is, therefore, not taken into account (see Appendix 12).

§ 5. The Specific Heats of Moist Air

The specific heats of moist air can be calculated as the mean between the specific heat of dry air and of water vapor.

Thus, for an arbitrary process,

$$c=(1-s)\cdot c_d+s\cdot c_w; \tag{1}$$

for an isobaric one

$$c_p=(1-s)\,c_{pd}+s\cdot c_{pw}; \tag{2}$$

for an isothermic one

$$c_v=(1-s)\,c_{vd}+sc_{vw}. \tag{3}$$

we obtain from equation (2)

$$c_p=c_{pd}\left[1+s\left(\frac{c_{pw}}{c_{pd}}-1\right)\right];$$

therefore

$$\frac{c_{pw}}{c_{pd}}=\frac{0.445}{0.240}=1.9; \tag{4}$$

$$c_p=c_{pd}(1+0.9\,s)\approx c_{pd}(1+0.9\,w), \tag{5}$$

i.e., the specific heat [at constant pressure] of moist air c_p exceeds c_{pd} by at most 4% (i.e., varies from 0.24 to 0.25). Exactly in the same way

$$c_v=c_{vd}(1+s)\approx c_{vd}(1+w), \tag{6}$$

since

$$\frac{c_{vw}}{c_{vd}}=\frac{0.335}{0.171}=2.00.$$

It is useful to give some combinations of specific heats:

$$\varkappa=\varkappa_d(1-0.11\,s) \tag{7}$$

and

$$\lambda=\lambda_d(1-0.29\,s). \tag{8}$$

Thus, the presence of water vapor lowers the constants of Poisson's equation. For example, λ changes from 0.285 to 0.282 (see Appendix 7).

§ 6. Adiabatic Processes in Moist Air

The following sections will be devoted to the examination of the fundamental problem of the thermodynamics of moist air: the change in temperature due to pressure variations. These variations are mainly related to vertical movements in the atmosphere.

We will consider this process as being adiabatic. This means that we will disregard radiation, whose influence is very small, as well as turbulence, the influence of which is apparently quite considerable, though it can be quantitatively evaluated. The (moist) adiabatic process will evolve in a different and more complicated way than the dry-adiabatic process previously studied.

Since moist air eventually becomes saturated with the adiabatic decrease of pressure, a further decrease of the pressure results in the condensation of water vapor and afterwards in sublimation, if the air is cooled below 0° C.

During condensation and sublimation, latent heat is liberated, which retards the further cooling of the air.

The process is strictly reversible, if the products of condensation are not precipitated, but remain suspended, and are borne away by the ascending air, i.e., if the process takes place without moisture-exchange with the environment. Indeed, if the moist air and the products of condensation descend, the same amount of heat will be used up in vaporization of water or ice particles as the amount which was liberated during condensation in the course of the air ascent.

Adiabatic changes of state of moist air were first studied by Hertz in 1884. He suggested four successive stages for the dynamic cooling of moist air:

1) dry stage - the air is not saturated with water vapor;

2) rain stage - the air is saturated with water vapor, water drops being present at a temperature above 0° C. The water vapor condenses on ascent, and the water drops evaporate on descent;

3) hail stage - the air is saturated with water vapor, the water drops present in it freeze on ascent, and the ice melts on descent at a constant temperature of 0° C;

4) snow stage - the air is saturated with regard to ice, crystals of ice being present in it. The water vapor sublimates on ascent, and the crystals of ice evaporate on descent.

It is to be noted that the proposed terms — stages of rain, hail and snow are erroneous (particularly "the hail stage", because the precipitation of hail has nothing to do with this isothermic stage). However, this nomenclature has been kept to the present day.

Attention was drawn quite long ago to the fact that a process occurring in the atmosphere is often irreversible, because the greater part of the condensation products precipitate in the form of rain or snow. Such an adiabatic process, which is accompanied by a moisture exchange, is called a pseudoadiabatic process. Thus, for example, no hail would form, if all the condensed water is precipitated before the temperature reaches 0° C. However, the differences between the rain and snow stages, resulting from the precipitation of the condensation products, or of their remaining in the air, are so small that they can be ignored, naturally providing the reverse process is likewise ignored.

The adiabatic process will also be isentropic, if there is no precipitation; the derivation of the adiabatic equations will therefore be particularly simple in this case. The pseudoadiabatic, irreversible process will be accompanied by an increase of entropy during precipitation; therefore it is impossible to associate a definite entropy with the pseudo-adiabatic processes. We shall examine a strictly reversible adiabatic process, studying the change of state of a moist air mass, composed of 1g of dry air, and w_0 g of water vapor.

§ 7. Dry Stage—Unsaturated Moist Air

Moist air can be considered as a homogeneous mixture of two ideal gases - dry air and water vapor, i.e., as a monocomponent system, if saturation is not reached. For a given equation of the thermodynamic path $dq = 0$, the state of the system will be completely determined by one parameter, e.g., by p. Varying p arbitrarily, we obtain a well defined variation of state of moist air.

According to equations (12) and (8) § 16 Chapter II, we can write the following equation for the specific entropy of moist air

$$\varphi = c_p \ln T - R \ln p + \text{const.}; \tag{1}$$

and since the process is isentropic,

$$c_p \ln \frac{T}{T_0} - R \ln \frac{p}{p_0} = 0 \tag{2}$$

or

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\lambda}. \tag{3}$$

Equation (3) determines a curve, which can be plotted on an aerologic diagram. This curve is called an unsaturated adiabatic curve. Since $\lambda = \lambda_d(1 - 0.29s)$ varies together with the moisture s *, an infinite number of unsaturated adiabatic curves pass through any point of the diagram, corresponding to any given value of s*. Given the humidity s*, we will examine the corresponding adiabatic curve (3). This curve will intersect the isobar $p = 1000$ mb at a temperature of

$$\theta_s = T\left(\frac{1000}{p}\right)^{\lambda}. \tag{4}$$

We will compare this adiabate with the dry adiabate, passing through the point (T, p):

$$\theta = T\left(\frac{1000}{p}\right)^{\lambda_d} \tag{5}$$

forming the quotient:

$$\theta_s : \theta = \left(\frac{1000}{p}\right)^{\lambda} : \left(\frac{1000}{p}\right)^{\lambda_d} = \left(\frac{1000}{p}\right)^{-0.29\lambda w}, \tag{6}$$

we get

$$\theta_s = \theta\left(\frac{1000}{p}\right)^{-0.29\lambda w} \approx \theta\left(\frac{1000}{p}\right)^{-0,1\,w}. \tag{7}$$

It can be seen from equation (7) that moist air, when reduced to the standard pressure of 1000 mg, has a temperature somewhat lower than dry air, this departure becoming greater with increasing humidity. Thus, the adiabatic variation of the temperature of moist air is somewhat smaller than the temperature variation of dry air. This is due to the fact that the specific heat of water vapor is greater than that of dry air. But this difference is very small. Thus, for example, if air having a specific moisture of $s^* = 1$ g/kg is reduced from a pressure of $p = 500$ mg to standard pressure, then

$$\theta_s = 0.9993\,\theta,$$

at

$$\theta = 300°\,\text{K}$$

we obtain

$$\theta_s = 299°,8.$$

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* [The Russian text has w, which practically equals s – see eq. 12, § 2 – Translator's note].

Such accuracy is, of course, completely unattainable in meteorology. We shall therefore determine the potential temperature of moist air according to the same equation as for dry air

$$\theta_s \approx \theta = T\left(\frac{1000}{p}\right)^{\lambda_d}. \tag{8}$$

This is all the more necessary since the humidity-correction $\lambda - \lambda_d$, is contained within the limits of error of the determination of λ_d. Thus, we assume that the adiabatic process in moist, unsaturated air is represented by dry adiabates on the thermodynamic diagram. The dry stage will therefore be called the stage of the dry adiabatic process.

It is obvious that the dry-adiabatic process is characterized by the following conditions:

$$\theta = \text{const.}, \quad w = \text{const.}$$

The dry stage comes to an end, and the process develops either into the second stage, if $t > 0°$ C, or into the fourth, if $t < 0°$ C, when saturation is reached. Let us determine the conditions for this transition. We will denote the parameters of state, which correspond to the termination of the dry stage as follows: T_k - the temperature of condensation, p_k - the pressure of condensation E_k - the vapor pressure of condensation. The following 4 equations must be solved jointly in order to determine the point of condensation:

$$w = 0.622\frac{F_k}{p_k} = 0.622\frac{e}{p}, \quad \frac{T_k}{T} = \left(\frac{P_k}{P}\right)^{\frac{R}{C_p}},$$

$$e = rE, \qquad \ln\frac{E}{E_k} = -\frac{L}{R_w}\left(\frac{1}{T} - \frac{1}{T_k}\right),$$

whence we obtain the determining equation for T_k :

$$c_p \ln\frac{T_k}{T} + 0.622L\left(\frac{1}{T_k} - \frac{1}{T}\right) + R\ln r = 0, \tag{9}$$

and p_k is determined from the equation of the dry adiabate. The graphic method is the simplest way of solving such a system of equations. If the state of air is represented on the adiabatic diagram by a point, which we will call the reflection point, then the change of position of this reflection point on the diagram will correspond to a definite process. If the reflection point moves along the dry adiabatic curve, in a region where w is smaller than the values marked on the isograms, this will correspond to the dry adiabatic process characterized by the condition $\theta = \text{const.}$, $w = \text{const.}$ The dry adiabatic process will last until the reflection point, moving along the dry adiabate, intersects the isogram w. Thus, the point of intersection of the dry adiabate $\theta = \text{const.}$ with the isogram $w = \text{const.}$ will be the condensation point, or in other words, the characteristic point. The coordinates of the characteristic point are (T_k, p_k).

§ 8. The Rain Stage

Further decrease in the pressure, after reaching the condensation point, will lead to the condensation of water vapor, if $t > 0°$ C. We shall now consider moist air as a two-component system, composed of 1 g of dry air, and w_0 g of water in two phases: saturated water vapor, and water, sufficiently liquid to form drops. Since we consider this process to be reversible (there is no precipitation), the total mass of the system $(1 + w_0)$ will remain constant; the partial masses of the components of the system - 1g of the dry air and w_0 g of the water in its two phases will remain constant too. However, the partial masses of the water phases will vary; the temperature will also decrease with the decrease of pressure though more slowly than in the first stage, and the vapor will condense.

Since the process is reversible, the entropy of the system will remain constant. Let us form the equation for the specific entropy of saturated moist air.

The specific entropy of the dry air is equal to:

$$\varphi_d = c_{pd} \ln T - R_d \ln (p - \overset{*}{e}) + \text{const.}, \tag{1}$$

and the specific entropy of the water (liquid water + water vapor) is equal, according to equation (6) § 11, Chapter III to

$$\varphi_w = \frac{L\mu_w}{T} + c_f \ln T + \text{const.} \tag{2}$$

Thus, the total entropy of the mixture will equal

$$\Phi = (1 + w_0)\varphi = \varphi_d + w_0 \varphi_w = c_{pd} \ln T - R_d \ln (p - E) + \frac{Lw_0\mu_w}{T} + w_0 c_f \ln T + \text{const.},$$

where μ_w is the relative mass of water vapor in 1 g of water (see § 11, Chapter III).

Since $w_0 \mu_w = w_w$, we shall obtain for the entropy of saturated moist air:

$$\Phi = c_{pd} \ln T - R_d \ln (p - E) + \left(\frac{Lw_w}{T} + w_0 c_f \ln T\right) + \text{const.} \tag{3}$$

However, a reversible process is being considered; therefore the entropy will remain constant for a given mass of moist air. The value of this constant is determined from the initial values. Denoting the initial values by the subscript and carrying out some not very complicated transformations, we obtain

$$\frac{c_{pd} + w_0 c_f}{R_d} \ln \frac{T}{T_0} + \frac{1}{R_d}\left(\frac{Lw_w}{T} - \frac{L_0 w_{w0}}{T_0}\right) = \ln \frac{p - E}{p_0 - E_0}. \tag{4}$$

Equation (4) gives the desired relationship between p and T, which is called the condensation adiabate. E and L are, as we have seen, functions of the temperature only. This equation corresponds entirely to the equation of the dry adiabate (3) § 7 and is correct as long as the temperature exceeds 0° C.

In the study of the change of moist air temperature on transition from the first to the second stage we will put

$$T_0 = T_k, \quad p_0 = p_k, \quad E_0 = E_k, \quad L_0 = L_k.$$

§ 9. The Hail Stage

Hertz assumed that the drops of water begin to freeze, as soon as the temperature falls to zero, whereby the temperature will not decrease for a while with a further decrease of the pressure, because the latent heat of melting L_{lf} will be liberated during freezing of water. The hail stage remains, therefore, isothermic until all the water is frozen. A partial vaporization of the drops will also take place in this phase, because the actual vapor pressure will decrease continuously with the decrease of pressure at a constant temperature, which will lead to an uninterrupted vaporization of water drops. The hail stage will end when only ice crystals and saturated water vapor remain in the air.

This mechanism is different from the one which actually operates in the atmosphere. Clouds composed of supercooled drops are very often found in the atmosphere. Water in the atmosphere can be supercooled below 0° C, if crystallization nuclei are absent; in such a case the rain stage, at which supercooled water exists, can be represented by a condensation-adiabate; this adiabate makes a turn after the freezing point.

* [The Russian text has e, but E is meant, for the air is saturated. – Translator's note]

Therefore a two-component system corresponds to the hail state: dry air + water, the water being in a three-phase equilibrium. Let the amount of water vapor be w_w, of water w_f, of ice w_i, and collected $w_i + w_f + w_w = w_0 = \text{const.}$ It is obvious that the hail stage corresponds to the triple point, which we examined previously. Since the process is reversible, we can make use of its isentropic properties.

The specific entropy of the dry air is equal to:

$$\varphi_d = c_{pd} \ln T - R_d \ln (p - E) + \text{const.}$$

The specific entropy of water (water vapor + water + ice), according to equation (9) § 11, Chapter III, equals

$$\varphi_{w0} = \frac{L \cdot \mu_3}{T} - \frac{L_{if}\,\mu_1}{T} + \text{const.},$$

the entropy of moist air will then be

$$\Phi = (1 + w_0)\,\varphi = \varphi_d + w_0 \varphi_{w_0} = c_{pd} \ln T - R \ln (p - E) + \frac{L w_0 \mu_3}{T} - \frac{L_{if} w_0 \mu_1}{T} + \text{const.}$$

Taking into consideration that $\mu_3 w_0 = w_w$, $\mu_1 w_0 = w_i$, we will obtain

$$\Phi = c_{pd} \ln T - R \ln (p - E) + \frac{L \cdot w_w}{T} - \frac{L_{if} w_i}{T} + \text{const.}$$

or, as a result of the isentropic and isothermic nature of the process

$$[L (w_w - w_{w_0}) - L_{if} (w_i - w_{i0})] = R T_0 \ln \frac{p - E}{p_0 - E}.$$

This equation gives the relation between the amount w_i of the formed ice and the pressure p.

§ 10. The Snow Stage

A further decrease of pressure, after transformation of all the water into ice, will again give rise to a dynamic cooling of the air. The decrease of temperature will be accompanied by a sublimation of water vapor. The fourth stage is similar to the second one; therefore we can write the equation of the adiabate of sublimation - by analogy with equation (4) § 8 - as follows:

$$\frac{c_{pd} + w_0 c_l}{R_d} \ln \frac{T}{T_0} + \frac{1}{R_d} \left(\frac{L_{iw} \cdot w_w}{T} - \frac{L_{iw0} \cdot w_{w0}}{T_0} \right) = \ln \frac{p - E}{p_0 - E_0}. \qquad (1)$$

Since the latent heat of sublimation, as we have seen above, (§ 6, Chapter III) is constant, equation (1) can be written in this form:

$$\frac{c_{pd} + w_0 c_l}{R_d} \ln \frac{T}{T_0} + \frac{L_{iw}}{R_d} \left(\frac{w_w}{T} - \frac{w_{w0}}{T_0} \right) = \ln \frac{p - E}{p_0 - E_0}. \qquad (2)$$

The process goes over directly from the dry stage into the fourth one, by-passing the rain and hail stages, if the air in the dry phase is so cold and dry that saturation is reached only at negative temperatures.

In concluding the survey of the four stages of the adiabatic process in moist air, we should point out that no sharp demarcations between these phases actually exist. In fact, the dry phase can be prolonged (in particular at low, negative temperatures), since the single condition of complete saturation is not sufficient to start-off the process of condensation or sublimation of vapor, but the presence of a sufficient quantity of condensation or sublimation nuclei is required.

The following frequently observed phenomena demonstrate the correctness of this assumption: an absolutely clear, cloudless sky; suddenly an airplane appears at a height of 6 - 8 km and a newly-formed cloudy trail extends behind it; this trail, broadening gradually, can spread over the whole sky in 15-20 minutes.

It is obvious that the air was completely saturated or even supersaturated with water vapor before the appearance of the airplane, but there were no condensation nuclei. An enormous quantity of such nuclei, in the form of combustion products exhausted from the motor, enter the atmosphere when the airplane passes in this layer. Now all the conditions are present for rapid formation of clouds.

There are also no conditions in the atmosphere corresponding to the third, i.e., the hail stage. For the development of this stage it is necessary that the water drops freeze immediately after reaching a temperature of 0°C. The existence of a hail stage excludes the existence of supercooled drops in the atmosphere. In fact the existence of the latter is a common phenomenon in the atmosphere. The phenomenon of ice forming on the surface of an airplane, with frequently disastrous consequences, reminds us of their existence.

Thus, the hail stage is not real and the actual process develops in a more complicated way. Even when water drops start freezing, the further decrease of pressure does not take place isothermically at all, but is accompanied by a decrease in temperature; consequently instead of an equilibrium state of the water phases, we obtain a thermodynamically unstable state, leading to rapid condensation; therefore the decrease of the temperature below 0°C sometimes takes place along the condensation adiabate, sometimes along the sublimation adiabate, and it is difficult to indicate in each separate case precisely when does the transition from the condensation to the sublimation adiabate take place.

§ 11. The Pseudoadiabatic Process

An irreversible process takes place, and the entropy of the system increases, if there is complete or partial precipitation during condensation. Therefore, we can not make use of (a definite) entropy for deriving the equation of the pseudoadiabate. The only thing which can be done is to write the equation of the pseudoadiabate in a differential form, and then to integrate it numerically, because an exact integration of this equation is impossible. Let us restrict ourselves to the examination of the simplest case, when all the condensation products precipitate. As before we will consider $(1+w)$ g of moist air, and w g of saturated water vapor, the value of w being variable.

Let us convert the system under consideration from the state (p, T, w), into an infinitely proximate state $(p+dp,\ T+dT,\ w+dw)$ by means of expansion, dp, dT, dw being negative. A quantity of heat δQ, equal to

$$\delta Q = (1+w)\left(c_p\,dT - RT\frac{dp}{p}\right). \tag{1}$$

will be required to carry out this elementary process.

Since there is no heat exchange, this quantity will be obtained from the heat liberated during the condensation of dw g of water vapor, i.e.,

$$\delta Q = -L\,dw, \tag{2}$$

from this, by combining equations (1) and (2), we obtain

$$-L\,dw = (1+w)\left(c_p\,dT - RT\frac{dp}{p}\right). \tag{3}$$

This is a precise differential equation for the pseudoadiabate.

Since it is impossible to integrate equation (3) exactly, we shall have to be content with the differential form of the equation of the pseudoadiabate, simplifying

it somewhat. We can neglect the quantity w, since $w \ll 1$; furthermore, we can consider that $c_p \approx c_{pd}$, $R \approx R_d$.

We then obtain the following approximate equation of the pseudoadiabate:

$$-L\,dw = c_{pd}\,dT - R_d \cdot T\,\frac{dp}{p} \tag{4}$$

or, taking into consideration that

$$\frac{d\theta}{\theta} = c_{pd}\frac{dT}{T} - R_d\frac{dp}{p},$$

we obtain

$$-\frac{L\,dw}{T} = c_{pd}\frac{d\theta}{\theta}. \tag{5}$$

In the same way a precise differential equation of the condensation (or sublimation) adiabate can be worked out,

$$-L\,dw_w + c_f(w_0 - w_w)\,dT + c_w \cdot w_w\,dT = c_{pd}\,dT - R_d T\,\frac{dp_d}{p_d}, \tag{6}$$

which assumes the form (4), if the heat absorbed by the liquid water and the saturated water vapor is not taken into consideration.

Thus, the equations of the condensation (or sublimation) adiabates or pseudoadiabates assume the same form, if second order terms are disregarded.

Accurate calculations have shown that the pseudodiabatic process always causes a higher pressure (at the same temperature), than the strictly adiabatic process, although the numerical difference is so small that it can be neglected, except at temperatures close to 0° C.

The character of the adiabate practically does not depend on the precipitation or nonprecipitation of the condensation products. If we will take into account that the hail stage is unreal and that the actual freezing temperature is below zero, we arrive at the following important conclusions:

1) There is no practical necessity and no ground for investigating the hail stage.

2) It is not necessary to investigate the snow stage, because the beginning of sublimation at temperatures between 0 and 20° is hardly probable, and besides the rain and snow stages are not essentially different, particularly at low temperatures.

3) There is no practical necessity to examine the pseudoadiabates, because the direct process takes place according to the condensation adiabate, whereas the reverse process proceeds according to the dry adiabate(if the condensation products precipitate immediately), or, at first according to the condensation adiabate, and afterwards according to the dry adiabate (if the condensation products precipitate partially).

In practice the adiabatic process in moist air reduces to two stages: the unsaturated stage, which we call the dry stage, and the saturated one, which we shall call the moist stage. The corresponding equations, describing these stages, are the equation of the dry adiabate, and the equation of the moist adiabate, for which we can use the equation of the condensation adiabate.

§ 12. Dry and Moist Adiabates on Aerologic Diagrams

In order to represent quickly and visually the adiabatic variations in the atmosphere, diagrams are drawn and curves plotted on them according to the equations of the corresponding adiabates. We have already examined some such curves. We will elucidate how dry and moist adiabates appear on these diagrams.

The equations of the dry adiabates are given in Table 9 in coordinates of the corresponding diagrams.

Table 9

Equations of dry adiabates on the aerologic diagrams

	Stüve diagram	Emagram	Tephigram	Aerogram
Abscissa	T	T	$-T$	$\ln \frac{T}{T_0}$
Ordinate	$-\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}$	$\ln \frac{p_0}{p}$	$\varphi_d = c_p \ln \theta +$ + const.	$T \ln \frac{p_0}{p}$
Equation of dry adiabate	$y + x = 0$	$y = \frac{c_p}{R} \ln x +$ + const.	$y = y_0$	$y = \frac{c_p}{R} T_0 x e^x$
The slope, relative to the x axis	$y' = -1$	$y' = \frac{c_p}{R} \cdot \frac{1}{x}$.	$y' = 0$	$y' = -\frac{c_p}{R_0} T_0 e^x (1+x)$
Sign of curvature of the dry adiabate	$y'' = 0$	$y'' = \frac{c_p}{R} \frac{1}{x^2}$	$y'' = 0$	$y'' = -\frac{c_p}{R} T_0 e^x (2+x)$

It can be seen from the table that the dry adiabates are represented by straight lines only on Stüve diagrams and tephigrams.

The dry adiabates are represented on the tephigram by straight lines, which are parallel to the abscissa axis and approach each other in the region of high θ, they are represented on the Stüve diagram by equidistant straight lines, which are inclined at an angle of 45° to the abscissa. The dry adiabates are represented on the emagram by slightly distorted logarithmic curves, rising from the right to the left and convexly facing the abscissa. Their angle of inclination decreases with the increase of x(T). They converge together as y increases.

The dry adiabates are represented on aerograms in the form of slightly distorted exponential curves, rising from the right to the left and convexly facing upwards. They converge together as x and y increase.

The plotting of moist adiabates involves more laborious computations.

The moist adiabates, plotted according to equation (4) § 8, rise more sharply than the dry adiabatic curves on all the examined diagrams. This is explained by the condensation of the water vapor, which causes both a decrease of the specific humidity and the liberation of latent heat, which somewhat compensates for the change of temperature, caused by the adiabatic variation of pressure. This compensation becomes greater, as the condensation grows in extent. Therefore the adiabates rise more sharply at a high temperature and a low pressure, when the

specific humidity is higher, than at low temperature and high pressure.

The moist adiabates practically coincide with the dry adiabates for very low temperatures, when the specific humidity is small enough to be neglected.

When considering the form of any moist adiabate, we see that it first ascends quite sharply, intersecting a great number of dry adiabates, then its ascent becomes slower as p decreases, and finally it converges asymptotically to some well-defined dry adiabate.

§ 13. Isoenthalpic Processes. Mixing of Two Masses of Moist Air

We call isoenthalpic the processes taking place in an isolated mass of air, which are both adiabatic and isobaric, i.e., preserve a constant value of the enthalpy:

$$\mathcal{I} = \text{const.}$$

The isoenthalpic process, taking place in dry air (a homogeneous thermodynamic system), and characterized by the equation

$$\mathcal{I} = c_p T = \text{const.}, \tag{1}$$

leads to a trivial conservation of state, since not only the pressure, but also the temperature, and consequently the specific volume too, remain constant. An untrivial result is obtained in the case of an isoenthalpic process, taking place in a heterogeneous thermodynamic system. We will here examine two examples of an isoenthalpic process in moist air, which are of practical interest for meteorology, namely:

1) mixing of two masses of air

2) vaporization of water drops suspended in air.

It is obvious that both processes are irreversible and therefore accompanied by an increase of the entropy.

Let us imagine a thermodynamic system, composed of two different masses of moist air, mixing adiabatically at constant pressure. At the beginning of the process, before mixing, the system's state was characterized by the variables m_1, T_1, s_1, m_2, T_2, s_2, where m is the mass, T -the temperature, s -the specific humidity, where the subscripts 1 and 2 refer to the first and second air masses. The state of the system, after mixing, is characterized by the variables m, T, s. It is impossible to follow the development of this process owing to its irregularity. We can only compare the final and initial states of the system.

Since the process is isoenthalpic,

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2, \tag{2}$$

where $\mathcal{I}_1$, $\mathcal{I}_2$ are respectively the enthalpies of the first and the second masses before mixing, and $\mathcal{I}$ -is the enthalpy of the mixture.

Combining equations (1) and (2), and disregarding the rather small influence of water vapor on the specific heat of the moist air, we obtain:

$$mc_pT = m_1c_pT_1 + m_2c_pT_2,$$

hence

$$T = \frac{m_1T_1 + m_2T_2}{m_1 + m_2}. \tag{3}$$

From the definition of the specific humidity follows exactly that:

$$s = \frac{m_1s_1 + m_2s_2}{m_1 + m_2}. \tag{4}$$

Expressing the specific humidity by the vapor pressure through equation

$$s = 0{,}622\frac{e}{p}$$

and taking into consideration the isobaric nature of the process, we obtain

$$e = \frac{m_1e_1 + m_2e_2}{m_1 + m_2}. \tag{5}$$

Combining equations (3) and (5), we can write

$$\frac{e - e_1}{e_2 - e} = \frac{m_2}{m_1} = \frac{T - T_1}{T_2 - T}. \tag{6}$$

We obtain the following results when using the graphic representation of the air masses state in the e, T, co-ordinate system.

If the states of a mass of moist air are represented on the (e, T) diagram by points (Figure 28), then the state of any mixture of two air masses will be represented by a point situated on the straight lines connecting the points which correspond to the states of the mixture components. Thus, for example, if the point A represents the state of the first mass, and the point B - that of the second mass, then the state of the mixture will be represented by a point C, dividing the segment AB in the proportion:

$$\frac{AC}{BC} = \frac{m_2}{m_1} = \frac{e - e_1}{e_2 - e} = \frac{T - T_1}{T_2 - T}. \tag{7}$$

It can be seen from equations (5) and (4) that the point C is situated where the center of gravity of the two air masses would have been, were they concentrated at the points A and B.

This theory has some practical value. From the examination of the diagram it follows that by mixing two nonsaturated air masses, differing considerably in their temperatures, it is possible to obtain air which is saturated and even supersaturated with water vapor.

However, saturation and supersaturation take place only for very high humidities (approximately 95 % and higher), and at a temperature difference of about 10°. The process of mixing of two different air masses can, under favorable conditions, terminate in the condensation of water vapor, although the amount of condensed water is so insignificant that there is no precipitation even under the most favorable conditions. Therefore, the mixing of the air masses does not play an independent role during the processes of cloud formation; it can only accompany the the main process, i.e., the adiabatic cooling, which causes the formation of clouds.

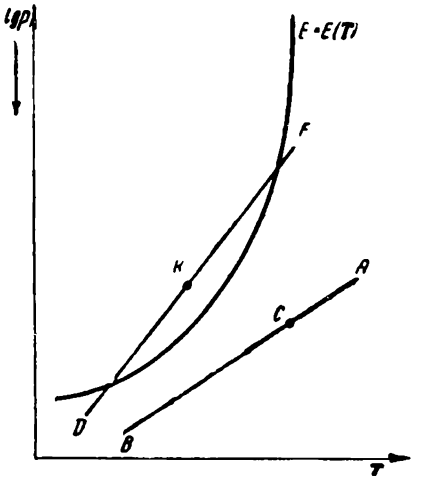

Figure 28. Determination of the water vapor pressure in a mixture of two air masses

§ 14. Isoenthalpic Evaporation of Water Drops. The Equivalent Temperature

Let us imagine a two-component thermodynamic system, composed of 1 g of absolutely dry air, and of water drops suspended in it. Such a system will not be in equilibrium. If the process is isoenthalpic, thermodynamic equilibrium will be reached either after the evaporation of all the water, with the formation of nonsaturated moist air, or after evaporation of a part of the water only, as a result of which saturated moist air with water drops suspended in it is formed. This process is irreversible. We shall first examine the first case.

The enthalpy of the system in its initial, unstable state, consists of the enthalpy of the dry air, and the enthalpy of the water drops:

$$\vartheta_0 = c_{pd}T_0 + c_f \cdot wT_0, \tag{1}$$

where c_f is the specific heat of liquid water.

In its final, stable state the enthalpy of the system is composed of the enthalpy of dry air at the temperature T, and the enthalpy of w g of unsaturated water vapor:

$$\vartheta = c_{pd}T + c_{pw} \cdot w \cdot T + Lw. \tag{2}$$

The process proceeds isoenthalpically, $\vartheta = \vartheta_0$ and hence

$$c_{pd}T_0 + c_f wT_0 = c_{pd}T + c_{pw} \cdot w \cdot T + Lw. \tag{3}$$

Since the enthalpy is determined up to an arbitrary constant, and we are not interested in the enthalpy itself but in its variation, and since $w \ll 1$, we can now ignore the following terms in equation (3):

$$c_f \cdot w \cdot T_0 \quad \text{and} \quad c_{pw} \cdot w \cdot T;$$

we then obtain

$$T_0 = T + \frac{Lw}{c_{pd}}. \tag{4}$$

The simplest way to determine unambiguously the enthalpy of moist air having a temperature T and a humidity w, is to revert to the temperature T_0 which the dry air had had before it was brought to the state (T, w) by the isoenthalpic evaporation of water drops suspended in it. This temperature is called the equivalent temperature.

In the determination of this temperature, generally no use is made of the described evaporation process, which can be easily reproduced by experiment, but of a reverse process of isoenthalpic condensation, which is impossible to realize in practice, or even in principle.

The temperature the air reaches when all the vapor contained in it condenses, is called the equivalent temperature and the liberated heat is used to heat this air at a constant pressure.

The following definition accords better with the nature of the considered phenomenon: the equivalent temperature is the temperature which absolutely dry air had had before its conversion, as a result of the isoenthalpic evaporation of the water drops in it, into the existing moist air having a temperature T and a humidity w.

It follows from equation (4) that the equivalent temperature T_e is determined by the equation

$$T_e = T + \frac{L \cdot w}{c_{pa}}. \tag{5}$$

The value Lw/c_{pa} is called the equivalent difference.

It follows from the above considerations that the equivalent temperature does not vary during isoenthalpic evaporation. The equivalent temperature is the maximum temperature, which the air had before the isoenthalpic evaporation of the water in it, and which it cannot regain.

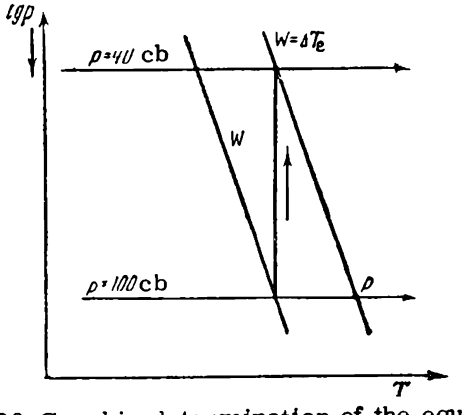

Figure 29. Graphic determination of the equivalent difference on the emagram

Expressing w in g/kg, we can consider the coefficient of w as being approximately constant and equal to:

$$\frac{L}{c_p \cdot 1000} \cong \frac{600}{0.24 \cdot 1000} = 2.5;$$

then the value of the equivalent temperature will be determined approximately by the equation

$$T_e = T + 2.5\,w. \tag{6}$$

At a pressure close to the normal, use can also be made of another approximate equation, which can be obtained by assuming that:

$$w \cong 622\,\frac{e}{p} \approx 622\,\frac{a}{p}\,,$$

where a is the absolute humidity, and p - the pressure, measured in millimeters Hg. Assuming p = 760 mm, we obtain:

$$T_e \cong T + 2a. \tag{7}$$

It can be seen from equations (6) and (7) that each gram of water vapor, upon its condensation increases the temperature of 1 kg of air, by 2.5° C, and the temperature of one cubic meter under normal pressure by 2° C.

It is clear from the above that the equivalent temperature cannot be measured, but only calculated. Various nomograms and rules are used for its computation. A special scale, for example, is marked on the emgaram, in order to compute equivalent difference, which gives the equivalent temperature after being added to the molecular temperature. The equivalent difference can be determined on any aerological diagram with the help of the isograms, according to the following reasoning: Rearranging the expression for the equivalent difference

$$\Delta T_e = 2{,}5\,w = \frac{10}{4}\cdot w \text{ or } w = \frac{40}{100}\,\Delta T_e,$$

it is easy to show that the graphic determination of ΔT_e is equivalent to the determination of the humidity of saturation, corresponding to an initial fixed relative humidity of 40%, and according to the actual humidity (Figure 29). For this purpose it is necessary to mark the point of intersection of the isogram w and the isobar p = 100 cb, then to ascend along the isotherm up to its intersection with the isobar p = 40 cb, where the value w is read off. This is the practical value of the equivalent difference.

§ 15. The Temperature Read by the Wet-bulb Thermometer

Let us assume that the quantity of evaporated water has been sufficient to saturate the air completely with water vapor. According to equation (5) § 14, if we denote the temperature of the saturated moist air by T' and by w', the saturation humidity at this temperature, we obtain:

$$T_e = T' + \frac{L'w'}{c_{pd}} = T + \frac{Lw}{c_{pd}} \tag{1}$$

or, disregarding the variation of L,

$$T' = T_e - \frac{Lw'}{c_{pd}}\,, \tag{2}$$

therefore

$$T' = T - \frac{L\,(w' - w)}{c_{pd}}\,. \tag{3}$$

The temperature shown by the wet bulb thermometer is denoted by T'. Thus, the temperature registered by the wet bulb thermometer is that temperature gain of the air which results from an isoenthalpic evaporation of the water it contains

until saturation is reached. It follows from this definition, that the wet bulb temperature is that minimum temperature to which air can be cooled during isoenthalpic evaporation. It is reached at complete saturation and cannot be decreased anymore, because a further isoenthalpic evaporation is impossible. Therefore, the wet bulb temperature is constant, for a given mass of air, during isoenthalpic evaporation.

It can be seen from equation (2) that the wet bulb temperature and the equivalent temperature determine each other unambiguously.

It follows from the above that the wet bulb temperature, and the equivalent temperature, are invariant characteristics of the air mass. During the isoenthalpic evaporation of water drops in it.

Isoenthalpic evaporation can often be observed in the atmopshere, when drops formed in the upper layers of the atmosphere evaporate before reaching the earth, while falling through the lower layers of unsaturated air. This process is isobaric and adiabatic in regard to the air, and thus isoenthalpic.

Since the enthalpy is well defined, both by the wet bulb temperature, and by the equivalent temperature, we see that during the isoenthalpic mixing of two air masses, which have different temperatures but equal wet bulb temperatures (equivalent temperatures), the mixture retains the previous wet bulb temperature (equivalent temperature). Equations (2) and (3) show that the calculation of the wet bulb temperature is more complicated than the calculation of the equivalent temperature. On the other hand the wet bulb temperature can be easily measured; this measurement is based on the following assumptions:

It is known that a well ventilated wet bulb thermometer indicates a certain quite definite temperature after reaching equilibrium. This means that after the wet bulb thermometer has reached a stationary state, the quantity of heat it absorbs from the surrounding air equals its heat loss by evaporation from the muslin. This first assumption is obvious and does not require further explanations.

In addition the air surrounding the thermometer is considered to be saturated at the temperature of the wet bulb thermometer. The correctness of this assumption cannot be based on theoretical considerations. Only the results of measurements can corroborate or refute it.

Isoenthalpic evaporation is an irreversible process. Therefore the entropy of the system will increase and not remain constant. This can be confirmed in the simplest way by calculating the increase in entropy of the air which surrounds the moistened bulb of the thermometer. At the beginning of the process the system is composed of $(1+w)$g of unsaturated moist air, and of $(w'-w)$g of water at a temperature T: At the end of the process it is composed of $(1+w')$g of moist air, saturated at the temperature T'. The quantity of heat transferred by the dry air, and consumed for evaporating water at the temperature T' equals:

$$\delta q = (c_{pd} + w c_{lw})(T - T').$$

Thus, the increase of the water's entropy is equal to:

$$\delta \varphi_w = \frac{\delta q}{T} = (c_{pd} + w c_{pw}) \frac{T - T'}{T'}.$$

The entropy of the moist air before evaporation is

$$\varphi = (c_{pd} + wc_{pw}) \ln T - R \ln (p - E) + \frac{Lw}{T}.$$

The entropy of the same $(1 + w)$ g of moist air after evaporation becomes equal to:

$$\varphi' = (c_{pd} + wc_{pw}) \ln T' - R \ln (p - E') + \frac{L'w'}{T'}.$$

The loss of entropy in the original air will therefore equal

$$\delta\varphi = \varphi - \varphi' = (c_{pd} + wc_{pw}) \ln \frac{T}{T'} + w \left(\frac{L}{T} - \frac{L'}{T'}\right).$$

The total increase of the entropy:

$$\delta\varphi_w - \delta\varphi = (c_{pd} + wc_{pw}) \left(\frac{T - T'}{T} - \ln \frac{T'}{T}\right) + w \left(\frac{L'}{T'} - \frac{L}{T}\right)$$

or, developing $\ln \frac{T'}{T}$ in a series and retaining only terms of up to second order, we obtain

$$\delta\varphi_w - \delta\varphi = (c_{pd} + wc_{pw}) \cdot \frac{1}{2} \left(\frac{T - T'}{T'}\right)^2 + w \left(\frac{L'}{T'} - \frac{L}{T}\right) + \dots$$

Thus, the entropy has increased as a result of the process. However, the magnitude of this increase is so small that it can be disregarded and the process practically considered as isentropic. Thus, the increase of the entropy is only about 0.001 cal/g·degree at $T - T^\circ = 10^\circ, T' = 290$ and $w = 12$ g/kg

§ 16. The Pseudotemperature of the Wet Bulb Thermometer

The following propositions concerning the wet bulb temperature may be stated.

1) The enthalpy of an air body, not saturated with water vapor, is equal to the enthalpy of the same air body, when saturated at the wet bulb temperature, less the entropy of the additional liquid water necessary to saturate this air. This proposition is the result of the two assumptions stated above.

2) The entropy of the air body unsaturated by water vapor is equal to the entropy of the same air body, saturated at the wet bulb temperature, minus the entropy of the liquid water necessary to saturate the air. This second proposition is only approximative, as we have indicated above. The divergence from the isentropic process can be substantial at high temperatures.

3) The dry adiabatic curve, which passes through the point A, determined by the temperature T of the dry thermometer, the moist adiabatic curve, which passes through the point C, determined by the temperature T' of the wet bulb thermometer, and the isogram, which passes through the point D, determined by the temperature τ of the dew point, intersect at the same point B, which is the point of condensation, or the characteristic point (Figure 30). In order to prove this proposition let us consider the air body which is represented by the point A. Let us assume that its dew point is at D. The line BD is an isogram. The point representing the dry adiabetic process will follow the dry adiabatic curve AB, and the corresponding dew point will follow the isogram DB. The air body becomes saturated, and its temperature becomes equal to the dew point, when the point representing the process passes into B. The point B is a condensation or a characteristic point.

We will now examine a body of saturated moist air, represented by the point C,

which is determined by the adiabatic curve passing through B. Its temperature is equal to the temperature T' of the wet bulb thermometer, and the mixing ratio equals W'. If the air body rises according to the moist adiabatic process, the point representing it will follow the moist adiabatic curve, and the air body will contain w g of saturated water vapor and $(W'-w)$ g of liquid water, when it reaches the point B. The air body which moved along CB will, at the point B, have the same temperature T, and the same humidity w, as that reached by the body which has moved along BA. Let us assume that the water precipitates in the form of rain. We will let the air body move down along BA. Since the process is adiabatic, the initial entropy of the body coming from the point C will be equal to the entropy of the body at the point A + the entropy of $(W'-w)$ g of the liquid water, which was precipitated at the point B. But the entropy of the saturated air body at the beginning of the process is, according to the second proposition, equal to the entropy of the unsaturated air body at point B plus the entropy of $(W'-w)$ g of water, necessary for its saturation. It follows from this, that the wet bulb temperature at the beginning and at the end of the process must be the same, i.e., that the temperature of the air body A, indicated by the wet bulb thermometer, is the temperature of saturation of the air body B.

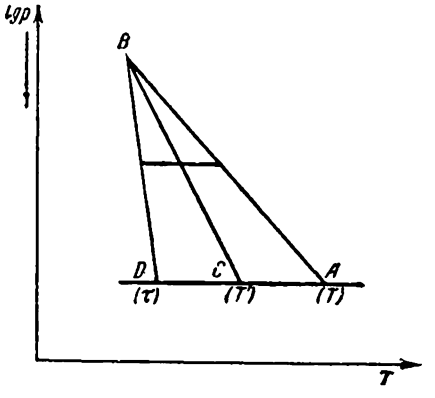

Figure 30. The dry adiabatic curve, the moist adiabatic curve corresponding to the wet bulb temperature, and the isogram intersect at the condensation point

Thus, the dry adiabatic curve AB, which passes through the point C, corresponding to the wet bulb temperature, and the isogram which passes through the point D, corresponding to the dew point, intersect at the condensation point B.

The above demonstration cannot be regarded as completely rigorous. Thus, for example, the temperature T_k, at which water is removed from the system, does not exactly equal the temperature indicated by the wet bulb thermometer. Besides, this demonstration is based on the not very exact second proposition. The wet bulb temperature, when determined by the method indicated and called the pseudotemperature of the wet bulb thermometer, must, therefore, be distinguished from the measured wet bulb temperature.

It follows from the third proposition that the pseudotemperature of the wet bulb thermometer is determined on the aerologic diagram in the following way. The body of moist air A moves upwards according to the dry adiabatic process, up to the condensation level B, and then descends according to the moist adiabatic process to the previous isobar. It will reach it at a temperature T_{sw}, which is called the pseudotemperature of the wet bulb thermometer for the air body A.

§ 17. The Thermohygrometric Characteristics of Air Masses

The principles of the modern analysis of air masses are based on the fact that tropospheric circulation involves voluminous gas masses. The physical conditions inside these masses vary rather slowly, but the variations are sharp at their interfaces.

In the study of the origin and the development of the air masses, we observe that some physical properties of air masses vary quite rapidly in time. Other properties, however, remain almost unchanges for prolonged periods. Therefore conservative" and "nonconservative" properties of the air masses can be distinguished.

Strictly speaking, it is impossible to discern even one physical invariant. When speaking of a physical invariant we mean that the variation of this magnitude during some definite process and within a definite period of time does not appreciably exceed the error limits of the observations. First of all use can be made of the physical invariants of air masses in order to identify such air masses. But the invariants can, in addition, also serve for studying the transformations the air masses undergo. If any invariants change essentially, it means that a process occurs in the air mass, which changes its properties substantially and transforms it into a different air mass, having other properties.

The dry, moist and pseudoadiabatic processes, isobaric heating and cooling, as well as isobaric evaporation, are the most important processes which must be taken into account when investigating the atmosphere, and which are amenable, even if only in part, to theoretical analysis. A great number of different thermohygrometric characteristics have been introduced into meteorology in recent years, in order to find the most rational invariants of air masses, and they can be either easily measured, or easily computed. We indicate the most important of them:

1) The dew point τ ;
2) The equivalent temperature T_e;
3) The equivalent-potential temperature θ_e;
4) The potential - equivalent temperature θ'_e ;
5) The pseudopotential temperature θ_{se} ;
6) The pseudo-equivalent temperature T_{se} ;
7) The wet bulb temperature T_w ;
8) The wet bulb pseudotemperature T_{sw} ;
9) The wet bulb pseudopotential temperature θ_{sw} ;

Some of the thermohygrometric characteristics have already been examined by us in the foregoing. We will now examine briefly the remaining characteristics.

§ 18. The Equivalent - Potential Temperature

It is usual to term equivalent-potential or equipotential temperature θ_e that temperature gain of air when it is adiabatically reduced to standard pressure with a subsequent condensation of all the water vapor, the liberated latent heat being used up for heating this air at a standard pressure. This definition, although generally accepted, is unsatisfactory because it is based on an impossible physical process. The following definition is more exact:

The equivalent-potential temperature θ_e is that temperature gain of absolutely dry air at standard pressure before its transformation into moist air (p, T, w)

as a result of w g of water isoenthalpically evaporating in it, and its following adiabatic reduction to the pressure p and to the temperature T.

It is obvious that the equation determining the equivalent - potential temperature has the following form:

$$\theta_e = \theta + \frac{Lw}{c_{pd}} = T\left(\frac{1000}{p}\right)^{\frac{R_d}{c_{pd}}} + \frac{Lw}{c_{pd}}. \qquad (1)$$

The equipotential temperature is an invariant of the dry-adiabatic process, because each of the factors affecting it remains constant during this process. We will prove that the equipotential temperature is quasi-invariant during isoenthalpic evaporation. Let us assume that the temperature of the air reaches T', as a result of the isoenthalpic evaporation of w' water vapor:

$$T' = T - \frac{Lw'}{c_{pd}}; \quad \theta'_e = T'\left(\frac{1000}{p}\right)^{\frac{R_d}{c_{pd}}} + \frac{L(w+w')}{c_{pd}}$$

hence

$$\theta'_e - \theta_e = -\frac{Lw'}{c_{pd}}\left[\left(\frac{1000}{p}\right)^{\frac{R_d}{c_{pd}}} - 1\right],$$

it follows from this that the equipotential temperature will increase at $p > 1000$ mb, and decrease at $p < 1000$ mb, during isoenthalpic evaporation. It can, however, be seen from Table 10 that this variation is very small. It becomes greater, the greater the amount w of the evaporated water, and the smaller the pressure p.

Table 10

The variation of equipotential temperature during isoenthalpic evaporation

p	900	700	500	300	mb
$w = 1$ g/kg	0.08	0.25	0.35	1.25	°C

In order to explain this phenomenon, we shall calculate the amount of work carried out by the gas against external pressure when passing from θ_e to T' through T, and from θ'_e to T' through θ'. Since the work A carried out during the isobaric process equals $A = R(T_0 - T)$, and during the adiabatic one $c_v(T_0 - T)$, we have

$$A_1 = A(\theta_e \to \theta \to T \to T') = R(\theta_e - \theta) + c_v(\theta - T) + R(T - T') = R,$$
$$A_2 = A(\theta'_e \to \theta' \to T'') = R(\theta'_e - \theta') + c_v(\theta' - T'),$$

hence

$$A_1 - A_2 > 0.$$

Thus, the work of expansion will be greater at a higher temperature, the pressure of the difference between the amounts of work being measured by the temperature reduction during the isoenthalpic evaporation.

Since the moist adiabatic process can be regarded as an infinite succession of elementary infinitesimal adiabatic and isoenthalpic processes, the equipotential temperature is a quasi-invariant for the moist and pseudoadiabatic processes.

The calculation of the equipotential temperature leads to the calculation of the potential temperature and of the equivalent difference.

§ 19. The Potential - Equivalent Temperature

The potential-equivalent temperature θ_e^* is that temperature gain of the air, when all the water vapor contained in it is condensed, the liberated latent heat is used up in heating the air at constant pressure, and the air is then adiabatically reduced to standard pressure. However, the following definition is more exact.

The potential-equivalent temperature is that temperature, which the absolutely dry air had had when its pressure had been 1000 mb, before its transformation into moist air (p, T, w), which took place as a result of the adiabatic compression to pressure p and the following isoenthalpic evaporation of w g of water in it.

Therefore

$$\theta_e^* = T_e \left(\frac{1000}{p}\right)^{\frac{R}{c_{pd}}} = \left(T + \frac{Lw}{c_{pd}}\right)\left(\frac{1000}{p}\right)^{\frac{R_d}{c_{pd}}}. \tag{1}$$

The potential equivalent temperature is greater than the equivalent - potential one:

$$\theta_e^* - \theta_e = \frac{Lw}{c_{pd}}\left[\left(\frac{1000}{p}\right)^{\frac{R_d}{c_{pd}}} - 1\right] > 0. \tag{2}$$

This difference grows, as p becomes smaller, and w greater. It is a measure of the difference of the work of expansion carried out by air against external pressure during the change of state following the path $\theta_e^* \longrightarrow T_e \longrightarrow T$ and following the path $\theta_e \longrightarrow \theta \longrightarrow T$.

It follows from the above that the potential - equivalent temperature is strictly an invariant of isoenthalpic evaporation at $T_e = \text{const.}$, $p = \text{const.}$, and a quasi-invariant of all the adiabatic processes, while it increases somewhat with the decrease of pressure.

§ 20. The Pseudopotential Temperature

The pseudopotential temperature θ_{se} is the temperature gain of air when its pressure is raised pseudoadiabatically until the water vapor is completely removed and then brought adiabatically to the standard pressure.

We will work out an equation determining the pseudopotential temperature. Since the variation of state during the pseudoadiabatic compression differs negligibly from the variation of state during the most-adiabatic process, we may derive this equation using the euqation of the condensation adiabatic curve in the form of

$$\ln T - \frac{R_d}{c_{pd}} \ln (p - E) + \frac{1}{c_{pd}} \frac{Lw}{T} = \text{const.} = \frac{1}{c_{pd}} \varphi. \tag{1}$$

We introduce an auxiliary thermometric characteristic, called the partial potential temperature:

$$\theta_d = T \left(\frac{1000}{p_d}\right)^{\frac{R_d}{c_{pd}}}. \tag{2}$$

It is obvious that the partial potential temperature θ_d is the temperature gain of the air when brought adiabatically from the state in which the partial pressure of dry air is p_d, to the standard pressure. By comparing equation (2) with the equation determining the potential temperature, and neglecting the small difference between c_p and c_{pd}, we obtain:

$$\theta_d = \theta \left(\frac{p}{p-e}\right)^{\frac{R_d}{c_{pd}}} = \theta \left(1 + \frac{e}{p-e}\right)^{\frac{R_d}{c_{pd}}} = \theta (1 + 1.605\, w)^{\frac{R_d}{c_{pd}}}. \quad (3)$$

Expanding θ_d in a power series and retaining but the first term, we obtain:

$$\theta_d = \theta (1 + 0.46\, w). \quad (4)$$

Introducing the partial temperature, from equation (2) into equation (1) of the condensation adiabatic curve, we give the following form to the equation:

$$\ln \theta_d + \frac{Lw}{c_{pd}T} = \text{const.} \quad (5)$$

This equation is correct at the commencement of saturation, i. e., at the condensation point. It also remains correct in the case when all the water vapor condenses and w vanishes.

The quantity thus obtained

$$\theta_d = \theta_{se}$$

will be the pseudopotential temperature. (Rossby calls this quantity the equivalent-potential temperature).

Introducing θ_{se} in equation (5) we obtain:

$$\ln \theta_{se} = \ln \theta_d + \frac{Lw}{c_{pd}T} \quad (6)$$

or, transforming into exponential form

$$\theta_{se} = \theta_d \cdot e^{\frac{Lw}{c_{pd}T}}. \quad (7)$$

In order to obtain the equation determining the pseudopotential temperature of the unsaturated air from equation (7), it is necessary to introduce instead of T, not the virtual temperature, but the temperature of condensation, because for the dry adiabatic process

$$\theta_{se} = \theta_d e^{\frac{Lw}{c_{pd}T_k}} = \theta (1 + 0.46\, w)\, e^{\frac{L_k w}{c_{pd}T_k}}, \quad (8)$$

i.e., θ_{se} is a magnitude which is strictly invariant for the dry-adiabatic process (because θ, w, T_k are constant during this process).

It follows from the derivation of equation (8) that the pseudopotential temperature is also an invariant for the duration of the moist and pseudoadiabatic processes.

From the equations (1), (5), (6) we obtain

$$\varphi = c_{pd} \ln \theta_{se} + \text{const.}, \quad (9)$$

i.e., the pseudopotential temperature bears to the entropy of moist air the same relation as does the potential temperature to the entropy of dry air.

Isoenthalpic evaporation is an irreversible process. The entropy of moist air therefore increases during this process, and hence the pseudopotential temperature increases too. However, this increase is small. Thus, the pseudopotential temperature is quasi-invariant of isoenthalpic evaporation.

§ 21. The Pseudoequivalent Temperature

The pseudoequivalent temperature T_{se} is the temperature gain of the air when pseudoadiabatically compressed until all the water vapor is completely removed and then expanded adiabatically to the previous pressure. Since θ_{se} is obtained by reducing T_{se} to standard pressure, we have

$$\theta_{se} = T_{se}\left(\frac{1000}{p}\right)^{\frac{R_d}{c_{pd}}} = T_{se}\cdot\frac{\theta}{T}. \tag{1}$$

Combining equation (1) with equation (8) § 20 we obtain the equation determining the pseudoequivalent temperature for unsaturated moist air:

$$T_{se} = T\,(1 + 0.46\,w)\,e^{\frac{L_k w}{c_{pd}T_k}} \tag{2}$$

and for saturated moist air:

$$T_{se} = T\,(1 + 0.46\,w)\,e^{\frac{Lw}{c_{pd}T}}. \tag{3}$$

The pseudoequivalent temperature is not invariant during the adiabatic processes, and is quasi-invariant during isoenthalpic evaporation.

The calculation of the pseudoequivalent and the pseudopotential temperature by means of adiabatic diagrams is simple in principle: for this purpose it is sufficient to move from the reflection point along the adiabatic curve to the condensation point, and then along the moist adiabatic curve up to the pressure level, at which this curve will virtually merge with the corresponding dry-adiabate; then, to descend along this dry adiabate to the previous pressure (to determine T_{se}), or to the standard pressure (to determine θ_{se}). But in practice, such a method of determination often appears very difficult, because the merging of the dry and the moist adiabatic curves takes place beyond the limits of the diagram. The pseudoequivalent temperature is always greater than the equivalent one.

§ 22. The Wet Bulb Pseudopotential Temperature.

The wet bulb pseudopotential temperature θ_{sw} is the temperature gain of the air, when adiabatically compressed to the level of condensation and then, after being humidified, expanded moist-adiabatically to standard pressure.

If the air body corresponding to the point A rises to a higher pressure than that of condensation, its temperature will vary according to the moist adiabate passing through the point B. Thus, if the air body contracts or expands dry - moist - or pseudoadiabatically, then its wet bulb pseudotemperature

just as its pseudoequivalent temperature, is determined by means of the moist adiabate passing through the point of condension B. θ_{se} and θ_{sw} remain constant during these processes, i. e., they are strictly invariant.

Let us derive the equation relating θ_{se} and θ_{sw} to each other. We will take the equation of the condensation adiabate in the form

$$\ln \theta_d + \frac{Lw}{c_p T} = \text{const.} \tag{1}$$

and apply it to an air body at standard pressure, and saturated with water vapor. Then $\theta_d = T = \theta_{sw}$ for this air body. The equation of the condensation adiabate can, therefore, be written as follows:

$$\ln \theta_{sw} + \frac{Lw(\theta_{sw})}{c_p \theta_{sw}} = \text{const.} \tag{2}$$

Since

$$\ln \theta_{se} = \text{const.},$$

we have

$$\ln \theta_{se} = \ln \theta_{sw} + \frac{Lw(\theta_{sw})}{c_p \theta_{sw}} \tag{3}$$

or, transforming to exponential form:

$$\theta_{se} = \theta_{sw} \cdot e^{\frac{Lw(\theta_{sw})}{c_p \theta_{sw}}}. \tag{4}$$

Since the exponent is rather small, this function can be expanded in a series, whose terms of orders higher than the first may be omitted without impairing the accuracy. Equation (4) will then have the form:

$$\theta_{se} = \theta_{sw} + \frac{Lw(\theta_{sw})}{c_p \theta_{sw}}. \tag{5}$$

It follows from the latter equation that the pseudopotential temperature θ_{se} is equal to the pseudopotential temperature of the wet bulb thermometer θ_{sw}, to which is added the temperature gain caused by the liberated latent heat of the saturated water vapor contained in the air at the pseudopotential temperature of the wet bulb thermometer. The equation (5) is similar to equation (1) § 15, linking the equivalent temperature T_e with the temperature of the wet bulb thermometer T_w. Thus, we may use θ_{se} or θ_{sw} to identify the air masses. However, the pseudopotential temperature of the wet bulb thermometer has definite advantages, in so far as it does not require the reduction of the air to zero pressure in order to remove the precipitating water.

§ 23. Comparison of the Thermohygrometrical Characteristics

We will now compare different thermohygrometrical characteristics (Figure 31). It follows, from the explanations given in the previous sections, that the following inequalities hold for saturated moist air:

$$T_k < \tau < T_{sw} < T_w < T < T_e < T_{se}. \tag{1}$$

The difference between T_w and T_{sw} is negligibly small, when compared with the measurements of humidity, particularly in a free atmosphere. It is equal to zero when the air is saturated. But then

$$T_k = \tau = T_{sw} = T_w = T < T_e < T_{se}. \tag{2}$$

The difference $T_{se} - T_e$ reaches a maximum in this case. The concepts of the dew point and the condensation point are meaningless if the air is perfectly dry. The value of T_{sw} becomes indefinite, while T_w retains a quite definite value. The difference $T_{se} - T_e$ vanishes. Thus for dry air

$$T_w < T = T_e = T_{se}. \tag{3}$$

For the potential temperatures similar inequalities hold: for unsaturated air (Figure 31,a)

$$\theta_{sw} < \theta < \theta_e < \theta_e^* < \theta_{se}, \tag{4}$$

for saturated air (Figure 31, b)

$$\theta_{sw} < \theta < \theta_e < \theta_e^* < \theta_{se}, \tag{5}$$

for perfectly dry air:

$$\theta = \theta_e = \theta_e^* = \theta_{se}. \tag{6}$$

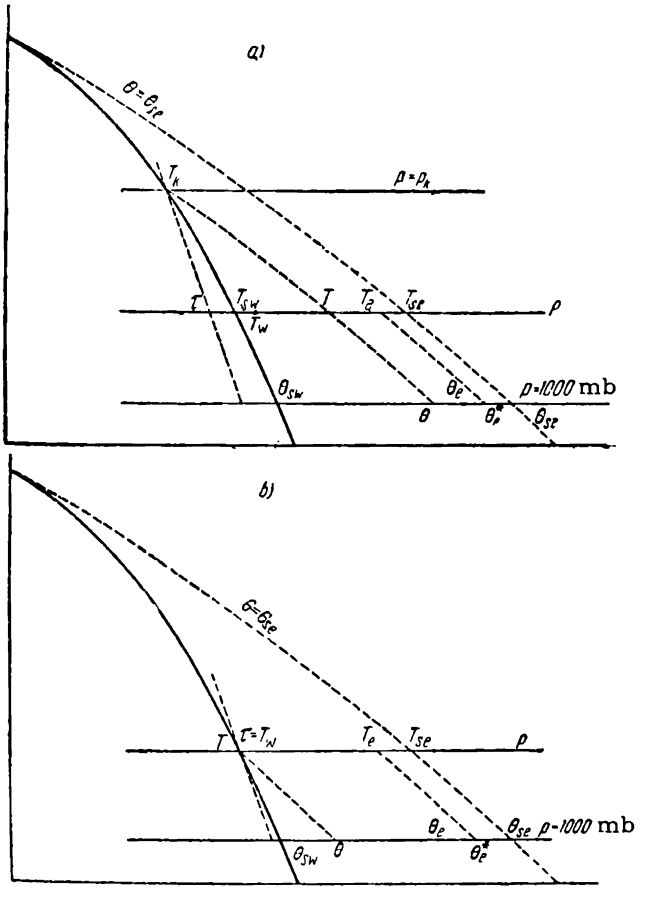

Figure 31. Thermohygrometric characteristics on the emagram: a) unsaturated air; b) saturated air

Table 11

The behavior of the thermohygrometric characteristics in different processes

Process	Dry-adiabatic		Moist-adiabatic		Pseudo adiabatic	Isoenthalpic	Isobaric	
Characteristic	Compression	Expansion	Compression	Expansion	Compression	Evaporation	Heating	Cooling
Molecular temp.	y	p	y	p	y	y	p	y
Relative humidity	p	y	п	п	п	p	y	p
Absolute humidity	y	p	y	p	y	p	пп	пп
Specific humidity	п	п	y	p	y	p	п	п
Mixing ratio	п	п	y	p	y	p	п	п
Dew point	пп	пп	y	p	y	p	п	п
Condensation point	п	п	п	п	y	p	y	p
Wet bulb temp.	y	p	y	p	y	п	p	y
Equivalent temp.	y	p	y	p	y	п	p	y
Wet bulb pseudo-temperature	y	p	y	p	y	пп	p	y
Pseudoequivalent temperature	y	p	y	p	y	пп	p	y
Equipotential temp.	п	п	ппп	ппп	ппп	ппп	p	y
Potential-equivalent temperature	ппп	ппп	ппп	ппп	ппп	п	p	y
Pseudopotential temperature	п	п	п	п	п	ппп	p	y
Pseudopotential wet bulb temperature	п	п	п	п	п	ппп	p	y

In Table 11 the behavior of different thermohygrometric characteristics is exhibited for the most important processes, namely: 1) dry-adiabatic compression and expansion, 2) moist-adiabatic compression and expansion, 3) pseudoadiabatic compression, 4) isoenthalpic evaporation, 5) isobaric heating and cooling. The following symbols are used in the table: p - the characteristic increases; y - decreases; п - is constant, пп - nearly constant.

Isobaric heating and cooling take place mostly in the lower layers of the atmosphere. From the table it can be seen that the characteristics of moisture are invariant in regard to these processes, i.e., the specific humidity, the mixing ratio and the dew point. The dew point has an advantage over the first two characteristics, because together with the temperature it also characterizes the degree of saturation of the air by water vapor. The dew point has a great practical importance for forecasts of fog formations and the first autumn frosts. Fog is formed, if the temperature drops below the dew point. This is correct both for solar fog, and for advection fog. If the temperature drops below 0° C, first autumn frosts are formed, provided that the dew point is lower than 0°C; but fog, preventing a further cooling, is formed if the dew point is higher than 0°C.

The dew point is quasi-invariant with respect to the adiabatic process. Any of the last four characteristics introduced in the table can be equally used in order to analyze results concerning the free atmosphere, where adiabatic processes play an important role. The pseudopotential wet bulb temperature is, however, to be preferred to the others, first because it is invariant for all the adiabatic processes, and second because its computation is carried out more simply by graphic methods than is the computation of the pseudopotential temperature (in the computation of the pseudopotential temperature it is necessary to extrapolate beyond the limits of the diagram, for the removal of the condensed water vapor); finally, the pseudopotential wet bulb temperature is closely related to the measured wet bulb temperature.

Chapter V

THE STATICS OF THE ATMOSPHERE

§ 1. The Fundamental Problem of Atmospheric Statics

In the present chapter we will investigate air in a static state relative to the earth's surface. Actually, the atmosphere is never completely at rest, and hence our assumption may appear to deviate excessively from reality. However, three circumstances support this premise.

First, this assumption provides a natural introduction to the more complicated problems of atmospheric dynamics; second, it enables to find more general laws for the pressure and mass distributions in the atmosphere.

Finally, as we shall later see, the statistical laws of the pressure and mass distribution seem to apply to the moving atmosphere too, if the motion is nearly horizontal. However, vertical motions taking place in the presence of orographic barriers can influence the pressure field quite considerably, and conclusions drawn from statics cease to be accurate in such a case.

The fundamental problem of atmospheric statics concerns the determination of the space distribution of the parameters of state, i. e., the pressure, the density, the specific volume and the temperature. The empirical data for such a study are given by meteorologic and aerologic observations. While the meteorologic observations concern a point fixed in space, the aerologic ones as a rule give only a sequence of values of the state parameter measured at about the same time, the coordinates of the points to which these values correspond remaining mostly unknown.

We now turn to the fundamental and practical problem of atmospheric statics: the determination of the height of a point on the basis of the vertical distribution of the parameters of state, ignoring their small horizontal variations, and referring the whole sequence of aerologic measurements only to the vertical direction.

§ 2. Mathematical Determination of the Parameters of State at a given Point

In the previous chapters we examined the physical properties of individual air bodies. Turning now to the examination of the totality of the individual physical air bodies of which the atmosphere is composed, we are first of all obliged to explain, in what way the state of air at a given spatial point can be characterized.

Mathematically we define the air density at any point as the limit towards which tends the ratio of an air-mass to the volume it occupies, as the latter decreases indefinitely:

$$\rho = \lim_{\Delta\tau \to 0} \frac{\Delta m}{\Delta \tau}. \qquad (1)$$

The mathematical definition of the concept of air density at a given point can cause difficulties due to the discreteness of gases, which are aggregations of separate molecules, separated from each other by a distance which exceeds their diameters. However, the immense number of molecules in a volume unit at normal conditions near the earth, allows to consider the atmospheric medium as being

continuous. Indeed, 1 cm^3 of dry and pure air contains $2.7 \cdot 10^{19}$ molecules under normal conditions, i. e., as much as $2.7 \cdot 10^7$ molecules per cubic micron. In order to comprehend the magnitude of this number, we should point out that more than 500 billion years are needed to count the molecules contained in 1 cm^3 of air, even if 100 molecules were counted in a minute without interrupting the operation by day or night. The density of the atmosphere decreases with increasing distance from the earth, therefore the number of molecules per volume unit becomes less and less. Thus, for example, 1 cm^3 contains only $3 \cdot 10^{11}$ molecules at an altitude of 800 km, i. e., 0. 3 [sic] molecules per cubic micron. It is obvious that air cannot be considered as a continuous medium at very great altitudes; it must be considered there as a discrete assemblage of separate molecules. The concept of air density at a point has no meaning in great altitudes, because the ratio (1) does not tend at these heights to any definite limit.

The mathematical expression for the pressure at a certain point

$$p = \lim_{\Delta f \to 0} \frac{\Delta P}{\Delta f}, \tag{2}$$

defined as the limiting ratio of the force acting on a surface element to the magnitude of the latter as it decreases indefinitely, also gives rise to fundamental difficulties. The force which produces the surface pressure is, according to the kinetic theory, the result of molecules colliding with the boundary surface enclosing the gas during their chaotic motion. It is obvious that the ratio (2) will not tend to a definite limit when the air is very rarefied, the mathematical concept of pressure at very great altitudes therefore has no meaning, nor has the mathematical concept of density. The mathematical definition of pressure is not difficult for the lower layers of the atmosphere on account of the huge number of molecules in a volume unit.

We can mathematically define the temperature of the air at a given point, as a value which is proportional to the limiting ratio of the kinetic energy of the molecules to the volume this air occupies as the latter indefinitely decreases. This definition of the temperature has no meaning at very great altitudes. The mathematical definition of temperature presents no practical difficulties for the lower layers of the atmosphere.

The air humidity at a given point can be defined mathematically as the limit of water vapor density

$$\rho_w = \lim_{\Delta \tau \to 0} \frac{\Delta m_w}{\Delta \tau}; \tag{3}$$

hence all the previous considerations, concerning the mathematical definition of the density at a point, can be applied to it.

The density, temperature and humidity of air are functions of the [spatial] coordinates of the point and of the time. These functions are, generally speaking continuous. They also have continuous derivatives. These functions and their derivatives with respect to the coordinates and the time may become discontinuous on some layers of the atmosphere.

The pressure is a function of position and time. For reasons which will be given below, the functions appearing in meteorology are considered to be continuous and have, generally speaking, continuous derivatives with respect to the [spatial] coordinates and the time. The spatial derivatives of the pressure can be discontinuous only on isolated layers of the atmosphere.

§ 3. Scalar Fields

All the parameters of state, i. e., the density, specific volume, pressure, temperature, and humidity, are scalar quantities. It is known that a scalar quantity, or simply a scalar, is the designation given to a quantity, which is determined by its numerical value only, be it positive or negative. All the parameters of state, including the temperature (if it is measured on the absolute scale) are essentially positive values.

We ascribe well-defined values to the physical scalar quantities under consideration. This means that any physical quantity, according to its definition, can have only one value for a given point in space at a given moment, under given real conditions.

A spatial distribution of scalar quantities is termed a scalar field. Equiscalar surfaces are drawn in order to visualize a scalar field.

An equiscalar (or isoscalar) surface is a surface, on which the considered scalar quantity has constant value. The equation of an equiscalar surface therefore has the following form:

$$a(x, y, z) = \text{const.} \tag{1}$$

As examples of equiscalar surfaces one can give the isobaric surfaces p = const., the isosteric surfaces v = const., the isopycnic [equal density] surfaces ρ = const., the isothermic surfaces T = const., etc. The equation of the equiscalar surface (1) does not include the time, if the field of the scalar quantity does not change with time. The equation of the equiscalar surface has the form:

$$a(x, y, z, t) = \text{const.} \tag{2}$$

if the scalar field does change with time.

The distribution of the parameters of state under real atmospheric conditions is so complicated that equations (1) and (2) cannot be expressed, in the general case, by means of any elementary function. We know nevertheless that equiscalar surfaces exist in the atmosphere, and we are even able to indicate some general properties of these surfaces. So, for example, we assume that the equiscalar surfaces are continuous and that they cannot be disrupted somewhere within the field's boundaries; these surfaces either stretch to infinity, or they close upon themselves, or terminate at the external or at the internal boundaries of the field. Since the scalar field is well defined everywhere, the equiscalar surfaces cannot intersect; otherwise the scalar quantity might have several values for one point. Nevertheless, some surfaces can still intersect each other.

It is useful to plot equiscalar surfaces for scalar values differing by one unit. Such surfaces

$$a = 0, 1, 2, 3, \ldots$$

we will call unit surfaces. These surfaces divide the scalar field under consideration into a sequence of unit layers. A proper choice of unit of the scalar quantity makes it possible to give the layer that thickness, which is most convenient for a clear and faithful representation of the field.

By selecting small enough units of measure, we obtain layers as thin as required, and in the limit infinitely thin layers.

Thus, we are able to say that between two arbitrary points of the scalar field $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$, that number of unit layers $N(\alpha)$ lie, which is sufficient to define the difference between the scalar values at these points.

$$\alpha(x_2, y_2, z_2) - \alpha(x_1, y_1, z_1) = N(\alpha). \tag{3}$$

Letting the unit layers become infinitely thick, only one layer occupies all space, and the number of layers between two arbitrary points is thus equal to zero. The scalar function in this case degenerates into a constant, and the equiscalar surfaces satisfying the condition α = const., becomes indefinite. Often two scalar functions are simultaneously given by

$$\alpha = \alpha(x, y, z)$$

and

$$\beta = \beta(x, y, z), \tag{4}$$

Their equiscalar surfaces intersect along doubly equiscalar curves (α, β). Each of the scalar quantities α and β under consideration, has a constant value along these lines. The unit equiscalar surfaces of one quantity divide the unit layers of the other quantity into tubes, whose sections have the form of curvilinear parallelograms. These unit tubes are also called solenoids.

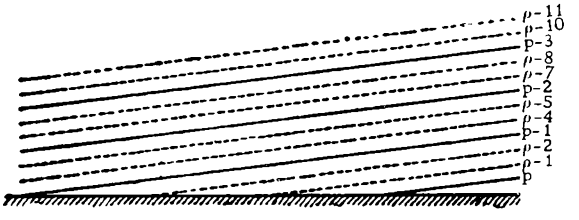

Figure 32. Barotropic density field

It follows from the foregoing discussion of equiscalar surfaces that the solenoids cannot terminate anywhere within the scalar field under consideration. The solenoids either extend to infinity, or close upon themselves. Therefore, if a closed contour is drawn one can speak of a well-defined number $N(\alpha, \beta)$, of solenoids encircled by it.

If a functional dependence

$$F(\alpha, \beta) = 0, \tag{5}$$

exists between the scalar quantities, then any equiscalar surface α = const., will, at the same time, be an equiscalar surface β = const., and vice-versa. Therefore, both scalar functions will have the same system of equiscalar surfaces, but the unit surfaces of one scalar function will not coincide with the surfaces of the other function. In such a case it is necessary to use different systems of enumeration for the equiscalar surfaces of the function α, and for these of the function β.

Figures 32 and 33 represent two fields: the density field, determined by the distribution of the density ρ, and the pressure field p. In the case represented in Figure 32, a functional dependence $F(\rho, p) = 0$ exists between ρ and p. The isothermic surfaces therefore coincide with the isobaric ones. Such a density field is called barotropic, i.e., determined by the pressure.

There is no well-defined relation between ρ and p in the case represented by Figure 33. The isobaric and the isosteric surfaces therefore intersect, giving rise to isobaric–isosteric solenoids. Such a field is called baroclinic.

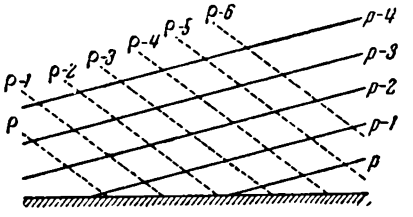

Figure 33. Baroclinic density field

§ 4. The Vector Field

A directed magnitude is called a vector. Analytically, a vector is represented by its components, with respect to three axes, perpendicular to each other. We will denote the unit vectors directed along the axes x, y, z, by i, j, k respectively (Figure 34). The vector A will then be represented in the following form:

$$A = A_x i + A_y j + A_z k.$$

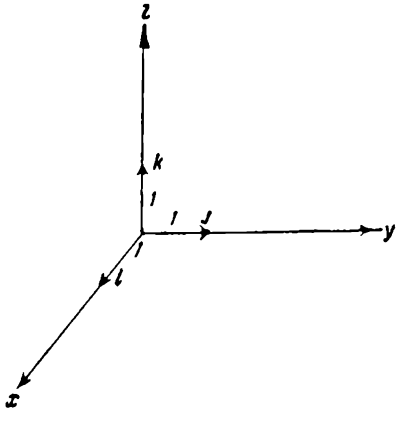

Figure 34. The unit vectors i, j, k.

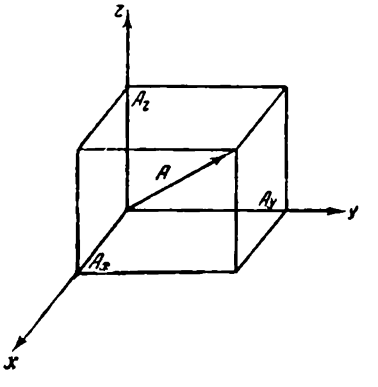

Figure 35. The vector A and its components A_x, A_y, A_z.

The vector field $A(x, y, z)$, i.e., the direction and the magnitude of the vector at each point in space, are completely determined by the fields of the three scalar quantities: A_x, A_y, A_z.

However, the direct representation of the vector field in the form of the three fields of its components in any coordinate system, does not give a graphical visualization of the vector field. The streamlines are of greater importance for a visual representation of the vectorial field. By a streamline is understood

a curve to which the field vector under consideration is tangent at each of its points (Figure 36). A surface spanned by streamlines is called a stream surface. If this surface has the form of a tube, it is called a steam tube (Figure 37).

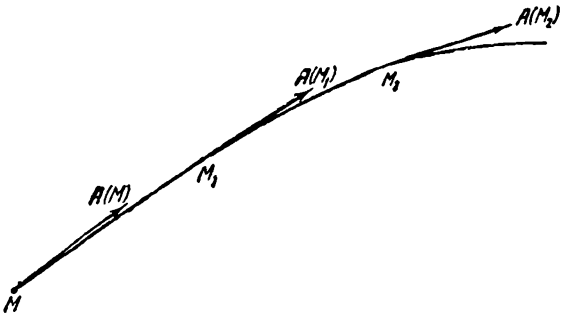

Figure 36. The streamline

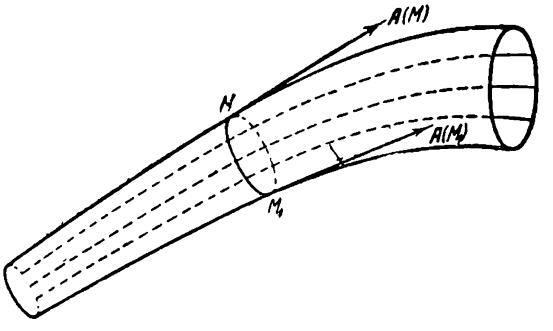

Figure 37. The stream tube

Since the field vector is tangent to the streamline, the differential equation of the of the streamline has the form:

$$\frac{dx}{A_x}=\frac{dy}{A_y}=\frac{dz}{A_z}. \tag{1}$$

The integration of these equations gives rise to two scalar functions: $\alpha(x,y,z)$ and $\beta(x,y,z)$. Doubly equiscalar curves of these functions:

$$\alpha(x,\ y,\ z)=\text{const.}\quad \beta(x,\ y,\ z)=\text{const.} \tag{2}$$

represent the streamlines.

Once the streamlines have been determined, it is only necessary to know the absolute value of the vector at each point in space, in order to represent the vector field completely. Setting

$$A=\sqrt{A_x^2+A_y^2+A_z^2},$$

we obtain the equiscalar surface

$$A(x,\ y,\ z)=\text{const.}, \tag{3}$$

which gives the absolute value of the vector at the points situated on this surface. Three scalar functions (2) and (3) give a visual representation of the streamline in a very general form.

Let us give another representation of the vector field, more lucid from the physical point of view. We should note that the solenoids (α, β) of the scalar functions (2) are the stream tubes of our vector field. The product of the absolute value of the vector by the cross section of the stream tube is called the flux of the vector through this section. In the most general case the vector flux changes from one section of the stream tube to another. But the flux density of the vector is a well defined scalar quantity at any spatial point and the vector field under consideration can be characterized both by the system of these stream tubes (2) and by the corresponding surfaces of equal flux density

$$\gamma(x,\ y,\ z) = \text{const.} \tag{4}$$

if the system of the stream tubes has been determined.

If γ degenerates to a function of α or β, the surfaces (2) and (4) still do not intersect. The flow in this case has the same value for all the sections of the stream tube. Such a vector field can be clearly defined by the stream tubes, which are also, at the same time, tubes of equal flux of the vector. Such a vector is directed along the axis of the stream tube, and its value is inversely proportional to the cross section of the tube. α and β can be chosen in such a way that the unit stream tubes will, at the same time, be the tubes of unit vector flux. A vector field which possesses these characteristics is called a solenoidal or source-free vector field. The tubes of unit flux are also called the solenoids of the vector field.

The streamlines, represented either by the differential equation (1) or by equation (2), form a family of curves of very general form. These curve families generally do not possess orthogonal surfaces.

Let us examine a vector field where the streamlines are orthogonal to an arbitrary set of surfaces

$$\alpha(x,\ y,\ z) = \text{const.} \tag{5}$$

The orientation of any surface element in space is determined by the direction cosines cos (n,x), cos (n,y), cos(n,z) of the normal n to this surface, i.e., by magnitudes proportional to the partial derivatives $\frac{\partial\alpha}{\partial x}$, $\frac{\partial\alpha}{\partial y}$, $\frac{\partial\alpha}{\partial z}$ (Figure 38). The components A_x, A_y, A_z of the vector under consideration must be proportional to these derivatives.

The condition of orthogonality of the vector $\boldsymbol{A}$ to the surface α = const. can be written in the form:

$$\frac{A_x}{\frac{\partial\alpha}{\partial x}} = \frac{A_y}{\frac{\partial\alpha}{\partial y}} = \frac{A_z}{\frac{\partial\alpha}{\partial z}} = \frac{A_n}{\frac{\partial\alpha}{\partial n}} = \beta(x,\ y,\ z), \tag{6}$$

where $\beta(x,y,z)$ is some function of the co-ordinates (x, y, z). Thus, the field of vectors everywhere orthogonal to the surfaces α = const., is determined by two functions α and β. The vector's direction coincides with the direction of the normals to the surfaces α = 0, 1, 2 ...., and its absolute value is determined by the equation

$$A = \beta \cdot \frac{\partial\alpha}{\partial n} = \frac{\beta}{h}, \tag{7}$$

where h is the thickness of the α unit layer. If between the functions α and β some functional dependence

$$F(\alpha, \beta) = 0,$$

exists, the vector A is determined by one scalar only, because the vector's direction is perpendicular to the surfaces $\alpha = \text{const.}$, and $\beta = \text{const.}$

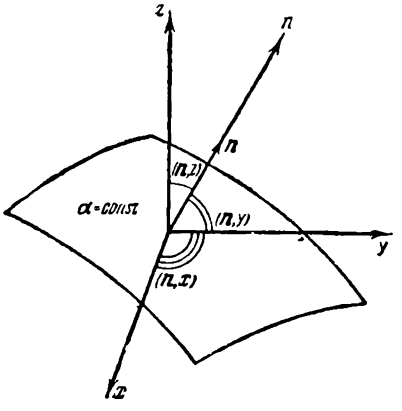

Figure 38. The spatial orientation of a surface element is determined by the direction of the normal n

The absolute value of the vector can be determined using the new enumeration, according to the equation

$$A = \frac{\partial \alpha}{\partial n} = \frac{1}{h}, \tag{8}$$

where h is the thickness of the unit layer. Such a vector field is called a potential field. This field is directly given by the unit layers of the scalar α (Figure 39). This potential vector is known in field theory as the gradient of the scalar function α, denoted by the symbol ∇ or grad, so that

$$A = \nabla\alpha = \operatorname{grad}\alpha = \frac{\partial \alpha}{\partial x} \cdot i + \frac{\partial \alpha}{\partial y} \cdot j + \frac{\partial \alpha}{\partial z} \cdot k. \tag{9}$$

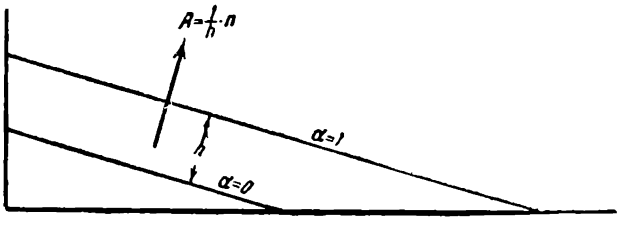

Figure 39. Potential vector field

The gradient of the scalar α is thus, according to field theory, that vector which at each point of the scalar field indicates the direction of the steepest increase of the scalar α; its value is equal to the derivative of the scalar in this direction.

However, in meteorology this vector is called the ascendent of the scalar field. Meteorology designates by the gradient G the vector which is numerically equal to the ascendent but directed in the opposite sense, i. e., in the direction of the most rapid decrease of the scalar α, so that

$$G = -A = -\nabla\alpha = -\text{grad}\,\alpha. \tag{10}$$

We shall use mainly the meteorological meaning of the word gradient in the following.

§ 5. The Force of the Earth's Gravitation

In atmospheric statics we consider the atmosphere both in its entirety, and its separate parts, as a mechanical system in equilibrium under the action of forces exerted on it. Two kinds of forces operate in the atmosphere: the external, which act on the elementary air mass from without, and the internal forces, caused by the interaction of the air masses themselves. Terrestrial gravitation is the most important force in the atmosphere; we will consider this force as the sole external force. The sole internal force in the static case is the pressure.

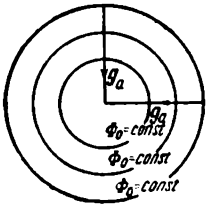

Figure 40. The force of gravitation is directed towards the center of the sphere. The equipotential surfaces are spherical

Let us assume that an observer, situated on an immobile earth, measures the force of the earth's gravitation. This force, acting on a mass unit located at a distance r from the earth's center, is directed towards this center and its value is equal to

$$g_a = K \cdot \frac{M}{r^2}, \tag{1}$$

where K is the universal constant of gravitation, M – the mass of the earth. The force of graviation is directed towards the earth's center (Figure 40). Therefore if it has a potential, its equipotential surfaces will be spherical. Denoting the distances along the common normals between the equipotential surfaces by δr, in case of the existence of a gravitational potential we obtain

$$\delta\Phi_a = g_a \cdot \delta r. \tag{2}$$

Substituting for g_a its value from equation (1) we find:

$$\delta\Phi_a = K \cdot \frac{M}{r^2} \cdot \delta r = -\delta\left(K \cdot \frac{M}{r}\right), \tag{3}$$

by which the existence of a potential Φ_a is proved:

$$\Phi_a = C - K \cdot \frac{M}{r}. \tag{4}$$

Fixing the constant of integration C by the condition that the potential should vanish at the pole, we obtain

$$\Phi_a = K \cdot M \left(\frac{1}{a_p} - \frac{1}{r} \right). \tag{5}$$

The gradient of the gravitational potential is the force of gravitation, referred to a mass unit, i.e.,

$$g_a = -\nabla \Phi_a. \tag{6}$$

It is obvious that equation (6) can be applied to an arbitrary mass distribution inside the earth, because the gravitational potential exists in this case too.

§ 6. Pressure

Generally speaking, not only volume forces but surface forces too act on an air parcel. The pressure is the sole surface force which acts on the air when it is at rest.

The pressure inside the gas is, according to Pascal's law, transmitted equally in all directions if the gas is in equilibrium.

If the pressure is equal at all points of the surface, then the force acting on a surface element $\delta\sigma$ will have a value of $p\delta\sigma$, and will be directed at the interior of the volume which the gas occupies, along the normal to this surface element.

We will now consider the dynamic significance of the pressure gradient (− grad p).

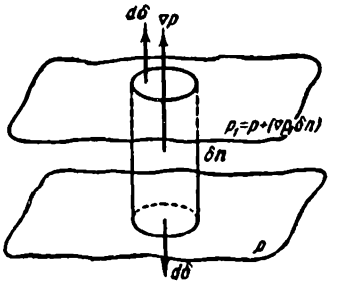

Figure 41. The pressure gradient is a force acting on a unit volume

Let us imagine an elementary cylinder in an isobaric field, the bases of which lie on the isobaric surfaces, and its axis directed along the common normal to these surfaces (Figure 41). Let us assume that the pressure on the bottom is equal to p, and on the top to $p' = p + (\operatorname{grad} p,\ \delta n)$, where $\delta n = \delta n \cdot n_1$, δn being the length of the cylinder, and n_1 - the unit vector in the direction of the pressure ascendent.

If $d\sigma$ is a vector, the value of which is equal to the area of the cylinder's base, and whose direction coincides with the outgoing normal to the surface at the cylinder base, then the forces acting on the bases of the cylinder will be:

on the bottom the following force acts:

$$-p\,d\sigma = p \cdot d\sigma \cdot n_1,$$

and on the top:

$$-p'd\sigma' = -[p + (\operatorname{grad} p,\ \delta n)]\,d\sigma \cdot n_1.$$

The sum of these two surface forces

$$dF = -p\,d\sigma - p'd\sigma' = -(\operatorname{grad} p,\ \delta n)\,\delta\sigma n_1 = -|\operatorname{grad} p| \cdot \delta n \cdot \delta\sigma n_1$$

is a force, which pushes the cylinder in the direction of the pressure gradient. The product $\delta n \cdot \delta\sigma$ is the elementary volume $d\tau$. Thus,

$$\frac{dF}{d\tau} = -\operatorname{grad} p. \tag{1}$$

Hence, the pressure gradient is a force which acts on a unit volume:

$$[\operatorname{grad} p] = [ML^{-2}T^{-2}].$$

We obtain the force per unit mass by multiplying the pressure gradient by the specific volume v, and denoting it by b, we obtain:

$$b = -v\nabla p = -v \operatorname{grad} p \tag{2}$$

or

$$b = -\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\operatorname{grad} p. \tag{3}$$

§ 7. The Equation of Absolute Motion of a Particle which is at Rest Relative to the Earth's Surface

We refer absolute space to the earth's center, which we consider to be immobile, disregarding the motion of the earth around the sun, and together with the sun in the universe.

The earth turns on its axis from west to east with a constant angular velocity ω. Therefore each particle, at rest with respect to the surface of the rotating earth, will move in absolute space. The trajectory of the particle in absolute space will lie on the circumference of the circle of latitude.

Since the particle under consideration moves on the circumference with a constant velocity, its tangential acceleration is equal to zero, and the total acceleration, equal to the centripetal acceleration , lies in the plane of the circle of latitude and is directed towards the earth's axis.

The absolute total acceleration of a particle, at rest with respect to the earth' s surface, is equal to the centripetal acceleration of transport of the particle. Denoting this acceleration by j_e, we obtain:

$$j_e = -\omega^2 \cdot R. \tag{1}$$

The inertial force will be the same as the centrifugal force:

$$I_e = -j_e = \omega^2 R, \tag{2}$$

directed along the radius of the circle of latitude towards the circumference (Figure 42).

The angular velocity of the earth's rotation is constant. Therefore the centrifugal force has a potential of

$$\Phi_e = -\frac{1}{2}\omega^2 R^2 \text{ and } I_e = -\nabla\Phi_e. \tag{3}$$

It is obvious that the equipotential surfaces of the centrifugal force are coaxial cylinders where the common axis coincides with the axis of rotation of the earth (Figure 43).

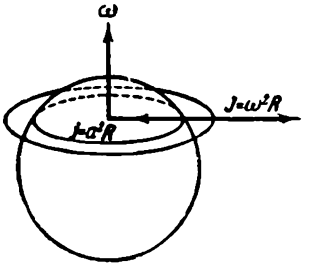

Figure 42. The centrifugal force is directed along the radius of the circle of latitude, from the center towards the circumference.

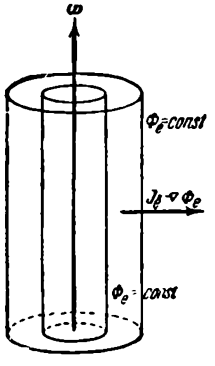

Figure 43. The equipotential surfaces of the centrifugal force are circular coaxial cylinders.

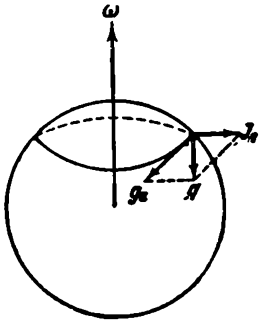

Figure 44. Gravity is equal to the vector sum of the gravitational and centrifugal forces.

The equation of absolute motion of the particle with respect to the earth's surface therefore has the following form

$$j_e = b + g_a \quad \text{or} \quad b + g_a + I_e = 0. \tag{4}$$

Equation (4) simultaneously describes the relative equilibrium, but is written in the absolute system of coordinates. The same equation, written in the

relative system of coordinates has the form:

$$\boldsymbol{b} + \boldsymbol{g} = 0, \tag{5}$$

where $\boldsymbol{g}$ is the gravity acceleration. We see, combining equations (4) and (5), that the pressure gradient appears to the observer from absolute space as being compensated by two forces $\boldsymbol{g}_a$ and $\boldsymbol{J}_e$, while to the observer from relative space – only by the gravity $\boldsymbol{g}$. Since the pressure gradient remains the same both in the absolute and in the relative system, we have (Figure 44)

$$g = g_a + J_e. \tag{6}$$

§ 8. The Field of Gravity

It is simpler to represent the field of gravity by means of level surfaces of gravity or, in other words, by equipotential surfaces of gravity.

A level surface of gravity is a surface to which the force of gravity is normal at each of its points. An example os such a level surface are the face of a calm sea.

It is obvious that the work done by the force of gravity when the position on a level surface is changed is equal to zero. In order to lift a unit mass by the height dz from one level surface to another along any path element ds it is necessary to counteract gravity and to carry out work, equal to

$$dw = (g, ds) = g\,dz. \tag{1}$$

According to the law of conservation of energy, this work cannot be destroyed. It shows up as a change in the potential energy of the particle which is determined by the position of the latter in the field of gravitation, and called the potential of gravity or, for brevity, the geopotential Φ. It is obvious that $dw = d\Phi$ or

$$d\Phi = g\,dz. \tag{2}$$

$d\Phi$ is a total differential. The level surface of gravity is, at the same time, an equipotential surface $\Phi = \text{const.}$, and the gravity – a gradient of the geopotential $\boldsymbol{g} = -\nabla\Phi$. Since the force of gravitation and the centrifugal force are potential vectors, gravity is a potential vector too. Therefore, equation (6) § 7 can be written in the following form:

$$-\nabla\Phi = -\nabla\Phi_a - \nabla\Phi_e,$$

i. e., the potentials Φ, Φ_a, Φ_e are linked by the relationship:

$$\Phi = \Phi_a + \Phi_e. \tag{3}$$

Consequently:

$$\Phi = KM\left(\frac{1}{a_p} - \frac{1}{r}\right) - \frac{1}{2}\,\omega^2 R^2. \tag{4}$$

It is easy to obtain the surfaces of equal potential Φ by drawing the spherical equipotential surfaces Φ_a and the cylindrical equipotential surfaces Φ_e and uniting these surfaces graphically.

Since all these surfaces are surfaces of revolution, it is sufficient to consider a meridional section.

The concentric circles in Figure 45 represent the lines of intersection of the meridional planes with the equipotential surfaces of gravity, whereas the vertical lines are lines of intersection of the meridional planes with the equipotential surfaces of the centrifugal force field. The value of the angular velocity of earth's rotation is magnified tenfold for the sake of clarity, and hence the slope of the geopotential level surfaces also appears much increased. The geopotential level surfaces are drawn on a geopotential scale of unit 10^7 m^2/sec^2.

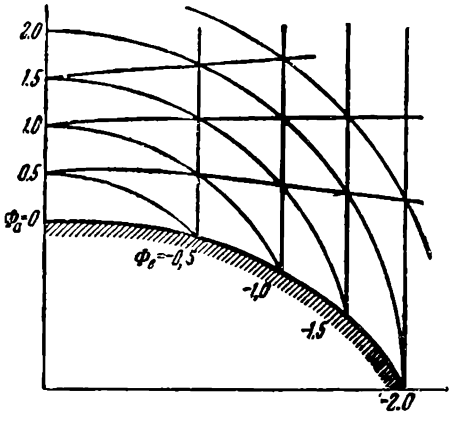

Figure 45. The graphic composition of equipotential surfaces

The following relation, which is obtained by the integration of equation (2)

$$\Phi = \int_0^z g\,dz, \tag{5}$$

exists between the altitude of a point above sea level and its geopotential, assuming that the geopotential vanishes at sea level.

Since

$$g = g_0 \frac{r^2}{(r+z)^2},$$

we have

$$\Phi = g_0 r^2 \int_0^z \frac{dz}{(r+z)^2} = g_0 \frac{z}{1+\frac{z}{r}}. \tag{6}$$

We obtain the equation of the equipotential surface

$$z = \frac{\Phi}{g_0 - \frac{\Phi}{r}} = \frac{\Phi}{g_{45}(1 - 0.0026\cos 2\varphi) - \frac{\Phi}{r}}. \tag{7}$$

by assigning a definite value to Φ .

It can be seen from the equation (7) that the level surfaces of gravity pass at different altitudes which depend on the latitude, namely: they descend and approach each other at the pole, but lift and move apart at the equator. The horizontal surfaces do not parallel each other at different altitudes, but are inclined from

the equator towards the pole, although the angle of this inclination is very small. We will evaluate the difference in altitude of these equipotential surfaces at the equator and at the pole, because this difference characterizes the closeness of these surfaces. Up to terms of the second order we obtain:

$$\frac{z_{0^\circ} - z_{90^\circ}}{z_{45^\circ}} = \frac{g_{90^\circ} - g_{0^\circ}}{g_{45^\circ}} = 0.0052 = 0.52\%.$$

For example, the equipotential surface passing above the equator at a height of 1,000 m sinks at the pole to an altitude of 995 m.

It follows from the above that the dynamic condition of a material particle is not clearly defined by its geometric position over sea level. If we consider two particles located at the same height above sea level, they may happen to be situated on different level surfaces of gravity, and therefore have different potential energies.

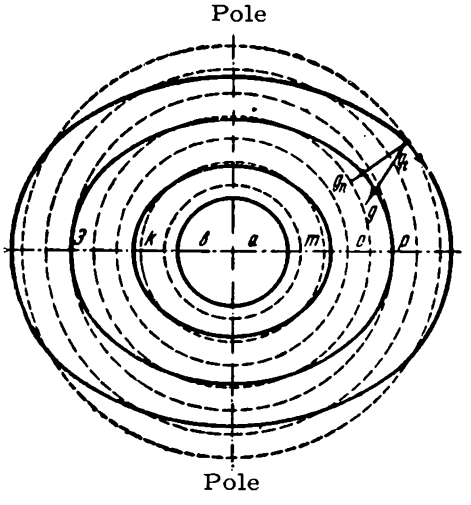

Figure 46. The surfaces of equal altitude and the equipotential surfaces

It is sometimes more convenient to use the equipotential surfaces rather than surfaces of equal altitude, because the gravity is always perpendicular to the equipotential surface, while it has a tangential component g_t with respect to the surface of equal altitude (Figure 46). A ball placed on the equipotential surface will remain in a state of equilibrium, but on the surface of equal altitude it will roll from the pole to the equator.

Therefore it is sometimes more convenient to characterize the position of a point not by its geometrical altitude, but by the value the geopotential assumes there, the so-called dynamic altitude.

The geopotential unit is the value in the CGS system of the work which must be carried out in order to displace a mass unit by 1 cm against a force field, having an intensity of 1 cm/sec^2. This unit, however, is too small and therefore inconvenient for practical use. The geopotential unit of the MTS system is 10^4 times greater. In geophysics use is made of a still greater geopotential unit, called the dynamic meter, in order to obtain a simpler and more convenient relationship between the geometric altitude and the geopotential. A dynamic meter is work which must be spent to displace a unit of mass by 1 meter against

a force field having an intensity of 10 m/sec^2 (a rounded value of the gravity acceleration). The dynamic meter and the geopotential unit are related in the CGS system by the relationship:

$$1 \text{ dyne} \cdot \text{meter} = 1000 \times 100 = 10^5 \text{ cm}^2/\text{sec}^2.$$

It must be kept in mind that the dynamic meter is not a unit of length, but of specific energy. In order to emphasize this, some special nomenclatures have been proposed for the dynamic meter. The most widespread designation is berk (in honor of Bjerknes). The other geopotential units are derived from the berk; most widely used are:

0.1 berk (dynamic meter) = 1 dynamic decimeter = 1 deciberk

10 dynamic meters = 1 decaberk
100 dynamic meters = 1 hectoberk
1000 dynamic meters = 1 dynamic kilometer = 1 kiloberk.

The deciberk is the geopotential unit in the MTS system.

It is not difficult to see that the following relationship, derived from equation (6), exists between the dynamic altitude Φ, measured in berks, and the geometric altitude z, measured in meters,

$$\Phi = \frac{g_\varphi}{10} \frac{z}{1 + \frac{z}{r}} \quad \text{or} \quad \Phi = \frac{g_\varphi}{10} z (1 - 1.57 \cdot 10^{-7} z). \tag{8}$$

The correction given by the quadratic term $1.57 \cdot 10^{-7} z^2 \frac{g_\varphi}{10}$ is very small. Thus, at an altitude of z = 1000 m it amounts to merely 16 berks. In this way the following approximation may be accepted:

$$\Phi = 0.98 z \text{ berk} \tag{9}$$

or conversely

$$z = 1.02 \Phi \text{ meter} \tag{10}$$

By substituting (10) in the quadratic term of equation (8) and solving it with respect to z we obtain a more exact equation:

$$z = \frac{10}{g_\varphi} \cdot \Phi (1 + 1.60 \cdot 10^{-7} \Phi). \tag{11}$$

However in the majority of cases use can be made of the approximate equations (9) and (10) for the conversion of dynamic into geometric altitudes and vice versa. Tables, calculated on the basis of the exact equations (8) and (11) were drawn up by Bjerknes, which make it possible to convert dynamic altitudes quickly and exactly into geometric altitudes, and vice versa. We should notice, that 1 berk equals approximately 1.02 meter, and 1 meter - 0.98 berk. Therefore, the number of berks which determine the dynamic altitude of a point is less by approximately 2% than the number of meters which determine its geometric altitude.

The practical advantage of dynamic as compared with geometric altitudes lies in the fact that it appears possible to relate the variation of gravity to the variation of the dynamic altitude, which measures the elevation of the

material particle. Many relations which contain dynamic altitudes therefore have a simpler form than similar relations containing geometric altitudes.

§ 9. The Pressure Field and the Field of Gravity

We will now study the simultaneous action of pressure and gravity on an air parcel. The fields of these forces can be best visualized by the isobaric and equipotential surfaces. We will make use of the equation of equilibrium in relative coordinates, in order to clarify the mutual position of these surfaces. The relative acceleration of the air parcel must equal zero in the case of relative equilibrium,

$$\boldsymbol{b} + \boldsymbol{g} = 0. \tag{1}$$

But

$$\boldsymbol{b} = -v\nabla p; \quad \boldsymbol{g} = -\nabla\Phi,$$

hence

$$\nabla\Phi = -v\nabla p \tag{2}$$

or

$$\nabla p = -\rho\nabla\Phi. \tag{3}$$

Equation (3) is called the fundamental equation of atmospheric statics.

The vectorial equations (2) and (3) characterize graphically the fields of density, pressure and geopotential for a system in equilibrium.

Each of these equations expresses a formal equivalence of the solenoid and potential vectors. In so far as such an equivalence exists, α must be a function of p, and ρ —a function of Φ.

If we denote the thickness of the isobaric and of the equipotential unit layers by h_p and h_Φ, we obtain:

$$|\nabla p| = \frac{1}{h_p} \text{ and } |\nabla\Phi| = \frac{1}{h_\Phi} \tag{4}$$

and

$$v = \frac{|\nabla\Phi|}{|\nabla p|} = \frac{h_p}{h_\Phi}, \tag{5}$$

$$\rho = \frac{|\nabla p|}{|\nabla\Phi|} = \frac{h_\Phi}{h_p}. \tag{6}$$

Thus, each isobaric unit layer contains, in an equilibrium system, v equipotential layers, and vice versa, each equipotential layer contains ρ isobaric unit layers.

We may illustrate the obtained result by some examples, employing real units.

The unit of pressure in the CGS system is the bar, as was stated before

which equals 1 dyne per cm^2. The pressure at sea level is approximately equal to one million bars, or to one meteorological bar. From this fundamental unit we obtain the units used in practice– decibar, centibar, millibar. The centibar is the MTS unit of pressure. The deciberk is the MTS unit of geopotential.

Therefore it can be said, using the equations (5) and (6) and applying the MTS system, that the thickness of an isobaric layer, corresponding to a pressure difference of 1 centibar, is equal to v deciberks. Conversely, a layer one deciberk thick contains ρ isobaric layers. The lower layer, at a normal specific volume of air has a dynamic thickness of 800 deciberk, or 80 berk (Figure 47)

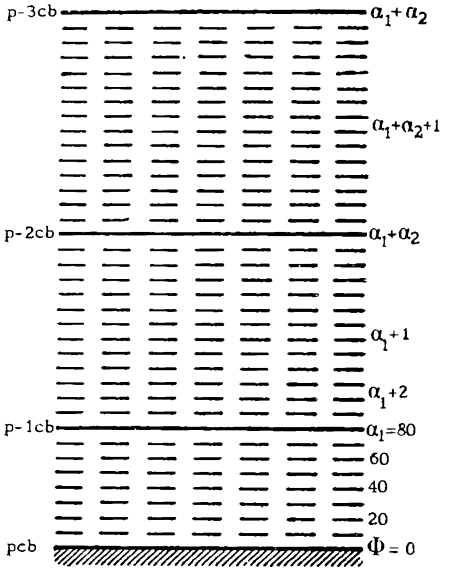

Figure 47. Each isobaric unit layer contains, in an equilibrium system, v equipotential layers

"Dynamic barometric step" is the term given to the dynamic altitude change (which may be an ascent or a descent) that gives rise to a change of one unit of pressure. In other words, the dynamic barometric step is the dynamic thickness of a unit isobaric layer. It can be seen from equation (5) that the barometric step is equal to the specific volume of air.

If the equation of state

$$f(\rho, p, T) = 0$$

does not contain other spatially inconstant parameters of state, the invariability of p and ρ on a given surface also entails the invariability of T. Consequently the isothermic surfaces coincide with two other isosurfaces in the case of equilibrium.

Therefore it can be said that the state of equilibrium is characterized by a complete coincidence of the isobaric, isosteric, isopycnic and isothermic surfaces, all these three sets of surfaces being horizontal at all points of the atmosphere.

The atmosphere is called barotropic (Figure 32), when its state is such that the isosteric surfaces coincide with the isobaric ones. Such a state does not usually occur in the atmosphere, and the isosteric surfaces intersect the isobaric. This atmosphere is called baroclinic (Figure 33).

Using this terminology one can say that the atmosphere is barotropic when it is in a state of equilibrium. However, this condition is necessary but not sufficient. We shall see further that some motions are also characterized by a coincidence of the isobaric and isothermic surfaces, but all these surfaces are horizontal in an equilibrium system.

§ 10. The Barometric Equation for a Polytropic Atmosphere

The differential relationship between the pressure and the geopotnetial was examined in the previous section, where it was found that the isobaric and the isosteric surfaces are horizontal when in a state of equilibrium. The distributions of the pressure and density in a resting atmosphere will now be examined. The distributions of pressure and density in the entire atmosphere are known, if their distribution along the vertical is determined; the latter is obtained by integrating the hydrostatic equation of the atmosphere which has the form:

$$dp = -g\rho\, dz. \tag{1}$$

However we will first consider the pressure difference:

$$p - p_0 = \int_{z_0}^{z} g\rho\, dz. \tag{2}$$

The expression $g\rho\, dz$, after the integral sign is the weight of the elementary air column of unit cross section and height dz. Thus, it can be seen from equation (2) that the pressure difference between two levels is equal to the weight of the air column of unit cross section, enclosed between these levels. From this a very important conclusion, which will be used repeatedly further on, can be drawn: the pressure at any level of the atmosphere is equal to the weight of the air column, of a unit cross section, which extends from this level up to the upper limit of the atmosphere.

The fundamental equation of atmospheric statics contains three variables p, ρ, Φ or p, ρ, z. It is obviously impossible to integrate such an equation in its general form. A particular case will therefore be examined. The daily aerological ascents determine simultaneously the values of pressure, temperature and humidity. Since the value of ρ is not measured, it is more expedient to substitute for it measureable magnitudes obtained from Clapeyron's equation for moist air:

$$\rho = \frac{p}{RT_v}. \tag{3}$$

The hydrostatic equation will then have the form:

$$\frac{dp}{p} = -\frac{g}{RT_v}\, dz, \tag{4}$$

from which the barometric equation in its general form is obtained by integration

$$z - z_0 = -\frac{R}{g}\int_{p_0}^{p} T_v \frac{dp}{p}. \tag{5}$$

The empirical data obtained by aerological soundings show that the changes of the (virtual) temperature with the height is so erratic that it is impossible to express this variation by any elementary function of altitude and pressure. The calculation of the integral (5) is therefore usually carried out by graphical methods. The same empirical material shows, however, that the atmosphere can always be divided into layers, the temperature of which depends linearly on the height. The barometric equation for the atmosphere with a linear distribution of temperature will first be examined, before we go on to explain the calculation of the integral (5) by graphic methods. Only the virtual temperatures of the moist air will be considered here, because the problems of moist air statics are solved in the same way as those for dry air, when the virtual temperature is introduced. The subscript v of the virtual temperature will therefore be omitted, for brevity, and the temperature will be simply designated by T.

Thus, we imagine the atmosphere to have a linear temperature distribution in altitude

$$T = T_0 - \gamma z, \tag{6}$$

where $\gamma = -\frac{\partial T}{\partial z}$ is the lapse rate, i. e., the decrease of temperature for a unit of height

$$[\gamma] = [\theta L^{-1}].$$

By substituting equation (6) in the fundamental hydrostatic equation (4), and by separating the variables, we obtain:

$$\frac{dz}{T_0 - \gamma z} = -\frac{R}{g}\frac{dp}{p},$$

from which, by integrating from p_0 to p and from 0 to z, we find

$$\ln \frac{T_0 - \gamma z}{T_0} = \frac{R\gamma}{g} \ln \frac{p}{p_0},$$

or, transforming to exponential form:

$$\frac{T_0 - \gamma z}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{R\gamma}{g}}. \tag{7}$$

The relation (7) between altitude and pressure is known as the barometric formula for a polytropic atmosphere, because the dependence of T on p expressed by this relation, is completely similar to the polytropic equation if the index of the polytrope is related to γ by the equation

$$k = \frac{g}{g - R\gamma}. \tag{8}$$

The state of the polytropic atmosphere is thus completely defined either by one parameter of state, or by the altitude, and consequently also by the value of the potential at the point under consideration if, apart from that, the index k of the polytrope is known (which is equivalent to the assignment of a temperature gradient) and likewise the state of the air on the initial (zero) level. Table XII summarizes the formulas which determine the state of a polytropic atmosphere:

Table 12

Formulas which determine the state of a polytropic atmosphere

	$f(p)$	$f(z)$	$f(T)$	$f(\rho)$
$p=$	p	$p_0\left(1-\frac{\gamma z}{T_0}\right)^{\frac{g}{R\gamma}}$	$p_0\left(\frac{T}{T_0}\right)^{\frac{g}{R\gamma}}$	$p_0\left(\frac{\rho}{\rho_0}\right)^{\frac{g}{g-R\gamma}}$
$z=$	$\frac{T_0}{\gamma}\left[1-\left(\frac{p}{p_0}\right)^{\frac{R\gamma}{g}}\right]$	z	$\frac{T_0-T}{\gamma}$	$\frac{T_0}{\gamma}\left[1-\left(\frac{\rho}{\rho_0}\right)^{\frac{R\gamma}{g-R\gamma}}\right]$
$T=$	$T_0\left(\frac{p}{p_0}\right)^{\frac{R\gamma}{g}}$	$T_0-\gamma z$	T	$T_0\left(\frac{\rho}{\rho_0}\right)^{\frac{R\gamma}{g-R\gamma}}$
$\rho=$	$\rho_0\left(\frac{p}{p_0}\right)^{\frac{g-R\gamma}{g}}$	$\rho_0\left(1-\frac{\gamma z}{T_0}\right)^{\frac{g-R\gamma}{R\gamma}}$	$\rho_0\left(\frac{T}{T_0}\right)^{\frac{g-R\gamma}{R\gamma}}$	ρ

Since the temperature of the polytropic atmosphere is a linearly decreasing function of the altitude, it reaches the absolute zero T = 0°K at a determined ultimate altitude, i.e., the limit altitude of the polytropic atmosphere:

$$z=\frac{T_0}{\gamma}. \tag{9}$$

If $\gamma>0$, the substitution of the value given by equation (9) in the formulas of Table 12 gives

$$p=0; \quad \rho=0.$$

Thus we find that the polytropic atmosphere must terminate at the altitude determined by equation (9), assuming that the gas laws retain their validity down to the absolute zero.

§ 11. The Homogeneous, Adiabatic and Isothermic Atmospheres

Some most important instances of the polytropic atmosphere will now be examined. An atmosphere is called homogeneous, if its density does not vary with the altitude, i.e., ρ = const. It follows from Table 12 that for a homogeneous atmosphere

$$g-R\gamma=0 \quad \text{or} \quad \gamma=\frac{g}{R}=0.0342\ ^\circ/_{м}, \tag{1}$$

and the formulas of Table 12 assume a simpler form:

$$\rho=\rho_0=\text{const.}, \quad T=T_0-0.0342\,z; \quad p=p_0-g\rho_0. \tag{2}$$

The [limit] altitude of a homogenous atmosphere at $T_0 = 273^{\circ}K$ will be

$$H = 7991\ м. \tag{3}$$

Such a significant drop of temperature with height can be only encountered under exceptional conditions, for example, over a chimney during firing, or over the crater of an active volcano. A considerable drop of temperature can also take place in a very thin layer of air over an intensely heated soil surface. The equilibrium of the atmosphere will be unstable under these conditions. The density will increase with the height, and the equilibrium will become more unstable, if $\gamma > g/R$. The concept of the homogeneous atmosphere is very helpful in theoretical studies.

An atmosphere with a dry-adiabatic lapse rate

$$\gamma = \frac{g}{c_{pd}} = 0.00985\ ^{\circ}/м. \tag{4}$$

is called an adiabatic atmosphere.

The fundamental formulas of Table 12 have the form

$$T = T_0 - \frac{g}{c_{pd}} z; \quad \frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{Rd}{c_{pd}}}; \quad p = p_0 \left(1 - \frac{g}{c_{pd}} \frac{z}{T_0}\right)^{\frac{c_{pd}}{Rd}}. \tag{5}$$

for the adiabatic atmosphere.

The [limit] altitude of the adiabatic atmosphere is

$$H = \frac{c_{pd} \cdot T_0}{g};$$

at $T = 273^{\circ}$ $H \approx 28{,}000$ meters.

The overlapping of one level on another is determined approximately by the equation $z_2 - z_1 \approx 100(T_1 - T_2)$, if the temperatures T_1 and T_2 of the adiabatic atmosphere are given for two levels.

The atmosphere which has a constant temperature T = const., and hence $\gamma = 0$, is called an isothermic atmosphere. All the formulas of Table 12 become indefinite for the isothermic atmosphere. By solving these indeterminate equations, say, for the barometric formula, according to the well known L'Hospital's rule, we obtain

$$z = \frac{RT}{g} \ln \frac{p_0}{p}. \tag{6}$$

It is obvious that this formula can be obtained more easily by a direct calculation of the integral in equation (5) § 10 for the isothermic atmosphere. Thus, the state of the isothermic atmosphere is well defined by the relations given in Table 13.

It can be seen from Table 13 that the [limit] altitude of the isothermic atmosphere is infinite, since both the pressure and the density decrease exponentially, asymptotically approaching zero.

Table 13

Equations determining the state of an isothermic atmosphere

	$f(p)$	$f(z)$	$f(\Phi)$	$f(\rho)$
$p=$	p	$p_0 e^{-\frac{gz}{RT}}$	$p_0 e^{-\frac{\Phi}{RT}}$	$\frac{p_0}{\rho_0}\cdot\rho$
$z=$	$\frac{RT}{g}\ln\frac{p_0}{p}$	z	$\frac{\Phi}{g}$	$\frac{RT}{g}\ln\frac{\rho_0}{\rho}$
$\Phi=$	$RT\ln\frac{p_0}{p}$	gz	Φ	$RT\ln\frac{\rho_0}{\rho}$
$\rho=$	$\frac{\rho_0}{p_0}\cdot p$	$\rho_0 e^{-\frac{gz}{RT}}$	$\rho_0 e^{-\frac{\Phi}{RT}}$	ρ

§ 12. The Barometric Mean Temperature

The foregoing integration of the fundamental hydrostatic equation for the polytropic atmosphere, is chiefly of theoretical importance. In practice the temperature is usually obtained in the form of a very complicated function of the pressure, as a result of aerological soundings. It may, therefore, be too tedious to divide the atmosphere each time into polytropic layers, to calculate their lapse rates and then to determine the relation between the pressure and the altitude, according to the polytropic equation. The calculations become more complicated, the oftener the lapse rate undergoes a change. The barometric equations for the isothermic atmosphere, on the other hand, are very simple and convenient for calculations. The mean barometric temperature is therefore introduced, in order to enable the use of the barometric equation for the isothermic atmosphere in the case, when the temperature is not constant along the vertical: this mean barometric temperature is determined in the following way. The temperature of the fictitious isothermic layer, at which the thickness of the layer and the difference between the pressures at its boundaries equal, respectively, these values for the real atmosphere, is called the barometric mean temperature:

$$p = p_0 e^{-\frac{g}{R}\int_0^z \frac{dz}{T}} = p_0 e^{-\frac{gz}{RT_m}},$$

where T_m is the mean barometric temperature. Hence

$$T_m = \frac{z}{\int_0^z \frac{dz}{T}}. \tag{1}$$

Let us show that the barometric mean temperature T_m is not identical with the arithmetic mean temperature T'_m, even in the case of a strictly linear variation of the temperature T_m:

$$T'_m = \frac{T + T_0}{2} = T_0 - \frac{\gamma z}{2}.$$

Indeed

$$T_m = \frac{z}{\int_0^z \frac{dz}{T}} = \frac{-\gamma z}{\ln\left(1-\frac{\gamma z}{T_0}\right)}. \qquad (2)$$

In order to compare T_m and T'_m let us express them by means of infinite series:

$$\left.\begin{aligned} T_m &= \frac{T_0}{1+\frac{1}{2}\left(\frac{\gamma z}{T_0}\right)+\frac{1}{3}\left(\frac{\gamma z}{T_0}\right)^2+\frac{1}{4}\left(\gamma\frac{z}{T_0}\right)^3+\dots}, \\ T'_m &= \frac{T_0}{1+\frac{1}{2}\left(\frac{\gamma z}{T_0}\right)+\frac{1}{4}\left(\frac{\gamma z}{T_0}\right)^2+\frac{1}{8}\left(\frac{\gamma z}{T_0}\right)^3+\dots}. \end{aligned}\right\} \qquad (3)$$

Since the denominator of the barometric mean temperature is greater than that of the mean arithmetic temperature, we have

$$T_m < T'_m. \qquad (4)$$

The arithmetic mean temperature will be equal to the barometric mean temperature only in the case, when the terms of the second and higher orders can be neglected. It is, however, not always possible to neglect these terms. Assuming T = 273° and allowing different values for the lapse rate $\gamma = -\frac{\partial T}{\partial z}$ we obtain:

$\gamma z =$	100	90	80	70	60	50	40	30	20	10°
$T_m =$	219.2	225.0	230.7	236.3	241.8	247.2	252.5	257.7	262.9	268.0
$T'_m =$	223.0	228.0	233.0	238.0	243.0	248.0	253.0	258.0	263.0	268.0
$T'_m - T_m =$	3.8	3.0	2.3	1.7	1.2	0.8	0.5	0.3	0.1	0.0

The error in determining the barometric mean temperature may lead to overestimation of the altitude, if it is to be determined according to the measured pressure, and to underestimated values, if the pressure is determined according to the barometric equation. Calculation has shown that the error due to the pressure being determined by the arithmetic mean temperature (instead of the barometric mean temperature) exceeds the usual accuracy of measurements which is ± 0.5 mb, for layers, having a thickness less than 5,000 meters and only at very high atmospheric lapse rates. The error is appreciable when the layer is higher, and it is therefore more expedient to use polytropic equations for altitudes above 5,000 meters, the use of this equation becoming obligatory for altitudes of about 10,000 meters.

The introduction of the arithmetic mean temperature into the barometric equation, instead of the barometric mean temperature, can lead to appreciable errors even for relatively thin layers if the temperature inside these layers varies nonlinearly.

§ 13. The Determination of the Absolute and Relative Geopotentials of the Isobaric Surface

Barometric equations are used to solve a great number of important practical problems. We will examine one of the most important – the determination of the absolute and of the relative geopotentials of an isobaric surface. The results of aerological ascents, which give the simultaneous values of the pressure the temperature and humidity at certain points of the atmosphere, form a basis for these computations, which are necessary for plotting pressure, contour maps. The absolute geopotential Φ_p of the isobaric surface p = const, is known as its dynamic altitute above sea level. By relative geopotential $\Phi^p_{p_0}$ of the isobaric surface p = const. we designate the geopotential (the dynamic altitude) over any other surface p_0 = const.

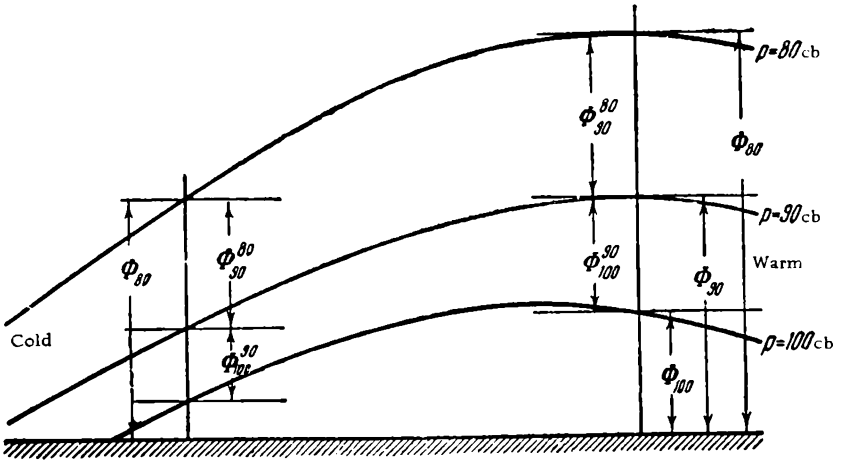

Figure 48. The absolute and the relative geopotentials of the isobaric surfaces

Let us explain on what the absolute geopotential of a given isobaric surface Φ_p depends. By introducing the geopotential in the barometric equation and solving for Φ_p, we obtain

$$\Phi_p = RT(\ln p_0 - \ln p), \tag{1}$$

where T is the barometric mean virtual temperature of the layer extending from sea level up to the isobaric surface. The absolute geopotential of the isobaric surface thus depends on T and p_0, standing in direct proportion to them, i. e., the absolute geopotential increases with the increase of the pressure p_0 at sea level, and with the increase of the mean virtual temperature T. The increase of T and p will, however, differently influence the absolute geopotential Φ_p of different isobaric surfaces. Let the absolute geopotential vary together with the variation of T and p_0. Then

$$\Delta\Phi_p = RT\frac{\Delta p_0}{p_0} + R\ln\frac{p_0}{p}\Delta T. \tag{2}$$

the absolute geopotential of all the isobaric surfaces will vary equally as a result of the variation of the pressure at sea level, if the mean temperature remains constant, whereby at

$$\Delta p_0 = 1 \text{ mb}, \quad p_0 = 1000 \text{ mb}, \quad T = 273^\circ \text{K},$$
$$\Delta\Phi_p = 8 \text{ berk.}$$

The absolute geopotential of all the isobaric surfaces can change under any other conditions which may be realized at sea level, within the limits from 6 to 9 berks, when the ground-level pressure varies by 1 mb.

The absolute geopotential of different isobaric surfaces varies differently - depending on the altitude of the isobaric surfaces - if the pressure at sea level remains constant and only the mean temperature of the layer changes. Thereby the geopotential of higher isobaric surfaces varies more than does the geopotential of lower surfaces. We should notice that the value of the mean temperature does not in itself influence the change of the absolute geopotential. The equation

$$\Delta\Phi_p = R \ln \frac{p_0}{p} \Delta T \qquad (3)$$

gives the changes of the absolute geopotential of the isobaric surface corresponding to the variation of the mean temperature of the layer. It has been proved by calculations that "the thermic contribution" to the absolute geopotential of the isobaric surfaces up to p = 800 mb, at $\Delta T = 1^\circ$ is smaller, compared to the "dynamic contribution" at $\Delta p_0 = 1$ mb, if the pressure at sea level is about 1,000 mb. The thermic contribution exceeds the dynamic one, beginning with p = 700 mb and higher and thus the higher the isobaric surface is, the greater the geopotential increase (Figure 49). It can be seen from equation (3) that an error of 1°C in the determination of T for very thick layers, can cause a major error in the determination of Φ_p. The necessity for a particularly careful determination of the mean virtual temperature of the layer is thus made evident. However, the mean temperature of a very thick layer cannot be determined with sufficient accuracy, in spite of all precautions. The subdivision of this layer into thinner layers by intermediate isobaric surfaces, with the implied determination of the relative geopotentials of these isobaric surfaces and their summation is, therefore, the only method which enables to enhance the accuracy of determination of the absolute geopotential.

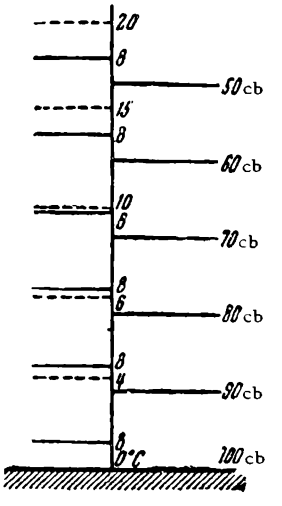

Figure 49. All the isobaric surfaces rise equally when the pressure increases by 1 mb. The higher the isobaric surface the more it rises, when the mean temperature of the layer increases by 1°C

The relative geopotential $\Phi^{p}_{p_0}$ is determined by the equation

$$\Phi^{p}_{p_0} = RT \ln \frac{p_0}{p}. \tag{4}$$

The relative geopotential $\Phi^{p}_{p_0}$ of the isobaric surface p=const. with respect to the surface p_0 = const. therefore depends only on the mean temperature of the isobaric layer, namely, the higher the temperature, the greater is the relative geopotential, and vice versa.

§ 14. Analytical Computation of the Geopotential of an Isobaric Surface

Let us explain V. Bjerknes's method for the computation of the geopotential of the isobaric surfaces.

The surfaces corresponding to the main pressure, i.e., to the pressure expressed in whole decibars (in whole hundred of millibars): 1,000, 900, 800, 700, 600 mb etc., are called by Bjerknes the main isobaric surfaces. The layer of the atmosphere enclosed between the main isobaric surfaces is called a complete main layer. The relative potential of two adjacent main isobaric surfaces is determined by the equation

$$\Phi^{p_k}_{p_i} = RT \ln \frac{p_i}{p_k}. \tag{1}$$

V. Bjerknes drew up Table 9M with the help of equation (1); from this table the relative geopotential of the main isobaric surfaces or the dynamic thickness of the main layers, depending on the mean temperature T of the layer, for the values p_i/p_k = 10/9, 9/8, ..., 2/1 and for t from -109°C to + 49° C. can be obtained. We can at once find the dynamic thickness of the layer in berks, when T is obtained beforehand (by means of any aerologic diagram) for the mean isobaric layer, determined from Table 9M. However, the pressure over the earth's surface naturally does not belong to a main layer. On the other hand, it often appears necessary to determine the geopotential not of the main isobaric surfaces, but of other, intermediate surfaces. The layer included between the main isobaric surface and any other isobaric surfaces, separated by whole hundreds of millibars from the main surface on both its sides, is called the partial mean layer.

The construction of other tables by V. Bjerknes becomes clear after the following transformations of the barometric equation for the geopotential. By introducing the temperature measured in degrees Celsius, instead of the absolute temperature, we obtain

$$\Phi^{p}_{p_i} = R\,273 \ln \frac{p_i}{p} + R \cdot 273 \ln \frac{p_i}{p} \cdot \frac{t}{273} \tag{2}$$

where p_i is the main pressure, and p - an arbitrary pressure. The first term is the relative geopotential $\Phi^{p}_{p_i}$ at 0° C:

$$(\Phi^{p}_{p_i})_0 = R\,273 \ln \frac{p_i}{p}. \tag{3}$$

Equation (3) can easily be tabulated, since the geopotential it contains depends only on p. Bjerknes drew Table 10M by means of equation (3), by assuming the

main pressures to equal 1, 000, 900 ... 100 mb and by setting the values of p at one millibar intervals, beginning with 100 mb below p_i, up to 100 mb above p_i. Table 10 M gives the dynamic thickness of the layer in berks for each pair of values of p_i and p.

Table 11 M drawn up by Bjerknes, is computed by means of equation (3), p_i being assumed to be equal to 1, 100 mb in order to avoid negative altitudes. The pressure p has all values from 1, 099 mb to 1 mb. Table 11 M thus gives the dynamic thickness Φ^{p}_{1100} of the layer from 1,100 to p at t = 0°C. By using this table for two levels p and p', we can obtain, by subtraction, the dynamic thickness between the levels p and p' at 0° C. Table 11 M is used for the computation of the heights of aerologic ascents. The second term of equation (2) gives the correction for the temperature

$$\Delta\Phi = (\Phi^{p}_{pi})_0 \frac{t}{273} \tag{4}$$

and can also be easily tabulated. It is to be noticed that the sign of the correction term always coincides with that of t. Table 12 M of Bjerknes gives the temperature correction for all the altitudes, given in Tables 10 M and 11 M.

Thus, only three tables are necessary to calculate the geopotential, according to Bjerknes' method: one (9 M) for the determination of the relative geopotential between two main isobaric surfaces, the second (10 M) – for the determination of the relative geopotential between the arbitrary and the main isobaric surfaces at 0° C, and the third (12 M) – for the temperature correction of the data determined according to Table 10 M. Bjerknes worked out a series of auxiliary tables to calculate the virtual temperature. It is, however, more expedient to carry out these computations graphically, by means of some diagram, an emagram for instance, as was shown above.

Since the greatest error in determining the absolute and the relative geopotentials is caused by a wrong determination of the mean temperature of the layer, we may, in order to avoid this error, subdivide the layer under consideration into thinner layers, determining first the dynamic thickness of each layer separately, and then adding up the results obtained. The determination of the relative geopotential of the main isobaric surface will then amount to the determination of the thickness of the main isobaric layers, using Table 9 M, and adding up the results.

The determination of the absolute geopotential of some main isobaric surfaces necessitates two operations:

1. Determination of the dynamic height of a 1, 000 millibar surface at sea level. This is done by using Table 10 M and the correction Table 12 M.

2. Determination of the dynamic thickness of each main layer, which is read from Table 9 M.

The obtained results are added together. The sum gives the absolute geopotential of the main isobaric surface. It is sometimes necessary to determine the absolute geopotential of some arbitrary (not main) isobaric surface, i. e., the dynamic height of some point in the atmosphere, for example, the height of ascent of radiosonde apparatus; in this case a third operation is added to the above-mentioned two operations: the determination of the dynamic height of the point situated over the upper main isobaric surface. This operation is carried out by using Table 10 M and the correction Table 12 M.

§ 15. Graphic Representation and Computation of Dynamic Altitudes

It is particularly convenient to use graphs for the computation of dynamic altitudes. For this purpose many special diagrams have been plotted. However, there is no necessity for special diagrams, since all these computations can be carried out by means of any of the energetic diagrams considered previously. The results of aerologic soundings are used for the calculation of altitudes, namely, the results of the analysis of meteorograms and radio-soundings.

The results of radio-sounding give a series of simultaneous values of p, T, r, by which a series of simultaneous values of the pressure and of the virtual temperature T_v can be found by means of some aerologic diagram. These values are plotted on the aerologic diagram as points, which are afterwards connected by broken lines. The broken line thus obtained represents the state of the vertical column of the atmosphere. This curve is called the stratification curve.

The stratification curve should not be confused with the curves, plotted on the aerologic diagrams and corresponding to the temporal change of state of any individual air parcel during any particular process.

Thus, let us assume that a stratification curve is given. The barometric equation in the form:

$$\Phi_a^b = -R\int_a^b T d\ln p. \tag{1}$$

forms a theoretical base for plotting and for calculating the dynamic altitude on the aerologic diagram. This expression makes possible a simple graphical interpretation of the emagram. Let us imagine (Figure 50) that the stratification curve is plotted on the emagram from the level a to the level b. Let F represent the area limited by the stratification curve, by the isobars p_a and p_b and by the isotherm T = 0° K. Then

$$F = \int_0^l T d(-\ln p). \tag{2}$$

Combining equations (1) and (2) we obtain

$$\Phi_a^b = R \cdot F. \tag{3}$$

Thus, the dynamic thickness of the atmospheric layer Φ_a^b is proportional to the area F. Since it was shown in § 17 Chapter II that the emagram is an energy diagram, it is obvious that this conclusion will be correct in regard to any energy diagram, for example, for the tephigram and aerogram. But this will not apply to the nonenergetic diagrams, e.g., the Stüve diagram.

Thus, the following rule can be formulated: the dynamic thickness of the isobaric layer confined between the isobars p_a and p_b is equal (up to the constant factor, which depends on the scale of the diagram) to the area on the energetic diagram which is limited by the stratification curve, by the isobars p_a and p_b, and by the isotherm T = 0° K. Figure 50 shows the dependence of the

dynamic thickness of the layer on the temperature: the hotter the layer, the greater is the area limited by the stratification curve, and the greater the dynamic thickness of the layer. This result is quite comprehensible from the physical point of view. Warm air is lighter than cold.

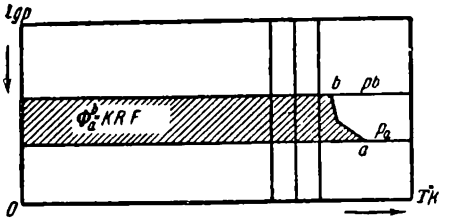

Figure 50. The dynamic altitude of an isobaric layer is represented on the emagram by an area of a region limited by the stratification curve, by the isotherm T = 0°K, and by the isobars p_a and p_b

Since the pressure difference at the limits of the layer represents the weight of a unit air column, then, for the same weight, the hotter air layer must be thicker than the cold one. The dynamic thickness of the layer is represented on the tephigram and on the emagram by an area which is limited by a stratification curve, by the isobars p_a and p_b, and by the isotherm T = 0°K.

The dynamic altitude can be given a different graphic representation on the tephigram and on the aerogram. For this purpose we will examine in what way the geopotential position is related to the value of the entropy.

Since $d\varphi = c_p \frac{dT}{T} - R\frac{dp}{p}$ и $dp = -\rho\, d\Phi$,

we have

$$d\varphi = c_p \frac{dT}{T} + \frac{d\Phi}{T}; \quad d\Phi = T d\varphi + c_p dT, \tag{4}$$

from which

$$\Phi = \int T d\varphi + c_p T + \text{const.} \quad \text{or} \quad \Phi_b - \Phi_a = \int_a^b T d\varphi + c_p (T_b - T_a). \tag{5}$$

The term $\int_a^b T d\varphi$ in Figure 51 is represented by the area limited by the stratification curve, by the zero isotherm, and by two isentropic curves φ_a and φ_b. The term $c_p(T_b - T_a)$ expresses the heat, which must be used up, in order to raise the temperature from T_b to T_a at a constant pressure; it is therefore equal to $\int_a^c T d\varphi$, i.e., to the area limited by the isobar p_c, by the isentropic curves φ_a and φ_b as well as by the zero isotherm. Thus, the dynamic thickness of the isobaric layer is proportional to the area limited by the stratification curve, by the isobar p_c, by the isentropic curves φ_b and φ_c, and by the zero isotherm.

However, the calculation, either on the emagram or on the tephigram, of the area which corresponds to the dynamic thickness of the layer, is impracticable because such a calculation requires the existence on the diagram of the isotherm T = 0°K, which necessitates an undue increase in the dimensions of the diagram.

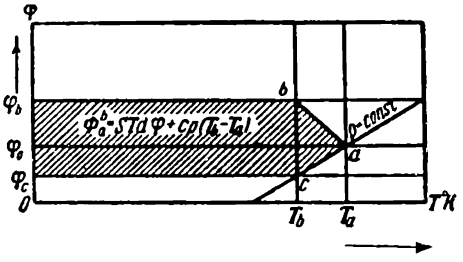

Figure 51. The dynamic altitude Φ_a^b of the isobaric layer is represented on the tephigram by the area limited by the stratification curve, by the isobar p_a, by the isentropic curves φ_b, φ_c and by the zero isotherm

The aerogram does not suffer from this disadvantage. In fact, the area on the aerogram which is limited by two isobars p_a and p_b, and by two isotherms T and T = 0°K, is proportional to the distance between the isobars, measured along the isotherm T.

Let us prove that this is really so. It follows from the structure of the aerogram that

$$x = k_2 \ln \frac{T}{T_0} = k_2 \ln T + k_3; \qquad y = k_1 \cdot T \ln \frac{p_0}{p},$$

where k_1 and k_2 are the positive constants, depending on the scale of the diagram, and $k_3 = -k_2 \ln T_0$.

From these equations we obtain:

$$T = e^{\frac{x - k_3}{k_2}}; \quad y = k_1 \ln \frac{p_0}{p} \cdot e^{\frac{x - k_3}{k_2}}.$$

The area F between the isobars p_a, p_b, from T = 0°K to T, is determined by the equation

$$F = \int_{-\infty}^{x} y\,dx = k_1 \ln \frac{p_0}{p} \int_{-\infty}^{x} e^{\frac{x - k_3}{k_2}} dx = k_1 k_2 \ln \frac{p_0}{p} e^{\frac{x - k_3}{k_2}} = k_2 y.$$

It can be seen from this equation that the area F bounded by the isobars p_a and p_b, measured from an infinitely distant line x = - ∞, which corresponds to the zero isotherm, and the isotherm T , is always proportional to the distance between the isobars, measured along the isotherm. In order to determine the dynamic thickness of the isobaric layer, it is therefore necessary to plot an isotherm in such a way that the areas f_1 and f_2 , enclosed between the stratification curve and this isotherm, are of equal magnitude.

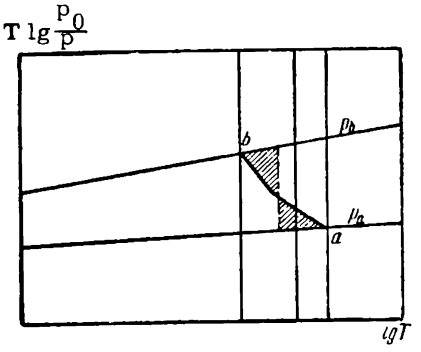

Figure 52. The dynamic thickness of the isobaric layer, determined by means of an aerogram

Such an isotherm can be approximately plotted in the majority of cases. The dynamic thickness of the isobaric layer will then be represented by the length of the segment of the isotherm T, enclosed between the isobars p_a and p_b. It follows from this that the aerogram is especially convenient for a quick graphic determination of altitudes. On other energetic diagrams only the mean temperature of the layer is represented graphically, whereas the corresponding dynamic thickness of the layer is determined according to the tables of Bjerknes.

§ 16. The Relation between the Altitude of the Tropopause and the Temperature of the Lower Troposphere

It was proved shortly after the discovery of the stratosphere that the height of the lower limit of the stratosphere (the tropopause) is not constant, but changes both in time, and from place to place. Numerous aerologic ascents have shown that the tropopause is higher in summer than in winter, and it is higher in the tropics than in polar latitudes. This fact, as we will soon show, agrees with the barometric equation.

Let us imagine an air layer, which extends from the terrestrial surface up to an altitude of about 20 km (Figure 53). Let us suppose that the pressures at the upper and the lower boundaries of the layer are given and have constant values p_o and p_h. The pressure difference $p_o - p_h$ will give us the weight of a unit air column of the layer under consideration. The weight of this column can remain constant, although the distribution of the temperature may be different in it, i. e., various altitude distributions of temperature correspond to one value of pressure difference. The indefiniteness of the distribution of temperature will be maintained even when sufficiently strict conditions for temperature variations are laid down, i. e., when the troposphere is assumed to be polytropic, and the stratosphere – isothermic. The indefiniteness of the temperature distribution will lead to an indefiniteness in the position of the tropopause, which is a function of the temperature of the lower atmospheric layers.

We shall find the relation between the altitude of the tropopause and the temperature of the lower troposphere, assuming p_o, p_h, h, γ to be given, by basing ourselves on the barometric equation. Since the thickness of the troposphere is too great to permit the application of the barometric equation for an isothermic atmosphere to the whole tropospheric layer – substituting the barometric mean temperature by the arithmetic mean one, we resort to the barometric equation

for the polytropic atmosphere in the following form:

$$\ln \frac{p_0}{p_\zeta} = \frac{g}{R\gamma} \ln \frac{T_0}{T_\zeta}, \tag{1}$$

where ζ is the height of the tropopause, p_ζ is the pressure at the tropopause, $T_\zeta = T_0 - \gamma_\zeta$ the temperature at the tropopause, γ -the lapse rate.

The barometric equation for an isothermic atmosphere

$$\ln \frac{p_\zeta}{p_h} = \frac{g(h-\zeta)}{RT_\zeta}. \tag{2}$$

is correct for the lower layer of the stratosphere.

We shall obtain the sought relation between the altitude of the tropopause and the temperature of the lower troposphere, by eliminating the auxiliary magnitude p_ζ between equations (1) and (2)

$$\frac{1}{\gamma} \ln \frac{T_0}{T_0 - \gamma\zeta} + \frac{h-\zeta}{T_0 - \gamma\zeta} = C, \text{ where } C = \frac{R}{g} \ln \frac{p_0}{p_h}. \tag{3}$$

Equation (3) is solved by the method of successive approximations, or graphically.

The following values of the tropopause altitude at a given temperature of the lower layer of the troposphere are obtained, assuming p_o = 1013 mb, p_h = 53,4 mb, h = 20 km, γ = 0.6°/100 m, and are given in Table 14.

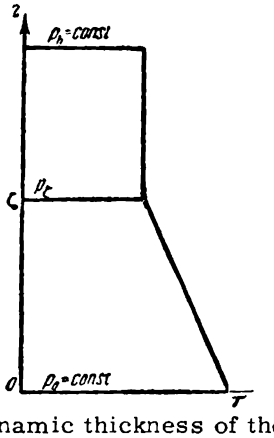

Figure 52. The dynamic thickness of the isobaric layer, determined by means of an aerogram

In the third line of the table are given the values of the stratospheric temperature, corresponding to the obtained tropopause altitude.

The stratification curves plotted with help of the data in Table 14, are represented in Figure 54. Both curves and the data of the table show that, in spite of a great variation of the lower troposphere, the pressure over the earth's surface usually varies little, because the stratosphere rises with the increase of the temperature of the lower troposphere, and its temperature decreases. Thus, a high and cold stratosphere corresponds to a warm troposphere, whereas a cold troposphere corresponds to a low and warm stratosphere. Certainly in the real

atmosphere we do not encounter a layer having a relatively constant pressure, just as the pressure over the earth's surface does not remain constant. A level exists in the stratosphere, however, at which the pressure changes considerably less than at other altitudes both higher and lower than this level. This level exists at an altitude of about 20 km. In addition, the average air pressure changes little with the latitude. Finally, the stratification curves, plotted on the basis of empirical data, agree well with the theoretical curves.

Table 14

The tropopause altitude and the tropospheric and stratospheric temperatures

T_0	— 10	0	10	15	20	24.2° C
ζ	5980	8440	11 390	13 220	15 550	20 000 m
T_ζ	— 46	—50	—58	—64	—73	—96° C

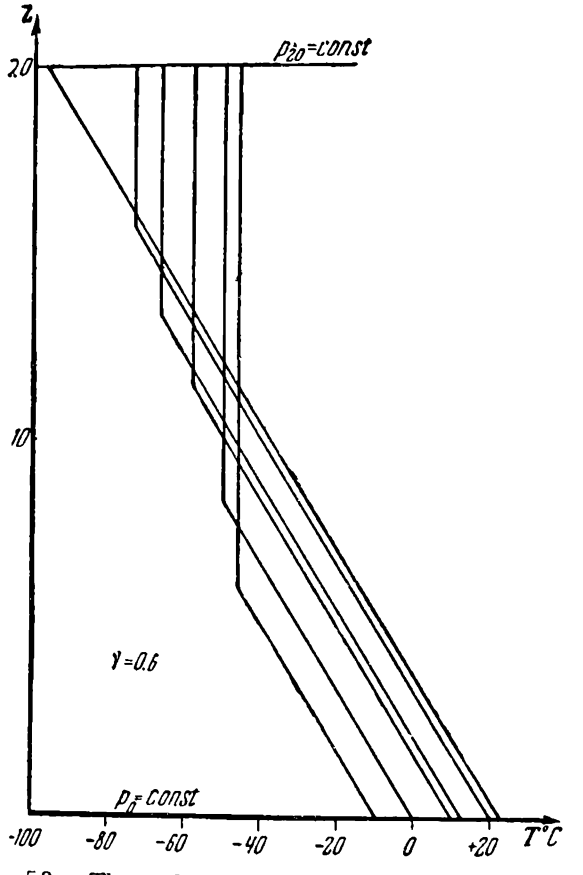

Figure 53. The relation between the altitude and the temperature of the tropopause

We can therefore assume that the following state is established in the real terrestrial atmosphere.

The influence of increasing lower tropospheric temperatures on ground level pressure is mostly compensated by the rise of the tropopause and by the decrease of the stratospheric temperature. This situation is a particular instance of an important principle of the statics of the atmosphere, i. e., the principle of temperature compensation: "warm below, cold above" and vice versa. This principle is very important for understanding the peculiarities of temperature stratification of stationary extratropical cyclones and anticyclones.

It was already proved at the end of the last century that stationary cyclones of the lower tropospheric layers are cold, while the stationary anticyclones are warm. This fact seemed puzzling. It was only possible to explain it after sufficient aerologic data had been accumulated, by establishing that, in spite of a warm troposphere, there is high ground level pressure in stationary anticyclones, due to the fact that the stratosphere rises to an exceptional height over the region of high pressure, and that it is extremely cold.

In the same way, in spite of a cold troposphere, a low pressure is observed in cyclones, due to the fact that the stratosphere descends to a very low level, and is extremely warm. Thus, the influence of the stratosphere is greater in this case than that of the troposphere.

§ 17. The Barometric Equation of Laplace

The determination of altitudes by means of barometric equations, the so-called "barometric levelling", came into use after it had been proved that the pressure decreases with increasing altitude.

The best known is Laplace's altitude equation. This equation can be easily derived from the barometric equation by expressing the virtual temperature in terms of the molecular temperature, measured in degrees Celsius, and of the humidity, making allowance also for the force of gravity:

$$\Delta z = 18400 \lg \frac{p_1}{p_2}(1+\alpha t)\left(1+0.378\,\frac{e}{p}\right)(1+0.0026\cos 2\varphi)\times \\ \times(1+3.14\cdot 10^{-7}h),$$

where Δz is the altitude difference in meters ($\Delta z = z_2 - z_1$); t – the arithmetic mean temperature of the air layer ($t = \frac{t_1 + t_2}{2}$); $\frac{e}{p}$ – the arithmetic mean of the ratio of the water vapor tension to the air pressure [$\frac{e}{p} = \frac{1}{2}\left(\frac{e_1}{p_1} + \frac{e_2}{p_2}\right)$]; h – the arithmetic mean height ($h = \frac{z_1 + z_2}{2}$). Detailed tables for the barometric determination of altitudes were worked out with the help of this equation.

The Laplace equation is used only for very exact computations. The inaccuracy of the data concerning the distribution of the temperature is so great that it is meaningless in this case to take into consideration the influence of temperature and of the change of the force of gravity, because they are considerably less important than the error in the temperature determination. However, if the mean values must be computed from a sufficiently large number of observations, then it is necessary to use more exact equations, because the errors in determining the vertical distribution of the temperature cancel out on averaging, in the majority of cases. The influence of the humidity and the change of the force of gravity must be taken into account in order to avoid a basic inaccuracy in the results.

§ 18. The Standard Atmosphere

For many practical purposes it is important to know some standard conditions for the distribution of pressure and density in the atmosphere. A knowledge of the distribution of pressure in the standard atmosphere, for example, makes it possible to construct an altimeter (a gauge of the altitude at which an aircraft is flying); this instrument actually measures the pressure, but it is calibrated in units of altitude of a standard atmosphere. The distribution of density in the standard atmosphere is used, for example, in aerodynamic computations.

The standard atmosphere is characterized by the following conditions:

1. The chemical composition of the air at all altitudes up to 15 km is identical, and the content by volume of the separate gases is the following:

Nitrogen	78.03%
Oxygen	20.99%
Argon	0.94%
Carbon dioxide	0.04%

2. The relative humidity is everywhere equal to 0%, i.e., the air is absolutely dry.

3. The pressure at sea level equals 760 mm Hg or 1,013.3 mb; the temperature is 15° C and the density 1.225 kg/m^3.

4. The air temperature varies linearly with the altitude up to 11 km, the lapse rate being equal to 0.65°/100 m (the standard troposphere). The temperature remains constant above 11 km (the standard stratosphere), i.e., the temperature is determined in relation to the altitude by the formulas:

$$T = 288° - 0.0065z \quad (z \leqslant 11 \text{ km}),$$
$$T = 216.5° \quad (z \geqslant 11 \text{ km}).$$

5. The change of pressure and of density in the standard troposphere is determined by the corresponding equation for a polytropic atmosphere:

$$\frac{p}{p_0} = (1 - 0.002257z)^{5.256},$$
$$\frac{\rho}{\rho_0} = (1 - 0.0022572z)^{4.256}.$$

6. The change of pressure and of density in the standard stratosphere is determined by the corresponding equations for an isothermic atmosphere:

$$p = p_{11} e^{-\frac{z-1100}{6340}} \quad ; \quad \rho = \rho_{11} e^{-\frac{z-11000}{6340}}.$$

7. The limit altitude of the standard atmosphere is determined by the following barometric altitude equations:

$$z = 44\,308\left[1 - \left(\frac{p}{p_0}\right)^{0.19028}\right] \quad (z \leqslant 11 \text{ km}),$$
$$z = 11\,000 + 6340 \ln\frac{p_{11}}{p} \quad (z > 11 \text{ km}).$$

Chapter VI

THE STABILITY OF STATIC EQUILIBRIUM

§ 1. Conditions of Quasistatic Equilibrium

In Chapter V we found the relationship between the characteristic variables of state and altitude in the course of our examination of the resting atmosphere.

In the present chapter the motion of a small air mass will be examined, assuming all the other air masses to be at rest, and that the motion takes place in a vertical direction. Such an elementary air mass will be called an individual air particle (or parcel). All the quantities which refer to such an individual air particle will be distinguished by the subscript "i", and those referring to the surrounding atmosphere – by the subscript "e".

We will begin the investigation of the processes accompanying the motion of such an air particle with the assumption that the motion under consideration takes place slowly enough for the pressure inside the air particle to stay equal to the pressure of the surrounding atmosphere, i. e., that at the same height

$$p_i = p_e = p, \quad \text{and also} \quad \frac{\partial p_i}{\partial z} = \frac{\partial p_e}{\partial z} = \frac{\partial p}{\partial z}. \tag{1}$$

These equations define the so–called quasistatic motion conditions of an air particle in the surrounding atmosphere. We assume, however, that the motion of the air particle is also too rapid to allow the remaining parameters of state to adjust to their values in the surrounding atmosphere, i. e., that at the same altitude, generally speaking,

$$\begin{aligned} T_i &\neq T_e; \quad \frac{\partial T_i}{\partial z} \neq \frac{\partial T_e}{\partial z}; \\ \rho_i &\neq \rho_e; \quad \frac{\partial \rho_i}{\partial z} \neq \frac{\partial \rho_e}{\partial z}; \\ s_i &= s_e; \quad \frac{\partial s_i}{\partial z} \neq \frac{\partial s_e}{\partial z}. \end{aligned} \tag{2}$$

It follows from equation (1) that the isobaric surfaces in the atmosphere remain unchanged during the motion of the air particle, whereas the isothermic, isosteric and other iso–surfaces will be deformed, i. e., disrupted or bent. In particular, those isobaric surfaces which are horizontal remain so, in spite of the displacement of the air particle from one isobaric surface to another, if all the parameters of state depend on the altitude only.

The condition of quasistatic motion, formulated above, is realized only for velocities much smaller than the velocity of sound (≈ 320 m/sec), so that this condition is actually fulfilled for vertical velocities encountered in the atmosphere.

§ 2. Dry–Adiabatic Temperature Variation of an Ascending Air Particle

The individual air particle which moves in a vertical direction will be subjected to the pressure prevalent in the surrounding atmosphere at the corresponding level. Our aim is to derive relations, which enable to determine the parameters of state of the air particle in any future state. These relations, known as the equations of change of state of the particle, have been examined in Chapter IV, in

which the change of state was determined by the pressure change. For many metereological purposes, however, it appears more expedient to examine the change of temperature and of other parameters of state in an ascending air particle not in relation to the pressure p, but to the altitude z. This is explained by the fact that the pressure inside the particle undergoes very great changes during its vertical motion. Indeed, the daily experience of the weather bureau shows that the vertical gradients of pressure are, on the average, 10, 000 times greater than the horizontal ones.

Naturally, the motions of air particles in the atmosphere do not take place adiabatically, but these conditions can be considered adiabatic to a first approximation, for motions extending over short distances and for sufficiently short durations.

The phenomenon of the foehn is undoubtedly the most striking example of adiabatic vertical motion in the atmosphere. Numerous observations carried out during the foehn show that in the falling air mass the temperature varies nearly adiabatically. Adiabatic curves are therefore used for the analysis of vertical motions in the majority of meteorological problems. The temperature of the air particle which is adiabatically displaced, is considered a function of altitude and, of course, of the initial values T_o and p_o.

However, one equation of state for an air particle is not sufficient in order to take account of the altitude, and it is also necessary to consider the equation of stratification of the surrounding atmosphere. This can be explained by the fact that the adiabatic temperature is a function of the pressure p. The same pressure for different atmospheric conditions can, however, be encountered at different altitudes. Thus, for example, the pressure $p = p_o/2$ is encountered at a height of 5, 500 m in an isothermal atmosphere, and at a height of 5, 000 m in an adiabatic one. Therefore, the same value of pressure, and consequently the same temperature of the ascending air particle, are encountered at a height of 5, 500 m in the first case and at a height of 5, 000 m in the second one. It should be emphasized that no calculation, no matter how elaborate, will give us the temperature of the adiabatically ascending air particle as a function of altitude, when using an adiabatic equation alone. It is therefore necessary to consider the stratification curve of the surrounding atmosphere as fully known; the variation of the temperature T_i of the independent air particle can be calculated only on this condition.

These preliminary remarks having been made, the sought relation for $T_i(z)$ can be derived.

Let us consider the differential equation, relating the change of temperature to the change of pressure during the dry-adiabatic process

$$\frac{dT}{T} - \frac{R}{c_p}\frac{dp}{p} = 0. \tag{1}$$

Since the change of pressure inside the air particle is equal to the change of pressure in the surrounding atmosphere and the latter, according to the fundamental equation of hydrostatics, is determined by the density of the same atmosphere, we have

$$dp = -g\rho_e\, dz, \tag{2}$$

where ρ_e is the density of the surrounding atmosphere. By substituting equation (2) in equation (1), and keeping in mind that the problem concerns the variation of the temperature T_i of the individual air particle, we obtain:

$$dT_i = \frac{R}{c_p} \cdot \frac{T_i}{p}(-g\rho_e dz),$$

and by substituting for ρ_e its value according to Clapeyron's equation $\rho_e = p/RT_e$, we finally obtain:

$$dT_i = -\frac{g}{c_p} \cdot \frac{T_i}{T_e} \cdot dz. \tag{3}$$

All the quantities of the right-hand side of equation (3), except dz, are essentially positive. The minus sign therefore indicates that the temperature of the air particle decreases: $dT_i < 0$, during the adiabatic ascent, when $dz > 0$, and the temperature of the air particle increases: $dT_i > 0$ during the adiabatic descent, when $dz < 0$.

An exact integration of equation (3) is impossible, primarily, because the variation of the temperature of the surrounding atmosphere is very irregular. If the temperature of the surrounding atmosphere always remains equal to the temperature of the adiabatically ascending air particle $T_i = T_e$, we have

$$\frac{dT_i}{dz} = -\frac{g}{c_p}. \tag{4}$$

The right-hand side of equation (4) is constant. Assuming that $g/c_p = \gamma_a$, we obtain

$$\gamma_a = 0.985, \qquad [\gamma_a] = [L^{-1}0].$$

This magnitude is called the dry-adiabatic lapse rate. The dry-adiabatic lapse rate is constant and equals $1^\circ/100$ m providing, of course, that the very small changes of the gravity acceleration both with altitude and latitude, and the variation of specific heat c_p with the temperature, are not taken into consideration. The atmosphere whose lapse rate equals $\gamma_a = 1^\circ/100$ m is called a dry-adiabatic atmosphere.

The dry adiabatic temperature variation of the ascending independent air particle does not stay constant, but depends on the ratio T_i/T_e.

It is obvious that the use of values of γ_a, measured to an accuracy of more than $1^\circ/100$m, would lead to superfluous precision.

The integration of equation (4) is very simple, if $T_i = T_e$; as a result of integration the following relation is obtained:

$$T_i = T_{i0} - \gamma_a z. \tag{5}$$

However, the integration of equation (4) can only be carried out for some exceptional cases, if $T_i \neq T_e$. Thus, for example, if the surrounding atmosphere is isothermal, i.e., T_e = const., we obtain:

$$T_i = T_{i0} e^{-\frac{\gamma_a z}{T_e}}. \tag{6}$$

If the temperature of the surrounding atmosphere varies linearly we obtain, separating the variables in equation (3) and integrating:

$$T_i = T_{i0}\left(\frac{T_{e0} - \gamma z}{T_{e0}}\right)^{\frac{\gamma_a}{\gamma}}, \tag{7}$$

where γ is the lapse rate of the surrounding atmosphere. Equation (7) shows that the temperature of the dry air particle ascending adiabatically in an atmosphere with a lapse rate γ depends not only on the initial temperature of the particle, but also on the distribution of temperature in the

surrounding atmosphere. The overheated air particle may ascend to different altitudes before cooling down to the temperature of the surrounding atmosphere, i.e., when

$$T_i = T_e = T_{e0} - \gamma z. \tag{8}$$

Relationship (8) makes it possible to determine this altitude. By substituting equation (8) by equation (7), we obtain, after a simple transformation:

$$z = \frac{1}{\gamma}\left\{T_{e0} - \left(\frac{T_{e0}^{\gamma a}}{T_{i0}^{\gamma}}\right)^{\frac{1}{\gamma a - \gamma}}\right\}. \tag{9}$$

§ 3. The Variation of the Humidity Characteristics with Altitude during the Adiabatic Ascent of Unsaturated Moist Air

From the definition of the specific humidity it follows that it remains constant, if there is no moisture exchange

$$s = s_0 = \text{const}. \tag{1}$$

The mixing ratio retains its exact value

$$w = w_0 = \text{const}. \tag{2}$$

The water vapor pressure does not change. Since e = 1.605 s· p, we obtain after logarithmic differentiation, owing to the constancy of s:

$$\frac{de}{e} = \frac{dp}{p}. \tag{3}$$

However, for the adiabatic process

$$\frac{dT}{T} = \frac{R}{c_p} \cdot \frac{dp}{p};$$

consequently

$$\frac{de}{e} = \frac{c_p}{R} \cdot \frac{dT}{T}, \tag{4}$$

or

$$\frac{de_i}{dz} = \frac{c_p}{R} \cdot \frac{e_i}{T_i} \cdot \frac{dT_i}{dz}, \tag{5}$$

whereby

$$\frac{c_p}{R}\frac{dT_i}{dz} = -3.5,$$

and hence

$$\frac{de_i}{dz} = -3.5\frac{e_i}{T_i}. \tag{6}$$

By integrating equation (4) we obtain

$$e_i = e_{i0}\left(\frac{T}{T_0}\right)^{\frac{c_p}{R}}. \tag{7}$$

It can be seen from equation (6) that the water vapor pressure changes very slowly during the adiabatic ascent and the change in the vapor pressure exceeds the accuracy of the humidity measurements only when the moisture content is considerable; the change in the vapor pressure with altitude is practically imperceptible when the moisture content is small.

We shall now determine how the saturation pressure E_i of a moist unsaturated air parcel varies with altitude during its adiabatic ascent. From the equation of Magnus follows

$$\frac{1}{E}\frac{dE}{dz} = -\frac{ab \ln 10}{(b+t)^2} \cdot \gamma_a$$

or

$$\frac{dE_i}{dz} = -\frac{4026.7}{(235+t)^2} \cdot E_i. \qquad (8)$$

Table 15 gives the values of $\frac{dE_i}{dz}$.

Table 15

The change in saturation pressure during adiabatic ascent of moist air

t_i	−30	−20	−10	0	10	20	30° C
E_i	0.373	1.27	2.85	6.10	12.26	23.38	42.42 mb
$\frac{dE_i}{dz}$	0.033	0.09	0.23	0.44	0.83	1.45	2.22 mb/100 m

It can be seen from Table 15that the change in saturation pressure is much greater than the change in the actual pressure of the water vapor. Therefore saturation is reached sooner or later in an adiabatically rising air particle, and then condensation commences.

We will now consider the change of the relative humidity during the adiabatic ascent of moist air. According to the definition

$$r = \frac{e}{E} \cdot 100^0/_0,$$

hence

$$\frac{1}{r} \cdot \frac{dr}{dz} = \frac{1}{e}\frac{de}{dz} - \frac{1}{E}\frac{dE}{dz}$$

or, when equations (6) and (8) are taken into consideration,

$$\frac{dr}{dz} = r\left[-\frac{3.5}{T} + \frac{4026.7}{(T-38)^2}\right]. \qquad (9)$$

Table 16 gives us the variation of the relative humidity during the adiabatic ascent of unsaturated moist air.

Thus, the relative humidity increases during adiabatic ascent and decreases during adiabatic descent, its variation becoming more pronounced for greater humidity. Indeed, the descending air currents (e. g. in the "foehns") are characterized by extraordinarily small values of the relative humidity. Layers having a relative humidity considerably less than 10% are often encountered in the free atmosphere. Condensation of water vapor eventually takes place in ascending currents.

Finally we consider the variation of the dew point during adiabatic ascent. According to the definition, the dew point is related to the actual water vapor pressure by the equation of Magnus

$$e = E_0 \cdot 10^{\frac{a\tau}{b+\tau}},$$

hence

$$\frac{1}{e}\frac{de}{dz} = \frac{4026.7}{(235+\tau)^2} \cdot \frac{d\tau}{dz}.$$

Or, according to equation (6)

$$\frac{1}{e}\frac{de}{dz} = -\frac{3.5}{T};$$

consequently

$$\frac{d\tau}{dz} = -0.0087\frac{(235+\tau)^2}{T}. \qquad (10)$$

Table 16

The change in the relative humidity of moist air during adiabatic ascent

$$\frac{dr_i}{dz}$$

T \ r	10	20	30	40	50	60	70	80	90	%
250	0.7	1.5	2.2	3.0	3.7	4.5	5.2	6.0	6.7	%/100 m
275	0.6	1.1	1.8	2.2	2.8	3.2	3.9	4.3	5.0	%/100 m
300	0.5	0.9	1.4	1.9	2.3	2.8	3.3	3.8	4.2	%/100 m

Table 17 gives the change of the dew point in an adiabatically ascending air parcel for different values of the temperature and of the difference $t - \tau$.

It can be seen from equation (10) and from Table 17 that the dew point falls during the adiabatic ascent of unsaturated moist air, though more slowly than does the temperature of the individual air particle, namely - by about 0.1° to 0.2° per 100 m, and the dew point changes more rapidly, the higher the temperature and the greater the saturation.

Table 17

The variation of the dew point with altitude during an adiabatic ascent

$t-\tau$ \ t	−50	−40	−30	−20	−10	0	10	20	30	40
1	0.13	0.14	0.15	0.16	0.17	0.18	0.18	0.19	0.20	0.21
10	0.12	0.13	0.14	0.14	0.15	0.16	0.17	0.18	0.19	0.20
20	0.10	0.11	0.12	0.13	0.14	0.15	0.15	0.16	0.17	0.18

Now it is also easy to find the level of condensation, i.e., that level at which the dew point equals the temperature of the adiabatically ascending air particle.

It is obvious that

$$t_k = t_0 - \gamma_a z_k; \quad \tau_k = \tau_0 + \left(\frac{d\tau}{dz}\right)_m \cdot z_k; \quad t_k = \tau_k,$$

where $\left(\frac{d\tau}{dz}\right)_m$ is the mean variation of the dew point with altitude. Hence:

$$t_0 - \frac{z_k}{100} = \tau_0 + \left(\frac{d\tau}{dz}\right)_m \cdot z_k$$

$$z_k = \frac{t_0 - \tau_0}{1{,}0 + \left(\frac{d\tau}{dz}\right)_m} = A(t_0 - \tau_0). \qquad (11)$$

This equation makes it possible to determine the altitude at which condensation begins. The value of A is determined by successive approximation. The values of A thus calculated are given in Table 18.

Table 18

The values of the coefficient A, for the determination of the condensation level

$t_0-\tau_0$ \ t	−50	−40	−30	−20	−10	0	10	20	30	40
1	115	116	117	118	119	121	122	123	124	125
10	114	115	116	117	118	120	121	122	123	124
20	113	114	115	116	117	118	120	121	122	123

The following empirical equation has been found for the determination of the condensation level

$$z_k = 122(t_0 - \tau_0). \qquad (12)$$

The accuracy of the approximate equation (11) depends greatly on the initial assumption that there is no moisture exchange with the surrounding air. It is obvious that such an assumption cannot be strictly correct. In order to compute the change of moisture resulting from a process of mixing with the surrounding air, not the mixing ratio at ground level is used, but its average value over the whole layer through which the air particle under consideration ascends.

§ 4. The Moist-Adiabatic Lapse Rate

In Chapter IV we derived a series of equations, which determine the dependence of temperature on pressure during the adiabatic ascent of moist saturated air. Let us now determine the dependence of the temperature on altitude from the equation of heat balance, which has the approximate form:

$$c_p dT - RT\frac{dp}{p} = -L\,dw, \qquad (1)$$

This expresses the fact that the work of expansion is not carried out entirely at the expense of the internal energy, but a part of this work is converted from the latent heat liberated during condensation. Let us eleiminate dw from equation (1). Since $w = 0.622\ \frac{E}{p}$, we obtain by differentiating:

$$dw = w\frac{dE}{E} - w\frac{dp}{p}. \qquad (2)$$

Substituting equation (2) in equation (1) we obtain:

$$c_p\,dT+\frac{Lw}{E}\cdot dE=Lw\frac{dp}{p}+RT\frac{dp}{p}$$

or

$$\left(c_p+\frac{Lw}{E}\frac{dE}{dT}\right)\frac{dT}{dz}=(Lw+RT)\cdot\frac{1}{p}\frac{dp}{dz}.$$

Since the process is quasistatic, we have

$$p_i=p_e;\quad \frac{\partial p_i}{\partial z}=\frac{\partial p_e}{\partial z}.$$

By assuming $\frac{1}{p}\frac{dp}{dz}=-\frac{1}{RT_e}$, we obtain:

$$\left(c_p+\frac{Lw}{E}\frac{dE}{dT_i}\right)\frac{dT_i}{dz}=-(Lw+RT_i)\frac{g}{RT_e},$$

hence

$$-\frac{dT_i}{dz}=\frac{g\cdot\frac{T_i}{T_e}+\frac{Lw}{RT_e}\cdot g}{c_p+\frac{Lw}{E}\cdot\frac{dE}{dT_i}}. \tag{3}$$

Thus, the change of temperature during the moist-adiabatic ascent is a variable magnitude, depending on many factors. Let us assume that the change of temperature with altitude is the same in the ascending air particle, as in the surrounding atmosphere

$$\frac{dT_i}{dz}=-\frac{dT_e}{dz};\quad T_i=T_e=T.$$

This change will be called the moist-adiabatic lapse rate and denoted by γ_a'. We then obtain

$$\gamma_a'=\frac{g+\frac{Lw}{RT_e}\cdot g}{c_p+\frac{Lw}{E}\cdot\frac{dE}{dT}}. \tag{4}$$

Assuming $w = 0.622\frac{E}{p}$, we obtain

$$\gamma_a'=\frac{g}{c_p}\frac{p+x}{p+y}=\gamma_a\frac{p+x}{p+y}, \tag{5}$$

where

$$x=0.622\frac{LE}{RT}.\qquad y=0.622\frac{L}{c_p}\frac{dE}{dT}.$$

It is obvious that the value of the moist-adiabatic lapse rate depends on the temperature and on the pressure. Since both temperature and pressure of the ascending air particle vary with altitude, the moist-adiabatic lapse rate is also a function of the altitude. Table 19 gives the values of x and y for different temperatures.

The isopleths of the values of the moist-adiabatic lapse rates, plotted according to the data cited in Appendix 13, are given in Figure 55. It can be seen from this figure that the value of the moist-adiabatic lapse rate decreases with the increase of temperature, and with decreasing pressure. Let us examine the process of adiabatic ascent of an air particle in order to understand this dependence correctly. During the dry-adiabatic ascent the air should cool down approximately by 1° for each 100 m, if the condensation of the water vapor were not taking place. The

Table 19

The values of x and y

	Ice		Water					
t	— 30	— 20	— 20	— 10	0	10	20	30
x	9	30	27	58	121	228	427	719 mb
y	50	152	141	360	682	1375	2209	3348 mb

latent heat liberated as a result of condensation of the water vapor slows down the process of dynamic cooling and the moist-adiabatic lapse rate is therefore smaller than the dry-adiabatic. The amount of liberated latent heat depends, however, on the amount of condensed water vapor. The value of the moist-adiabatic lapse rate therefore also depends on the amount of condensed moisture. It is obvious that with a decrease of temperature by 1°, the amount of condensed moisture will be:

$$dw = w\frac{dE}{E} - w\frac{dp}{p}.$$

Consequently it stands to reason that a greater amount of water vapor will be condensed at high temperatures, when dE is particularly great, as compared with low temperatures (on equal cooling), leading to smaller values for the moist-adiabatic lapse rate.

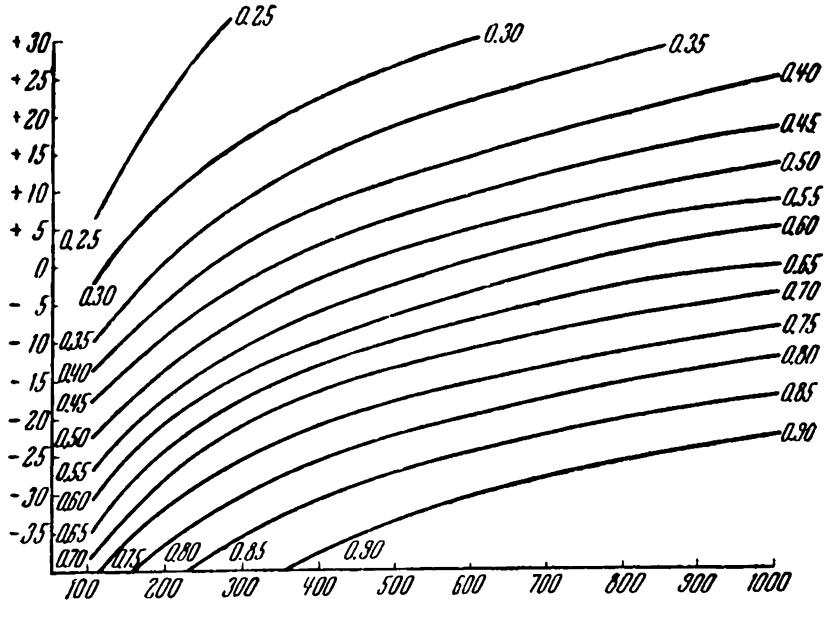

Figure 55. Isopleths of the moist-adiabatic lapse rate

The dependence of the moist-adiabatic lapse rate on the temperature is therefore inverse. For exactly the same reason, the amount of condensed moisture will be greater at a low pressure, than at a high one and the moist-adiabatic lapse rate will therefore be greater at a high pressure; the dependence of the moist-adiabatic lapse rate on the pressure will therefore be direct. Since the amount of the condensed water vapor decreases more and more as the ascent progresses, the moist-adiabatic lapse rate increases with increasing altitude of ascent, and asymptotically approaches the dry-adiabatic lapse rate.

In connection with the study of the temperature and pressure dependence of the moist–adiabatic lapse rate it must be emphasized that an exact and rigorous explanation of the causes of the adiabatic variation of temperature with the altitude is still not to be found in meteorological literature. Thus for example, Brunt says in his book on " Physical and Dynamic Meteorology " when deriving the equation for the moist-adiabatic lapse rate: " When a unit mass of dry air, containing its natural admixture of water vapor and liquid water, ascends the height z, the increase of the potential energy is compensated by the loss of internal energy from the air and water vapor, and also from the water present in it". Koshmider writes in his book "Dynamic Meteorology", as if arguing against Brunt: "The cooling of air during the adiabatic ascent has sometimes been explained by the fact that the temperature drop, i. e., the decrease in internal energy, is necessary for the increase of the potential energy. This is not exact, because in the place of the ascending quantity of air, another quantity must descend, and $\gamma = \gamma_\alpha$, when the equilibrium is indifferent: for example, no energy is required, and none is consumed, in a vertical displacement (if the molecular and eddy viscosities are not taken into account)". And then: "Adiabatic cooling can take place during an ascent without yielding any energy, and energy can be used up without an adiabatic cooling. Consequently, the energy consumed in ascent and the adiabatic cooling are not related to each other. "

Let us first consider the equation of thermal energy balance for dry air, in the form

$$dq = c_v dT + R\,dT - v\,dp. \tag{6}$$

in order to clarify this fundamental problem of atmospheric thermodynamics.

The first term $c_v dT$ of the equation represents the change of the internal energy $c_v T$ of the gas. Brunt, for some reason, incorrectly calls the enthalpy $c_p T$, internal energy. On the basis of this reasoning, he might have obtained a correct equation for the moist–adiabatic lapse rate, had he not fallen into this error. The second term RdT is the work carried out by the gas during its isobaric expansion against the external pressure $dw_p = RdT = pdv$. The expression RT = pv represents some store of potential energy, which a gas of a volume v possesses at constant pressure. This potential energy will be called the energy of expansion. It is obvious that the potential energy of expansion varies in the same way as the internal energy, namely, the potential energy of expansion increases with the increase of temperature and vice versa, the ratio of the energy of expansion to the internal energy remaining constant.

$$\frac{w_p}{u} = \frac{RT}{c_v T} = \frac{0.07}{0.17} = 0.4. \tag{7}$$

The two first terms of equation (6) can be considered together, by uniting them in the following way

$$c_v dT + R\,dT = c_p dT.$$

This is the change of the enthalpy, or of the heat content. Thus the enthalpy represents the sum of the internal energy and of the potential energy of expansion:

$$э = c_p T = c_v T + RT = u + w_p. \tag{8}*$$

---

* [Э stands everywhere for the enthalpy. – Translator's note].

The last term of equation (6) can finally be written as follows:

$$-v\,dp = -\frac{RT}{p}\,dp = -RT\,d\ln p. \tag{9}$$

But this is precisely the work of isothermal expansion:

$$dw = p\,dv = -v\,dp = -RT\,d\ln p. \tag{10}$$

It is easy to see that the work of isothermal expansion represents the change of potential energy of the independent air particle $d\pi$. Indeed:

$$d\pi = g\,dz - \frac{dp}{\rho} = -v\,dp. \tag{11}$$

Thus the equation of heat balance can now be written in the following form:

$$dq = du + dw_p + d\pi \tag{12}$$

or, for an adiabatic process

$$du + dw_p + d\pi = 0 \tag{13}$$

or, in short

$$d\vartheta + d\pi = 0. \tag{14}$$

Thus, the sum of the particle's potential energy of position and of the enthalpy is constant for an adiabatic process:

$$\vartheta + \pi = \text{const.}, \tag{15}$$

which could be written down without additional explanations – on the basis of the law of the conservation of energy. Thus, if the air particle ascends dry-adiabatically, its potential energy of position increases at the expense of the enthalpy, i. e., as a result of the simultaneous decrease of the internal energy and of the energy of expansion, the increase of the potential energy of position being made up of the contributions of the internal energy (71%) and of the energy of expansion (29%).

The potential energy of position is clearly defined both by the value of the geopotential corresponding to the particle's altitude, and by the ratio T_i/T_e, or more generally – by the ratio T_{v_i}/T_{v_e}; therefore

$$\pi = \frac{T_i}{T_e}\,\Phi. \tag{16}$$

Equation (16) can be regarded as a dry-adiabatic equation. The dry adiabate will be represented by a straight line, forming with the co-ordinate axes angles of 45°, if we agree to use equal units in plotting the enthalpy on the abscissa axis, and the particle's potential energy of position on the ordinate axis.

The equation of the condensation adiabate can also be written in a very concise form as a sum of the potential energy of position and of the enthalpy, which remains constant throughout the adiabatic process. In this case the moist-adiabatic curves are also represented in the $\vartheta - \pi$ coordinate system by straight lines, inclined at an angle of 45° to the to the coordinate axes. However, the expression for the enthalpy will, of course, be different for the moist adiabatic process. Whereas the enthalpy of dry air is proportional to the molecular temperature, the enthalpy of moist air is proportional to the equivalent temperature T_ϑ, therefore

$$\vartheta = c_p T_\vartheta + \text{const.} \tag{17}$$

If the very small terms are omitted from the differential equation of the condensation adiabate, it can be written in the following form:

$$c_p dT_\vartheta + \frac{T_l}{T_e} d\Phi = 0 \tag{18}$$

or when the wet bulb temperature T' is introduced instead of the equivalent temperature T_ϑ, in the form:

$$c_p dT' + d(L'w') + \frac{T_l}{T_e} d\Phi = 0. \tag{19}$$

Since the wet bulb temperature agrees with the actual temperature ($T'=T$) for a moist-adiabatic process, it is easy to obtain from equation (19) an equation, which enables to determine the value of the moist-adiabatic lapse rate. By assuming $T_i / T_e = 1$, we obtain:

$$c_p \frac{dT}{d\Phi} + \frac{\partial (Lw)}{\partial T} \cdot \frac{dT}{d\Phi} + \frac{\partial (Lw)}{\partial p} \frac{dp}{d\Phi} + 1 = 0,$$

hence

$$-\frac{dT}{d\Phi} = \frac{1}{c_p} \frac{1 + \frac{\partial (Lw)}{\partial p} \frac{dp}{d\Phi}}{1 + \frac{1}{c_p} \frac{\partial (Lw)}{\partial T}}.$$

But

$$\frac{\partial (Lw)}{\partial p} = -0.622 \frac{LE}{p^2}; \quad \frac{\partial (Lw)}{\partial T} = 0.622 \frac{L}{p} \frac{dE}{dT},$$

$$\frac{dp}{d\Phi} = -\rho, \quad \frac{dT}{d\Phi} = \frac{1}{g} \frac{dT}{dz},$$

therefore

$$\gamma'_a = \gamma_a \frac{p + x}{p + y},$$

where

$$x = 0.622 \frac{LE}{RT}; \quad y = 0.622 \frac{L}{c_p} \frac{dE}{dT}.$$

§ 5. Equilibrium Conditions for a Dry Atmosphere

The results obtained above can be applied to the analysis of the thermodynamic state of the atmosphere; in particular, to explain the decrease of temperature with altitude. These results, to be sure, specifically apply to the progressive change of state of an individual air parcel. But since some air parcels are usually ascending while others are descending, the same kind of adiabatic decrease of temperature must ultimately take place everywhere in the atmosphere as a consequence of this general mixing, which in fact eventually happens to every air particle in the course of its ascent.

The atmosphere is said to be in convective equilibrium, when the ascending adiabatically air particles encounter at all altitudes a surrounding atmosphere of the same temperature as their own.

The convective equilibrium in fact is but the first approximation to the actual state of the atmosphere. However it explains in the main the phenomenon of the air parcel's temperature falling as its ascent progresses, in which lies its importance. Helmholtz has already drawn attention to the fact that as a consequence of the existence of convective equilibrium at $t_0 = 0°C$ the temperature must

reach the absolute zero at an altitude of about 28 km, where the upper limit of the atmosphere should be found. Since this is not the case, the stratification of the atmosphere must be caused not only by adiabatic vertical motions of the atmosphere, but also by other processes.

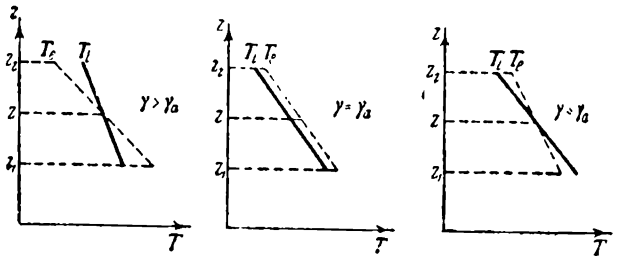

Figure 56. Analysis of the atmospheric equilibrium conditions in relation to the lapse rate.

a) $\gamma > \gamma_a$ -unstable equilibrium; b) $\gamma = \gamma_a$ -indifferent equilibrium; c) $\gamma < \gamma_a$ stable equilibrium.

Let us now explain in more detail the equilibrium conditions of the atmosphere, which follow from the foregoing argument. For clarity we shall use graphic representations both of the stratification curves and of the change of state curves.

In a T, z coordinate system let us plot a dry-adiabatic curve, i.e., a curve, which represents the change of temperature in a parcel of dry air, rising adiabatically (Figure 56). The dry-adiabatic curve will be a straight line, inclined to both axes at an angle of 45°, and rising from right to left, if the same unit is chosen for 100 m on the ordinate axis, and for 1°C on the abscissa axis. Side by side with the dry-adiabate a section of the stratification curve will be plotted, which characterizes the vertical temperature distribution in the surrounding atmosphere near some point at the level z. It is necessary to distinguish between the stratification curves and the curves of the change of state of the individual air parcels, because on this difference is based the entire analysis of atmospheric stability. We emphasize again that the curve of change of state (the dry-adiabatic curve in our case) gives the progressive variation of the temperature of the same air parcel, rising adiabatically, whereas the stratification curve gives the vertical temperature distribution in the surrounding atmosphere. The stratification curve can only be plotted on the basis of the actual temperature measurements at different altitudes, while the curve of state is plotted theoretically. The angle the stratification curve forms with the abscissa can vary from day to day, or when passing from one altitude to another, but the angle of inclination of the dry adiabate always remains the same (it is equal to 45° for the scale chosen).

Three cases must generally be distinguished in the stratification of dry air:

$$1)\ \gamma > \gamma_a;\quad 2)\ \gamma = \gamma_a;\quad 3)\ \gamma < \gamma_a.$$

For all the three cases we suppose that the atmosphere is in a state of rest; consequently, each air parcel, as well as the whole layer under consideration, characterized by the lapse rate γ, is in a state of equilibrium. The question arises whether this equilibrium will be stable, unstable, or indifferent, and under what conditions. We must answer this question in a manner as detailed as possible, because it is of great practical importance.

1) $\gamma > \gamma_a$. Let us consider an arbitrary dry air parcel at level z, having at this level the same temperature as the surrounding atmosphere. This air parcel

is in equilibrium. Let us suppose that equilibrium of the air parcel is spontaneously disturbed, and it rises adiabatically to the level $z_2 > z$. The temperature of the air parcel will vary according to the dry-abiabate while the temperature of the surrounding atmosphere will vary according to the stratification curve. Since the air parcel cools by γ_a, while the surrounding atmosphere cools by $\gamma > \gamma_a$, the ascending air parcel at the level z_2 will be hotter than the surrounding atmosphere. Since the pressures inside the individual air parcel and in the surrounding atmosphere are equal (quasistatic condition), the ascending parcel will be lighter than the surrounding atmosphere and will continue to ascend under the action of the hydrostatic force, i.e., it will no longer preserve the state of equilibrium in which it has been at the level z.

Let us assume now that the same parcel, which was on the level z, is thrown out of balance by some impulse and descends to the level $z_1 < z$. Its temperature increases by γ_a in accordance with this condition, while the temperature of the surrounding atmosphere increases by $\gamma > \gamma_a$. As a result, the air parcel at the level z_1 will be colder than the surrounding atmosphere; therefore, a downward force will act on it. The air parcel is displaced from its previous position of equilibrium at the level z.

Thus, a small vertical displacement of the individual particle is sufficient, in case $\gamma > \gamma_a$, to start it moving further and further away from the equilibrium position. Since we consider an arbitrary air parcel in the given layer, the equilibrium of the whole layer will be equally unstable at $\gamma > \gamma_a$. When $\gamma > \gamma_a$ the stratification of the atmosphere, or its equilibrium are said to be unstable. It stands to reason that the unstable stratification promotes the growth of vertical motions in the atmosphere (both of regular vertical currents and of irregular turbulent motion). A vertical exchange of different impurities (water vapor, dust, condensation nuclei, etc) evolves together with the growth of vertical movements in the atmosphere, and consequently, favorable conditions for the development of convective cloudiness are created.

2) $\gamma = \gamma_a$. The adiabatically moving dry air parcel will in this case have the same temperature as the surrounding air at any altitude, since the curve of change of state (dry adiabate) coincides with the stratification curve. The individual air parcel will therefore have the same density as the surrounding air at any level it reaches. When displaced from its initial position, the parcel is therefore not accelerated. It follows that the equilibrium of the whole layer is indifferent for $\gamma = \gamma_a$.

3) $\gamma < \gamma_a$. Applying the same reasoning it can be shown that any air parcel, displaced from its equilibrium level z will, for $\gamma < \gamma_a$, have a nonvanishing component of acceleration in the direction of z, which tends to bring this parcel into a state of equilibrium. The atmospheric equilibrium is thus stable at $\gamma < \gamma_a$. A stable stratification forestalls the growth of vertical movements (regular vertical currents and turbulent motions, as well as vertical transport of different impurities and admixtures are precluded and consequently cloud formation is also prevented). A particularly stable equilibrium forms in the isothermic ($\gamma = 0$) and inversion ($\gamma < 0$) layers. These layers are therefore called retarding layers.

It should be noted on the basis of the above consideration that lapse rates exceeding γ_a, can not be expected to occur in the atmosphere. Indeed, the smallest impulse causes a rearrangement of the layers having an unstable stratification. It has been proved by aerological observations that superadiabatic lapse rates are rarely encountered in the free atmosphere. The lapse rates can exceed the adiabatic limits significantly only during the hot summer days in the lowest, 1 meter thick

ground level layer. On the other hand, the following considerations indicate that such lapse rates can, nevertheless, be encountered. Until now nothing has been said about the behavior of air parcels in the neighborhood of the moving one, which were taken to be in a state of rest. In fact, an air parcel moving among the others will impart some impulse to them and dislodge them from their respective equilibrium positions. A definite energy, which is derived from the unstable stratification of the atmosphere, is necessary for this to occur. Thus, superadiabatic lapse rates must exist, and are in fact encountered in the atmosphere.

It was proved by observations that such lapse rates only slightly exceed the adiabatic lapse rate of a free atmosphere, not surpassing the value of γ by more than 1.2° C per 100 m.

The moist-adiabatic lapse rate has the same significance as a criterion of stability in relation to moist-adiabatic processes, as the dry-adiabatic lapse rate has for the dry-adiabatic changes of state. Namely, the equilibrium is unstable in relation to the moist-adiabatic changes of state, if $\gamma > \gamma'_a$; the equilibrium is stable, if $\gamma < \gamma'_a$; finally, it is indifferent, if $\gamma = \gamma'_a$. It is to be noted that these criteria are not universal, as are the criteria for dry air. The conditions of stability are realized only for impulses which are directed downwards. The air parcel will be heated dry-adiabatically on its descent, and will therefore acquire an acceleration directed towards its initial level. These conditions are realized for any impulse, directed upwards or downwards also if the air contains water drops, but such as evaporate so quickly that it is always saturated.

§ 6. Convective Acceleration. The Energy of Instability

We will now derive a quantitative expression for the vertical acceleration $\frac{d^2z}{dt^2}$, which is imparted to an independent air particle under the influence of the lifting force, resulting from a temperature difference existing between the air particle and the surrounding atmosphere. Let us denote by P_i the weight of the air particle, occupying a volume v and having a density ρ_i and a temperature T_i. We will denote by P_e the weight of an equal volume of the surrounding air, having the density ρ_e and the temperature T_e. The independent air particle immersed in the surrounding atmosphere will, according to Archimedes' principle, become lighter by the weight of that volume of surrounding air which it displaces, i.e., the weight of the air particle in the surrounding atmosphere will be equal to $P_i - P_e$

The lifting force F acting on the independent air particle will be directed upwards, if $P_i < P_e$:

$$F = P_e - P_i = gv(\rho_e - \rho_i).$$

This lifting force will impart to the air particle an acceleration $\frac{d^2z}{dt^2}$, inversely proportional to the mass $v\rho i$; consequently

$$\frac{d^2z}{dt^2} = g\frac{\rho_e - \rho_i}{\rho_i}. \qquad (1)$$

By substituting ρ from Clapeyron's equation $\rho = p\ RT$, and keeping in mind that $p_i = p_e = p$ according to the quasi-static condition, we obtain:

$$\frac{d^2z}{dt^2} = g\frac{T_i - T_e}{T_e}. \qquad (2)$$

It is obvious that the air particle will be accelerated upwards if $T_i > T_e$, and downwards, if $T_i < T_e$; the acceleration will be equal to zero, if $T_i = T_e$.

Let now $T_i = T_{io} - \gamma_a z$; $T_e = T_{eo} - \gamma z$.

Then

$$\frac{d^2z}{dt^2} = g\frac{T_{i0} - T_{e0}}{T_e} + g\frac{(\gamma - \gamma_a)z}{T_e}. \tag{3}$$

Thus the acceleration of the particle is caused:

1) by the initial heating of the particle relative to the surrounding air

$$\left(\frac{d^2z}{dt^2}\right)_1 = g\frac{T_{i0} - T_{e0}}{T_e}; \tag{4}$$

2) by the specific stratification conditions

$$\left(\frac{d^2z}{dt^2}\right)_2 = \frac{(\gamma - \gamma_a)z}{T_e}. \tag{5}$$

The first term of equation (3) remains constant during the entire motion of the air particle. The second term varies during this motion.

It can be shown by calculations that the acceleration due to the instability of the stratification is very small when there is saturation. But the acceleration due to heating is of the order of 1 cm/sec^2, the stratification accelerations being many tenfolds less. There is therefore no reason to expect significant vertical velocities in the absence of overheating and condensation. The acceleration due to the moist-adiabatic is greater than the acceleration due to the dry-adiabatic convection. Considerable vertical velocities can therefore be observed in clouds. We can, however, evaluate the orders of magnitude of these velocities in a more precise way.

Every air particle, as well as the whole layer, possesses in an unstable stratification a certain store of potential energy, the so-called energy of instability; this energy is transformed into kinetic energy or spent as work done against the forces of friction as soon as the unstable equilibrium is disturbed. The air particle in this case moves under the action of the lifting force without an external influx of energy, and the stored energy of instability is liberated and transformed into kinetic energy. In order to lift an air particle in a stable stratification, a force sufficient to overcome the residual weight of the air particle must be applied. The work this force will perform is converted into potential energy of the air particle. It is obvious that the work w, performed in elevating the particle and the energy of instability e are related to each other by

$$w = -e.$$

The energy liberated during the dry-adiabatic convection can now be calculated. It was shown above that a lifting force per unit mass expressed by the equation

$$F = g\frac{T_i - T_e}{T_e}.$$

acts on an air particle ascending through an atmosphere whose temperature T_e is likewise a function of altitude. The energy of instability, equal to

$$\Delta e = F\Delta z = g\frac{T_i - T_e}{T_e}\cdot\Delta z = \frac{T_i - T_e}{T_e}\cdot\Delta\Phi. \tag{6}$$

is liberated during the ascent of the air particle by a height Δz This energy is converted into kinetic energy which equals $c^2/2$ per unit mass. Hence

$$\frac{c^2}{2} = \frac{T_i - T_e}{T_e} \Delta\Phi$$

or

$$c = \sqrt{\frac{2(T_i - T_e)\Delta\Phi}{T_e}} . \tag{7}$$

The values of the vertical velocity c developed during the liberation of the energy of instability, and calculated according to the equation

$$c = \sqrt{\frac{2(T_i - T_e)\Delta\Phi}{300}} = 0.24\sqrt{(T_i - T_e)\Delta\Phi} \text{ m/sec}$$

are given in Table 20.

It can be seen from Table 20 that considerable vertical velocities can be reached only in the case, when the air particle remains for a long time warmer than its surrounding medium, and such conditions are possible only in the case of initial overheating and of unstable stratification.

Table 20

Vertical velocities, developed during liberation of the energy of instability

$\Delta\Phi$ \ $T_i - T_e$	+1	+3	+5	+10° C
100	2.4	3.4	5.4	2.5
300	3.4	4.8	7.6	10.5
500	5.4	7.6	12.0	16.5
1000	7.5	10.5	16.5	24.0

An unstable stratification is more probable in the case of water vapor condensation, than during an adiabatic ascent. It has already been noted that the values of the vertical velocities given in Table 20 are the maximal. In fact, it was assumed in the calculations that all the energy of instability is converted into kinetic energy. As a matter of fact, a part of this energy is used up to accomplish work against the forces of friction. It was also assumed that the ascent of the independent air particle takes place adiabatically. Actually, the temperature of the air particle is gradually equalized with the temperature of its surroundings, as a result of turbulent heat exchange.

Any of the energy diagrams previously examined can be used for graphic calculation of the energy of instability. Indeed, the elementary energy of instability is expressed by equation (6). This equation gives the amount of potential energy which is liberated during the disturbance of the unstable equilibrium, when a particle of given mass moves along an elementary path Δz . It is sufficient to integrate equation (6) from z_1 to z_2, in order to calculate the energy of instability which will be liberated during the motion of the particle along an arbitrary path extending from z_1 to z_2. It is impossible to compute the integral

$$e = \int_{\Phi_0}^{\Phi} \frac{T_i - T_e}{T_e} \cdot d\Phi \tag{8}$$

in the general case, but it can always be graphically evaluated. Assuming $d\Phi = -(RT_e/p)\,dp$, we obtain

$$e = \int_{p_0}^{p} (T_e - T_i)\, d \ln p. \qquad (9)$$

The integrand is represented on the emagram by the surface element, bounded by an elementary segment of the stratification curve, by the curve of change of state and by two isobars, whereas the entire integral is represented on the emagram by an area bounded by the stratification curve, by the curve of change of state, and by the two corresponding isobars.

The area ABC in Figure 57, bounded by the stratification curve and the dry and moist adiabates, is proportional to the work, which must be carried out to lift a mass unit to the level C. The area CDE, bounded by the stratification curve and by the moist adiabatic curve, is proportional to the energy of instability liberated during the ascent of a saturated air particle, the so-called energy of moist instability.

The difference between the areas CDE and ABC is proportional to the total energy of instability liberated during the motion of the air particle from the level A to the level having pressure p. But any energy diagram can be obtained from another such diagram by means of an area-preserving transformation. The energy of instability will therefore be represented on any energy diagram by an area, bounded by the stratification curve, by the curve of state, and by the isobars passing through the points corresponding to initial and final levels of the air particle.

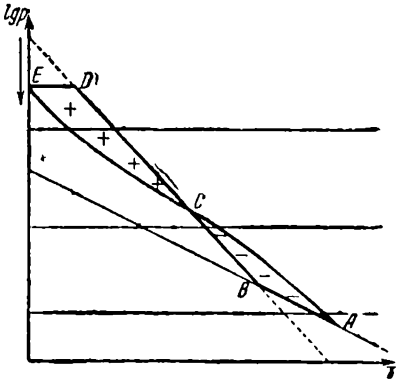

Figure 57. The area, enclosed between the stratification curve and the moist-adiabatic curve, is proportional to the energy of moist instability

Since this air particle has been chosen quite arbitrarily, the corresponding areas on the energy diagrams characterize the energy of instability per unit mass corresponding to a given stratification.

§ 7. The Potential Temperature as a Criterion of Dry Air Stability

The potential temperature is a good criterion of the dry air stability. The potential temperature of an air particle, as was proved in Chapter II, § 16, does not change during the adiabatic processes; the adiabatically moving air particle will therefore preserve its potential temperature in passing through different levels, whence

$$\frac{d\theta_l}{dz} = 0 \quad \text{и} \quad \theta_l = \text{const.} \tag{1}$$

We shall now determine precisely the atmospheric altitude variation of the potential temperature. By definition

$$\theta_e = T_e \left(\frac{1000}{p}\right)^{\frac{R}{c_v}}. \tag{2}$$

Computing the logarithmic derivative of θ_e with respect to z, we obtain:

$$\frac{1}{\theta_e}\frac{d\theta_e}{dz} = \frac{1}{T_e}\frac{dT_e}{dz} - \frac{R}{c_p}\frac{1}{p}\frac{dp}{dz}. \tag{3}$$

But, according to the fundamental hydrostatic equation

$$\frac{1}{p}\frac{dp}{dz} = -\frac{g}{RT_e}.$$

Consequently

$$\frac{d\theta_e}{dz} = \frac{\theta_e}{T_e}\left(\frac{dT_e}{dz} + \gamma_a\right).$$

assuming, as was done before, that $-\frac{dT_e}{dz} = \gamma$, we finally obtain:

$$\frac{d\theta_e}{dz} = \frac{\theta_e}{T_e}(\gamma_a - \gamma). \tag{4}$$

It can be seen from equation (4) that the potential temperature increases with altitude in case of a stable stratification ($\gamma < \gamma_a$), decreases in case of unstable stratification ($\gamma > \gamma_a$). It has already been noted that the potential temperature lapse rates can differ for the same stratification.

By assuming $\frac{\theta_e}{T_e} = \left(\frac{1000}{p}\right)^{\frac{R}{c_p}}$, we obtain:

$$\frac{d\theta_e}{dz} = \left(\frac{1000}{p}\right)^{\frac{R}{c_p}}(\gamma_a - \gamma). \tag{5}$$

Thus, the dependence of the potential temperature on the altitude at a given lapse rate, is not linear, but is essentially determined by the pressure and, through it, on the altitude itself,

The convective acceleration which is typical of the particular stratification, can also be expressed by the potential temperature. By substituting $\gamma - \gamma_a$ in equation (5) § 6, according to equation (4), we obtain

$$\frac{d^2z}{dt^2} = -\frac{gz}{\theta}\frac{d\theta}{dz}.$$

The potential temperature in many cases plays the same part in the compressible atmosphere, as does the molecular temperature in an incompressible, homogeneous fluid. One can easily convince oneself of this fact, by taking the potential temperature as a criterion of stability.

A method based on the following considerations is used, if it is necessary to determine rapidly the approximate value of the potential temperature. Since

a descending air particle is heated by almost 1° per 100 m, it reaches sea level with a temperature of $(T + \frac{z}{100})°$, when dropping from an altitude of z m. This value will be approximately equal to the potential temperature if the pressure at sea level is close to the standard one (1,000 mb):

$$\theta = T + 0.01\, z. \tag{6}$$

The particle's pressure must be reduced to standard pressure, if the pressure at sea level differs appreciably from the standard one; for this purpose a temperature increment of $\left(\frac{1,000 - p_0}{12}\right)°$ is added to the right-hand side of equation (6). Hence the definitive approximate value of the potential temperature will be

$$\theta = T + 0.01\, z + \frac{1000 - p_0}{12}. \tag{7}$$

The obtained estimate will be a closer approximation to the actual potential temperature, the less does the atmospheric stratification deviate from an adiabatic one. Otherwise an important difference can occur between the actual and the computed potential temperature. We have seen above that the temperature of an adiabatically ascending or descending air parcel depends not only on its altitude, but also on the stratification of the surrounding atmosphere.

The determining equation for the potential temperature at a given altitude can therefore be written:

$$T_i = T_{i0}\left(\frac{T_{e0} - \gamma_a z}{T_{e0}}\right)^{\frac{\gamma_a}{\gamma}}$$

or, assuming $T_i = T_e = T$ and $T_{i0} = 0$, and expanding in a series we obtain, retaining terms of order not higher than the second

$$\theta = T + \gamma_a z + \frac{\gamma_a}{\gamma}\left(\frac{\gamma_a}{\gamma} - 1\right) \cdot \frac{\gamma^2 z^2}{2T}. \tag{8}$$

The last term vanishes, and equation (8) goes over into equation (6) in the case of an adiabatic stratification $(\gamma = \gamma_a)$. It is proved by computations that use can be made of the approximate equation (7), when the potential temperature for altitudes of about 1,000 m is to be determined, but this equation is not sufficiently for altitudes of above 5,000 m.

§ 8. The Influence of Vertical Motion on Dry Air Stability

When large scale vertical motions take place in the atmosphere, entire layers of air rise and fall in such a way that they become subjected to different pressures and their horizontal cross section varies. The already mentioned phenomenon of the foehn can serve as an example of this process. In the case of anticyclones, a subsidence of vast air masses often takes place over a wide territory without a change of the relative position of the layers. The stratification of major air masses is altered in the course of such motions.

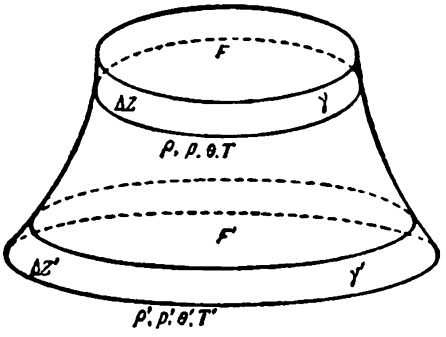

Figure 58. Change of atmospheric stability during the vertical displacement of great air masses

Let some air layer (Figure 58) descend adiabatically and spread out in such a way that its thickness diminishes. Let us assume in addition that there is no overturning and no mixing, i.e., that the particles, which were situated on the upper boundary of the layer before the descent began, remain there after downfall. Let the layer's initial cross section be F, its thickness Δz, the air density at the lower boundary ρ, , the pressure p, the potential temperature θ, , the molecular temperature T, the lapse rate γ, and the corresponding values after descent: F', $\Delta z'$, ρ', p', θ', T', γ'.

In addition let the potential temperatures be respectively $\theta + \Delta\theta$ and $\theta' + \Delta\theta'$ at the upper boundary of the layer. Then as a result of an adiabatic descent, we obtain $\theta = \theta'$, $\theta + \Delta\theta = \theta' + \Delta\theta'$ and consequently $\Delta\theta = \Delta\theta'$. Since the air mass remains unchanged, $\rho F \Delta z = \rho' F' \Delta z'$ and

$$\frac{1}{pF}\frac{\Delta\theta}{\Delta z} = \frac{1}{\rho' F'}\frac{\Delta\theta'}{\Delta z'}$$

or, in the limit

$$\frac{1}{\rho F}\frac{d\theta}{dz} = \frac{1}{\rho' F'}\frac{d\theta'}{dz'}.$$

Assuming that $\rho = \frac{p}{RT}$ and $\frac{d\theta}{dz} = \frac{\theta}{T}(\gamma_a - \gamma)$, we obtain after some reductions

$$\frac{1}{pF}(\gamma_a - \gamma) = \frac{1}{p'F'}(\gamma_a - \gamma'), \tag{1}$$

hence, finally

$$\gamma' = \gamma_a - \frac{F'p'}{Fp}(\gamma_a - \gamma). \tag{2}$$

It is obvious that the spreading out affects the layer in the same way as the descent, i.e., diminishes its thickness, and vice versa. For our purposes in the following, and also for simplification, we will consider F to be constant, in spite of the fact that its variations are actually as important as the variations of the pressure. Equation (2) can be written in a form more readily explained, by assuming Δp = p' - p:

$$\gamma' = \gamma - (\gamma_a - \gamma)\frac{\Delta p}{p}. \tag{3}$$

The case most often encountered is $\gamma < \gamma_a$, i.e., when the stratification of the layer is stable before its descent. $\gamma' < \gamma$ and the lapse

rate decreases, when $p > 0$, i. e., when the air parcel is falling, whereby γ' becomes zero and even negative, if Δp is large enough. Thus an isothermal process and even a temperature inversion can take place as a result of the subsidence and spreading out of air masses. Such a process often takes place in the center of stationary anticyclones, where extensive inversions of temperature are observed, called inversions of ascent or of subsidence according to their origin. Since the stratification becomes more stable the smaller the lapse rate as compared with the dry-adiabatic lapse rate, the result obtained can be formulated in the following way: the subsidence of air having an originally stable stratification increases the stability of the atmosphere; contrariwise, the stability decreases when a stable stratified layer rises, i. e., when $\Delta p < 0$.

The effect of ascending and descending motion is reversed in the rare cases where the stratification is unstable to start with, when $\gamma > \gamma_a$, the downward motion causes an increase of the lapse rate, i. e., increases the instability; but the ascending motion diminishes the lapse rate, and consequently diminishes the instability too.

If the lapse rate is initially equal to the adiabatic lapse rate $(\gamma = \gamma_a)$, it remains equal to it also after descent. This result brings us to the conclusion that an atmospheric state cannot lose its stability merely as a result of adiabatic vertical motions. The results obtained above are given in Table 21.

Let us have a look at Table 22, which gives the values of the lapse rate after the adiabatic descent of the air layer, and which was calculated on the basis of equation (3) for p = 500 mb.

It should be noted that although the above theory of formation of subsidence inversions seems plausible at a first glance, but nevertheless the intensity of the inversions obtained according to this theory are incompatible with the more intensive inversions observed in stationary anticyclones. Inversions of great intensity cannot develop at positive lapse rates, as Table 22 shows; but an adiabatic descent of several kilometers is necessary for their growth even at negative lapse rates, which is not very probable. It seems that the intensive anticyclone inversions have their origin in some other sources of a dynamic character, whereas the adiabatic descent only augments the effect of these dynamic factors.

Table 21

The change in atmospheric stability during vertical motions of large air masses

Stratification / Vertical motions	Stable $\gamma < \gamma_a$	Unstable $\gamma > \gamma_a$	Indifferent $\gamma = \gamma_a$
Subsidence ..... $\Delta p > 0$	$\gamma' < \gamma < \gamma_a$ Stability increases	$\gamma' > \gamma > \gamma_a$ Stability decreases	$\gamma' = \gamma = \gamma_a$ The stratification remains constant
Ascent ..... $\Delta p < 0$	$\gamma_a > \gamma' > \gamma$ Stability decreases	$\gamma_a < \gamma' < \gamma$ Stability increases	$\gamma' = \gamma = \gamma_a$ The stratification remains constant

Table 22

The values of the lapse rates γ' after a dry air subsidence

γ \ Δp	10	50	100	200	300	500 mb
0.8	0.796	0.78	0 76	0.72	0.68	0.60
0.6	0.592	0.56	0.52	0.44	0.36	0.20
0.4	0.388	0.34	0.23	0.16	0.04	— 0.20
0.2	0.184	0.12	0.04	— 0.12	— 0.23	— 0.60
0.0	0.020	— 0.10	— 0.20	— 0.40	— 0.60	— 1.00
— 0.2	— 0.224	— 0.32	— 0.44	— 0.68	— 0.92	— 1.40

§ 9. Polytropic Variation of Temperature with altitude

The temperature variation of an air parcel during the polytropic process will now be determined. From the equation of heat balance we have:

$$dT = \frac{R}{c_p - c_\pi} \cdot \frac{T}{p} dp, \tag{1}$$

from which we obtain, by introducing the altitude z through the fundamental equation of statics

$$\frac{dT_i}{dz} = -\frac{g}{c_p - c_\pi} \frac{T_i}{T_e}. \tag{2}$$

If the temperature of the ascending air parcel always equals the temperature of the surrounding atmosphere, we have

$$\frac{dT_i}{dz} = -\frac{g}{c_p - c_\pi}. \tag{3}$$

The magnitude

$$\frac{g}{c_p - c_\pi} = \gamma_\pi \tag{4}$$

may be called the polytropic lapse rate. Equation (4) and Table 23 show that the polytropic lapse rate increases together with c_π.

Table 23

Polytropic Lapse Rates

c_π	0.00	0.5	0.10	0.15	0.17	0.20	0.23	0.24
γ_π°/m	0.0100	0.0126	0.0171	0.0267	0.0342	0.0600	0.2400	∞

The case $c_\pi = c_v$ is of the greatest interest, i.e., when the density of the atmosphere does not change with altitude, and when the air parcel changes its temperature in such a way that its specific volume and density remain constant, while its temperature is constantly equalized with the temperature of the surrounding atmosphere. In such a case

$$\gamma_v = \frac{g}{c_p - c_v} = \frac{g}{R} = 3.42°/100 \text{ m},$$

i. e., the temperature must decrease by 3.42° per 100 m, if the density does not vary with the altitude. The density will increase or decrease with altitude, depending on whether the actual lapse rate will be greater or smaller than γ_v. This limiting lapse rate is called the lapse rate of "autoconvection". It has been assumed in the past that convection starts immediately and automatically without any external impulse, because the air density increases with altitude. However, this idea appears to be erroneous. According to the theory expounded above, the dry-adiabatic lapse rate of autoconvection is the limiting one (in the sense of stability) for dry air. According to contemporary views, an unstable equilibrium can exist in the atmosphere on any scale, if there are no disturbances to disrupt it. Lapse rates exceeding several times the dry-adiabatic lapse rate are, however, encountered very often in the lowest layers of the atmosphere. Their presence must be explained in some other way, but not exclusively by the absence of distrubances. A dry adiabatic lapse rate should exist in the atmosphere, if both the ascending and the descending air parcels were to change their state adiabatically. However, an air parcel situated in the lowest layer of the atmosphere in fact changes its state not at all adiabatically. Some other lapse rate must therefore serve as criterion of stability for the lowest layers of the atmosphere, which should be a polytropic and not a dry-adiabatic lapse rate. The value of this lapse rate is not constant – it varies both with altitude and time. This lapse rate is greater than the adiabatic one during the heating of the lower atmospheric layers, and smaller during cooling. Its value can be particularly great in the lowest layer, where the heating takes place nearly isobarically. It can be seen from Table 23 that the polytropic lapse rates can be of the same order of magnitude as those observed in the terrestrial surface layer of the atmosphere, Thus the great superadiabatic gradients observed in this lowest layer are still not indicative of a very unstable stratification, since the criterion of stratification stability must here be not the dry-adiabatic, but the polytropic lapse rate, the value of which can not be evaluated even approximately because it varies over a very wide range depending on the actual influx of heat to the air parcels in the lower layers of the atmosphere.

§ 10. The Criterion of Stability for Unsaturated Moist Air

The dry-adiabatic lapse rate also retains its significance as a criterion for moist air unsaturated by water vapor, if its composition is the same at all altitudes i. e., if the specific humidity of the atmosphere does not vary with altitude. The dry-adiabatic lapse rate will not, generally speaking, be a criterion of atmospheric stability, if the specific humidity varies with altitude. In order to verify this let us consider the equation for the convective acceleration in the form

$$\frac{d^2z}{dt^2} = g\frac{\rho_e - \rho_i}{\rho_i}. \tag{1}$$

Replacing ρ_i and ρ_e by their values, obtained from Clapeyron's equation:

$$\rho_i = \frac{p}{RT_{vi}}; \quad \rho_e = \frac{p}{RT_{ve}}, \tag{2}$$

we obtain

$$\frac{d^2z}{dt^2} = g\frac{T_{vi} - T_{ve}}{T_{ve}},$$

where T_{vi} and T_{ve} —are respectively the virtual temperatures of the independent air parcel and of the surrounding atmosphere, whence

$$T_{vi} = T_i(1 + 0.605\, s_i); \quad T_{ve} = T_i(1 + 0.605\, s_e). \tag{3}$$

In the equation (2) let us introduce the lapse rates

$$T_{vi} = T_{vi_0} + \frac{dT_{vi_0}}{dz} dz = T_{vi_0} - \gamma_a \Delta z,$$

taking into consideration that the process takes place without any moisture exchange with the surrounding medium, and that $s_i = s_0 = \text{const.}$ in this case. On the other hand, we have

$$T_{ve} = T_{ve_0} - \gamma_v \Delta z,$$

where $\gamma_v = -\frac{dTe}{dx}$ is the virtual temperature lapse rate in the surrounding air.

By putting $T_{vi0} = T_{ve0}$ we obtain

$$\frac{d^2z}{dt^2} = g\frac{\gamma_v - \gamma_a}{T_{ve}} \cdot \Delta z. \tag{4}$$

Thus, a virtual temperature lapse rate which equals the dry-adiabatic lapse rate, $\gamma_v = \gamma_a$ serves as criterion of stability for moist air.

The moist air is thus in a stable equilibrium, if $\gamma_v < \gamma_a$, in an unstable equilibrium, if $\gamma_v > \gamma_a$, and in an indifferent one, if $\gamma_v = \gamma_a$.

The virtual temperature lapse rate γ_v equals the molecular temperature lapse rate γ, if the content of water vapor does not vary with altitude. The virtual temperature lapse rate can differ essentially from that of the molecular temperature, if the specific humidity does vary with altitude.

Indeed, by differentiating equation (3) with respect to z, we obtain

$$\gamma_v = \gamma - 0.6T\frac{ds}{dz}. \tag{5}$$

Equation (5) shows that the conditions of stability may assume diverse forms for a humid atmosphere with a nonhomogeneous distribution of water vapor. The lapse rate can eventually reach superadiabatic values in the course of specific humidity growth with altitude, but the stratification remains stable. Such cases are rare, however. Sharp minima in the specific humidity altitude distribution $\left(\frac{ds}{dz} < 0\right)$ are more frequently observed in the atmosphere. The condition of stability then appears to be not the decrease, but the increase of the temperature with altitude (temperature inversion). This agrees with the well known fact that an inversion is often observed at the top of a cloud. It follows from the above analysis that the existence of inversion is necessary from the point of view of statics, if stability prevails. The cloudy layer will not be in a state of stable equilibrium, if the layer of drier air aloft is not hotter than the cloudy layer situated directly under it. Moving clouds are for the same reason often colder than the surrounding atmosphere at the same level, though the observed temperature difference is usually greater than that which can be explained by the foregoing analysis.

§ 11. Stability and Instability

The stability and instability of dry and unsaturated moist air have been considered in § 5, by proceeding from the conception of the equilibrium stability of a mechanical system. We have seen that it is easy to give an exact definition of stability and instability: air is in a stable equilibrium if the air parcel, being thrown out of the state of equilibrium by any exterior impulse and then left to proceed on its own, is retarded in the direction of the impulse and tends to return to its previous level. The equilibrium is unstable, if the air parcel is accelerated in the direction of the impulse and tends to move away from its initial level. Finally, the equilibrium is indifferent if the air parcel can remain at rest on any level and may be either accelerated or retarded.

However, the determination of the character of the stability of atmospheric stratifications on the basis of actual observations, seems less, owing to the following circumstances:

1. It was previously assumed that the independent air parcel moves vertically, while the remaining air remains at rest. The actual situation is different. Since air is a compressible and a continuous fluid, the displacement of an independent air parcel will immediately cause a compensatory flow in the surrounding medium; consequently, the density both of the rising air parcel and of the surrounding medium will be altered. Since the acceleration of the parcel depends on the difference between its density and that of the surrounding medium, the parcel's acceleration depends not only on the initial impulse and the initial stratification, but also on the kinematic conditions of the motion.

2. If the vertical motion causes a condensation of water vapor, the latent heat of evaporation liberated during this process causes additional changes in the density distribution. If the saturated air parcel, for instance, rises moist-adiabatically within the descending unsaturated air, and such cases do constantly occur in the atmosphere, the ascending saturated air will change its state moist-adiabatically, whereas the descending unsaturated air will do so dry-adiabatically. The heating of the descending air can be greater or less than the cooling of the ascending air, depending on the kinematic conditions.

3. It may happen that the stratification is stable with regard to small disturbances, and unstable with regard to sufficiently great ones, if the stability is slight.

The stability of atmospheric stratification is thus not only a problem of statics, but also of the dynamics of the atmosphere. Therefore, when solving the problem of the stability of the atmosphere, one cannot consider only the stratification of the atmosphere, but it is also necessary to make allowance for: 1) the nature of the disturbances, 2) the density changes which take place as a result of motion.

As we have seen in Chapter V, the state of equilibrium is characterized by the coincidence of the isobaric and the isosteric surfaces, as well as by their horizontal position. The atmosphere is baroclinic, and accelerations which tend to restore the barotropic state develop in it, if this condition is not realized.

In this way, isobaric-isosteric solenoids—positive in case of instability, negative in the case of stability—appear in the atmosphere when equilibrium is disturbed.

The conditions for stability were briefly examined above, assuming that the air parcel rises in a resting atmosphere. This assumption is quite accurate for those cases where the rising air parcel is small, and the compensating descending current consists of a vast air mass. The dry-adiabatic lapse rate is

then a criterion of stability for completely dry air, while the moist-adiabatic lapse rate is a criterion of stability for moist saturated air.

If the moist air is unsaturated, the dry-adiabatic lapse rate can be adopted as a reasonably accurate criterion of static stability, by considering it constant and using the virtual instead of the molecular temperature. The last condition is, however, not obligatory, because the change of humidity is usually insignificant, and may therefore be neglected. Only in rare cases of very rapid intermittent changes of humidity with altitude is it necessary to take into account its influence on the degree of stability of moist air.

These criteria, however, apply to the moist air only as long as the motion which began in an impulse does not take the parcel past the limit beyond which condensation starts in ascending parcels. Thus, although the criteria obtained for completely dry air are also applicable to unsaturated moist air when the impulses which disturb the air parcel's equilibrium are small, they cease to apply, if the impulse is large enough for the parcel to reach the level of condensation. Since atmospheric motions often transcend this level, new criteria applicable to strong impulses causing considerable displacements of moist air masses have to be elaborated. It is more convenient to study this problem by means of some aerologic diagram such as the emagram.

Let the curve ACED (Figure 59) represent the stratification curve, plotted according to the results of aerologic soundings. Further, let the curve ABCD represent the curve of change of state of the individual air parcel A. It can be seen from the diagram that the parcel remains colder than the surrounding atmosphere, until it has reached the level C, and it will therefore tend to regain its initial equilibrium position. The equilibrium will thus be stable with respect to not very great impulses. But the air parcel will be hotter than the surrounding atmosphere, if the applied impulse is so strong that the parcel ascends higher than the level C. The parcel will therefore acquire an upward acceleration, which will not diminish to zero until the parcel has reached the level D, where it will have the same temperature as the surrounding atmosphere. Thus, the equilibrium will be unstable with respect to impulses which are sufficiently great to cause the parcel to ascend to the level C; at a higher level, the air parcel acquires upwards acceleration due only to the specific stratification. This level is called the level of free convection. The level of free convection is different for various points of the stratification curve. The resistance to ascent, which must be overcome at the expense of the external energy (area ABC) in order to lift the parcel up to the level of free convection, is also different for the various points of the stratification curve; the quantity of energy liberated during free convection will also be different (area CED). It should be noted that the stratification of the layer ED is stable. The air parcel E will therefore encounter a resistance to its motion when it begins to move upwards. But the parcel A will continue to ascend when, having been raised to the level of free convection, it penetrates into this layer.

This example shows that latent instability can be realized by an upward impulse strong enough to raise the air parcel higher than a certain level. It is easy to see that this instability, in the case under consideration, cannot be realized by means of downward impulses, great as these may be.

Three essentially different cases of stratification may be distinguised, by comparing the slope of the stratification curve with the slope of the dry and moist adiabates: 1) absolute stability, 2) absolute instability, 3) conditional instability.

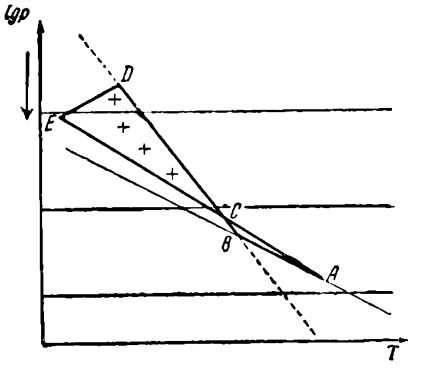

Figure 59. The stratification of the atmosphere is stable with respect to weak impulses, and unstable with respect to strong impulses

These cases will be examined separately.

1) Absolute stability is characterized by a lapse rate smaller than the moist-adiabatic one ($\gamma < \gamma_a'$). It can be seen from the curve Oa (Figure 60) that the stratification will in this case remain stable, regardless of the impulses applied to the air parcel and of the extent of the motions, caused by these impulses.

2) Absolute instability is characterized by a lapse rate greater than the dry-adiabatic one ($\gamma > \gamma_a$) . It can be seen from the curve Ob that the stratification will be unstable in this case, however small the applied impulses are, and wherever they may be directed.

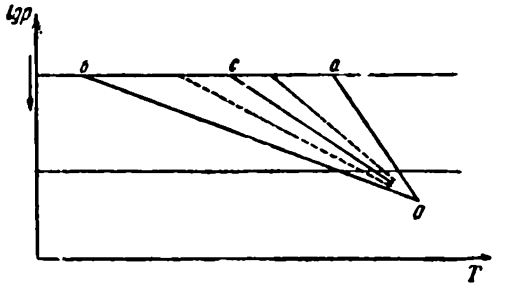

Figure 60. Different types of atmospheric stratification:

a) absolute stability ($\gamma < \gamma_a'$);
b) absolute instability ($\gamma > \gamma_a$);
c) conditional instability ($\gamma_a' < \gamma < \gamma_a$).

3) Conditional instability (the curve Oc) is characterized by the lapse rate of moist air, which is smaller than the dry-adiabatic, but greater than the moist-adiabatic lapse rate ($\gamma_a' < \gamma < \gamma$).

The realization of this conditional instability depends on the distribution of moisture. The following two types of conditional instability may be distinguished: the permanent and the latent instability.

Permanent instability. Let ABCD and A'B'C'D' (Figure 61) represent respectively the stratification curve and the distribution curve of the wet bulb temperature. Let the air parcel A ascend dry-adiabatically up to the condensation

level, and proceed moist-adiabatically after passing this level. A'aa' represents the moist adiabate or the curve of the pseudopotential wet bulb temperature of the parcel A. Another air parcel B will be now examined, and a curve B'b' of the pseudopotential wet bulb temperature plotted for it. It can be seen from Figure 61 that not a single curve of pseudopotential wet bulb temperature intersects the stratification curve. This means that any parcel ascending adiabatically from the layer under consideration will remain colder than the surrounding atmosphere. Such a stratification is permanent.

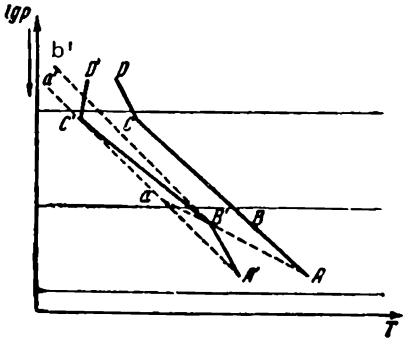

Figure 61. The permanent type of conditional instability

Latent instability. The stratification curve ABCDE and the distribution curve of the wet bulb temperature A'B'D' (Figure 62) will now be examined. Let us assume that the air parcel starts adiabatically on its ascent.

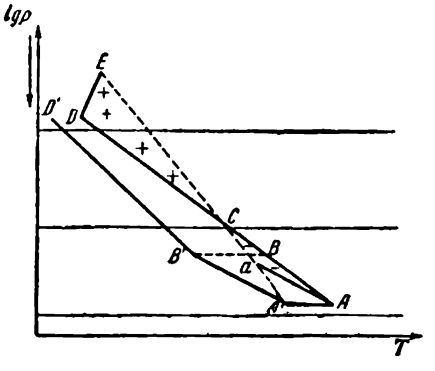

Figure 62. The latent instability. The curves of pseudopotential wet bulb temperature intersect the stratification curve

The parcel's temperature will vary dry-adiabatically up to level of condensation, and above this level according to the moist adiabatic curve, which passes through the point A'. Below the level C the parcel will remain colder than the surrounding atmosphere, and will therefore be stable with respect to small impulses. But if an impulse is applied, strong enough to send the air parcel ascending up to its level of free convection, then it will be found hotter than the surrounding atmosphere. An upward acceleration will therefore be imparted to it, which will not revert to zero until the parcel is raised to the level E, where its temperature will be found equal to the temperature of the surrounding atmosphere. The energy of instability, which has been latent in the stratification structure, is thus realized during the ascent of the air parcel. It can be seen from Figure 62 that latent instability exists below the level B. The stratification is permanent above this level, since there no curve of the pseudopotential wet bulb temperature can any longer intersect the stratification curve.

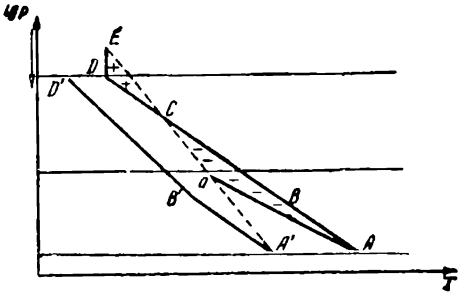

Figure 63. The pseudo-latent instability

The surface AaBC represents the amount of energy which is required to raise a unit of mass from A up to the level C of free convection. The area CDE represents the amount of the actually apparent "sensible" energy, referred to a unit of mass brought up from A. Two cases of latent instability must be discerned.

1) The area CDE is greater than the area AaBC (Figure 62). The amount of energy realized in this case is greater than the energy previously lost. An excess of energy is thus obtained, which is used up to overcome friction and also converted into kinetic energy. This may be called latent instability proper.

2) The area CDE is smaller than the area AaCB (Figure 63). In this case energy is lacking, and therefore must be taken from without to overcome the friction and for conversion into kinetic energy. This case may be called pseudo-latent instability.

The relation between the positive and the negative areas is an important factor in the forecasting of storms and other convective phenomena. However, it should be kept in mind that a positive area can only be obtained in the case when the negative area is so small that the applied impulse is sufficient to raise an air parcel up to the level of free convection.

Table 24 contains a summary of the various definitions and of the criteria of stability and instability, on the condition that the isolated individual air parcel ascends in a quiescent atmosphere.

Table 24

The criteria of stability, based on the assumption that the surrounding air is immobile

1. Unsaturated moist air; the moving parcel does not attain saturation:

   a) absolute stability..... $\gamma < \gamma_a$ or $\frac{d\theta}{dz} > 0$,

   b) indifferent equilibrium..... $\gamma = \gamma_a$ or $\frac{d\theta}{dz} = 0$,

   c) absolute instability ..... $\gamma > \gamma_a$ or $\frac{d\theta}{dz} < 0$.

2. Saturated moist air: the moving parcel retains its original saturation:

   a) absolute stability..... $\gamma < \gamma'_a$ or $\frac{d\theta_{se}}{dz} > 0$,

   b) indifferent equilibrium..... $\gamma = \gamma'_a$ or $\frac{d\theta_{se}}{dz} = 0$,

Table 24 (cont'd)

c) absolute instability ..... $\gamma > \gamma'_a$ or $\frac{d\theta_{se}}{dz} < 0$.

3. Unsaturated moist air: the moving parcel attains a state of equilibrium:

a) absolute stability ..... $\gamma < \gamma'_a$ or $\frac{d\theta_{se}}{dz} > 0$,

b) conditional instability ..... $\gamma'_a < \gamma < \gamma_a$.

b') the permanent type	The curves of the pseudopotential wet bulb temperature do not intersect the stratification curves (Figure 61).
b'') the latent type	Any curve of the pseudopotential wet bulb temperature intersects the stratification curve.
b''_1) pseudo-latent instability....	The positive area on the emagram is smaller than the negative one (Figure 62).
b''_1) proper latent instability....	The positive area on the emagram is greater than the negative one (Figure 63).

c) absolute instability ..... $\gamma > \gamma_a$ or $\frac{d\theta}{dz} < 0$.

§ 12. The Layer Method

In the course of our foregoing examination of the stability of atmospheric stratification we made the assumption that the individual air parcels move in a resting atmosphere. The real situation is different. Since atmospheric air is, broadly speaking, an incompressible and homogeneous fluid, the motion of air parcels will cause immediate compensatory flow in the surrounding atmosphere, and consequently also a change of state. Thus, the ascending air parcel is enveloped in a descending air current, whose state is in transition. The growth of cumuli is a typical and most important example of such a process, when a growing cloud rises moist--adiabatically in a dry-adiabatically downward-bound current of air. In the present section we shall explain how the previously obtained criteria are modified by making allowance for the presence of ascending or descending currents. In addition, not a single isolated air parcel will be examined, but an entire air layer; the method is therefore called the layer method (in contrast to the preceding air parcel method).

Thus, let us imagine any isobaric layer of the atmosphere, sufficiently extended in its level to include both ascending and descending currents within its limits. The following assumptions will be made:

1) No horizontal motions exist within the layer, and if they do they are of a kind that do not give rise to influx or redistribution of air masses within the confines of the layer under consideration.

2) The air is initially quiescent, and the density field is barotropic.

3) The change of state takes place adiabatically.

The first assumption indicates the fact that equal amounts of air are transported upwards and downwards through any isobaric surface, or through any isobaric unit layer, i. e.,

$$\int_{S'} \rho' w' ds' = \int_{S} \rho w \, ds, \tag{1}$$

where S' is the area occupied by the ascending currents; S – the area occupied by the descending currents; ρ' and ρ – the densities of the ascending and descending air; w' and w – the upwards and downwards vertical velocities.

Denoting by M' the mass of the ascending air contained in a unit layer, and by M – the mass of the descending air contained in the same layer, we can rewrite equation (1) in the form

$$M' w' = -M \cdot w, \tag{2}$$

where w', w are now the mean velocities of the ascending and descending air currents. The second assumption admits the coincidence of the isobaric and the isosteric surfaces, and consequently the absence of the isobaric–isosteric solenoids at the initial moment.

The third assumption means that the local temperature variations inside the layer are caused only by the convection of differently heated air masses, in which an adiabatic change of state takes place. The local rate of temperature variation is therefore given by the following equations:

$$\begin{aligned}
&\frac{\partial T'}{\partial t} = w' (\gamma - \gamma_a) && \text{in ascending unsaturated air} \\
&\frac{\partial T'}{\partial t} = w' (\gamma - \gamma'_a) && \text{in ascending saturated air} \\
&\frac{\partial T}{\partial t} = w \ (\gamma - \gamma_a) && \text{in descending unsaturated air} \\
&\frac{\partial T}{\partial t} = w \ (\gamma - \gamma'_a) && \text{in descending saturated air.}
\end{aligned} \tag{3}$$

Tne entire air layer will be set in motion, part of it moving upwards and the other part downwards, if an upward directed impulse is imparted to any mass of air in the layer. The isobaric surfaces remain horizontal as before, whereas the isosteric surfaces are usually deformed, and are thus brought to intersect the isobaric surfaces. In consequence of the local changes of temperature, heat is accumulated inside the layer; the rate of this accumulation equals

$$\frac{\partial Q}{\partial t} = c_p \left(M \frac{\partial T}{\partial t} + M' \frac{\partial T'}{\partial t} \right) = c_p (M + M') \frac{\partial T}{\partial t} + c_p M' \left(\frac{\partial T'}{\partial t} - \frac{\partial T}{\partial t} \right). \tag{4}$$

This accumulated heat as expressed in the right-hand side of equation (4) is divided into the part taken up for a "uniform" heating (or cooling) of the whole layer at the rate $\frac{\partial T}{\partial t}$, which is the heating (or cooling) due to the downward motion, and

that used for "additional" heating (or cooling) of the ascending air at a rate of $\left(\frac{\partial T'}{\partial t} - \frac{\partial T}{\partial t}\right)$

It is obvious that the second term, which can be called the solenoid-producing term, represents the amount of heat converted into kinetic energy. The equilibrium will be unstable if the ascending air is hotter than the surrounding descending air, and vice versa.

All the possible cases of air ascending in a descending current will now be examined:

1) Unsaturated air. The upward motion in this case, as can be seen from equation (3), causes a local heating, whereas the downward motion causes a local cooling, if $\gamma > \gamma_a$. A reverse process takes place if $\gamma < \gamma_a$.

By substituting equations (2) and (3) in equation (4) we obtain:

$$\frac{\partial Q}{\partial t} = -c_p\left(1 + \frac{M'}{M}\right) M'w' (\gamma - \gamma_a) + c_p \left(1 + \frac{M'}{M}\right) M'w'(\gamma - \gamma_a) = 0. \quad (5)$$

The vertical motions, as can be seen from equation (5), cannot cause either gain or loss of heat in the layer under consideration. They only cause a horizontal redistribution of heat, namely: an additional heating of the ascending air takes place at the expense of the uniform cooling of the layer, if $\gamma > \gamma_a$, and vice versa; the isotherms will be deformed, and the solenoid-producing term will have the form of

$$c_p\left(1 + \frac{M'}{M}\right) M'w' (\gamma - \gamma_a).$$

Thus, if an upward impulse is imparted to an air mass M' whose lapse rate is γ and which has initially been in a state of rest, the mass will gain an additional acceleration in the direction of the impulse, if $\gamma > \gamma_a$; acceleration will be absent if $\gamma = \gamma_a$, finally, the air mass will be retarded in the direction of the impulse if $\gamma < \gamma_a$.

Consequently, the criteria of stability previously deduced by the air-parcel method remain valid also for unsaturated moist air, when compensation currents are present in the surrounding atmosphere. However, this method gives a nonconstant value of the energy of instability.

Attention must be drawn to the fact that the sign of the solenoid-producing term does not depend on M and M'. This means that the unsaturated moist air responds equally to impulses of arbitrary magnitude.

2) Saturated air. By repeating the same reasoning for the case when the ascending and descending air masses are saturated, we shall arrive at the same conclusion, namely: the criteria of stability, as deduced for saturated air according to the air-parcel method, remain accurate even in the case, when the saturated air ascends adiabatically in a current of adiabatically descending saturated air.

3) The moist-adiabatic ascent of air through a dry-adiabatically subsiding current.

This case is of the greatest interest. It corresponds to the formation of cumuli (Figure 64). The symmetry which was characteristic for the first two

cases is not retained in this case, and equation (4) acquire the form

$$\frac{\partial Q}{\partial t} = c_p w' M' (\gamma_a - \gamma'_a) =$$
$$= c_p w' M' \left(1 + \frac{M'}{M}\right)(\gamma_a - \gamma) + c_p w' M' \left[\gamma - \gamma'_a - (\gamma_a - \gamma)\frac{M'}{M}\right]. \quad (6)$$

The left-hand side of equation (6) represents the heat liberated per unit time by the condensation of water vapor in the ascending air mass M', if the velocity of ascent is w'. This heating is directly proportional to the rate of ascent of the rising mass. It is also proportional to the difference $\gamma_a - \gamma'_a$ or to the angle between the dry and moist-adiabatic curves. This angle diminishes with decreasing temperature, since the two adiabatic curves converge when the temperature decreases.

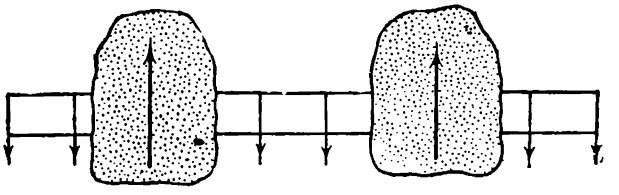

Figure 64. The moist-adiabatic ascent of air, accompanied by a compensating dry-adiabatic subsidence

The first term on the right side $c_p w' M' (1 + \frac{M'}{M})(\gamma_a - \gamma)$, which represents the uniform heating (cooling) of the whole layer, is directly proportional to the upward velocity of the saturated air and to its mass; it is also directly proportional to the difference $\gamma_a - \gamma$, i. e., to the angle between the stratification curve and the dry-adiabatic curve. The layer is uniformly heated, if This term is usually positive, an important part of the heat of condensation transferred to the layer being used for its uniform heating.

Finally the last term on the right-hand side of equation (6)

$$c_p w' M' \left[\gamma - \gamma'_a - (\gamma_a - \gamma)\frac{M'}{M}\right] \quad (7)$$

represents that part of the liberated heat, which can be converted into kinetic energy. The kinetic energy increases if this term is positive, which corresponds to unstable equilibrium; it decreases and the stratification is stabilized, if this term is negative. The stratification is therefore

$$\begin{aligned} &\text{unstable, if } \gamma > \gamma'_a + \frac{M'(\gamma_a - \gamma)}{M}, \\ &\text{indifferent, if } \gamma = \gamma'_a + \frac{M'(\gamma_a - \gamma)}{M}, \\ &\text{stable, if } \gamma < \gamma'_a + \frac{M'(\gamma_a - \gamma)}{M}. \end{aligned} \quad (8)$$

Thus, the nature of the stratification is not clearly defined by the lapse rate, but depends on a number of subsidiary conditions. All the possible cases of atmospheric stratification, beginning with most extreme, will now be examined.

1) It follows from equation (8) that the stratification is unstable, if $\gamma > \gamma_a$. This is correct for any relation between M and M'. Thus, the air layer in this

case responds similarly to disturbances of different magnitude, and the criterion of stability agrees with the criterion which was determined by the air parcel method.

2) It follows from equation (8) that the equilibrium is stable if $\gamma < \gamma'_a$, This is also correct for any relation between M and M', i. e., the behavior of the air does not depend on the horizontal extent of the disturbance. Thus, the criterion of stability agrees in this case too with that determined by the air parcel method.

3) It follows from equation (8) that for any relation between M and M', for $\gamma = \gamma'_a$, the equilibrium is stable and not indifferent, as was determined by the air parcel method.

4) $\gamma'_a < \gamma < \gamma_a$. This is the most frequent, and therefore the most important case from the practical point of view. It can be seen from equation (8) that the equilibrium can be of any kind, depending on the relation between M and M'. The condition (8) proves that the air will react differently to disturbances of a different horizontal extent. The air will be in an unstable equilibrium if the impulse imparted to it is strong enough, and in a stable equilibrium if the impulse is small. Thus, the air responds unequally to different impulses imparted to it from below. Such a stratification is called conditionally unstable

Chapter VII

RADIATION

§ 1. The Fundamental Laws of Radiation

Radiative transfer is a frequently observed form of energy exchange. Since only electromagnetic radiation will be treated in this book, the term "radiation" will only be used to describe the emission of electromagnetic waves by bodies and the transfer of radiant energy from one body to another.

Radiation and (internal) heat energy are quite different from each other. Radiation energy is transformed into heat energy only when material particles are present. The transfer of heat energy takes place only in a physical medium. Radiation energy can also be transferred in the absence of a physical medium - in a vacuum, meaning that none of the known material particles are present. In meteorology, however, we only deal with propagation of radiation energy in a physical medium — the atmosphere.

All electromagnetic waves propagate in the same medium with an equal velocity c. They only differ from each other in their wave length λ or in the corrosponding frequency of vibration ν . The following relation exists between λ and ν

$$\lambda = \frac{c}{\nu} .$$

The velocity of propagation of electromagnetic waves in the atmosphere is close to the velocity of light in vacuum, $c_0 = 2.9977 \cdot 10^{10}$ cm sec $^{-1}$.

Electromagnetic waves can be differentiated according to wave length. At present, the entire spectrum of electromagnetic waves is known, almost without exception, from 100,000 km to 10^{-11} cm. In meteorology we restrict ourselves to the examination of a comparatively small range, from 0.1 μ to 100 μ (μ = micron = 10^{-4} cm). The waves corresponding to this region are also called light waves, although only wave lengths of 0.4 to 0.7 μ directly stimulate our visual nerves. However, shorter waves can cause perception of light,though indirectly, when falling on certain substances. Such short light waves are called ultra-violet waves. Waves with $\lambda > 0.7$ μ are called infra-red waves.

According to the law of conservation of energy, radiation can only take place at the expense of other kinds of energy (heat, electrical energy, chemical energy). Light can therfore be emitted only by physical bodies, and not by geometric forms or surfaces. Radiant energy most frequently originates as a result of the transformation of the internal energy of solid, liquid or gaseous substances. Such radiation is called thermal radiation.

Two classes of light waves will be considered: "short-wave radiation", which belongs to the solar spectrum and extends approximately from 0.3 to 3.0 μ and "long-wave radiation ", which is emitted both by the terrestrial surface and the atmosphere, and lies between the limits of 3 μ and 30 μ . Actually these limits are approximate and not exact, since the region of the spectrum corresponding to

solar emission of light and heat extends farther on both sides of this interval. The energy lying beyond the limits just mentioned (which should be well remembered), comprises however, only 1 % of the total energy. The upper limit of 30 μ may be regarded as too low for a long wave radiation. The small amounts of water vapor present in the air completely absorb this radiation.

Radiant energy emanating from the body is irradiated in all the directions. The radiating body could lose all its store of internal energy if it did not receive energy from without. This does not happen in nature because energy emitted by other bodies flows in uninterruptedly and is absorbed by any body on which it is incident.

Every body emits energy under all conditions, and at any temperature, even the lowest. At the same time it continuously absorbs the flux of energy emitted by other neighboring bodies. Constancy of temperature, and therefore of internal energy, cannot persist in a state of static equilibrium since an uninterrupted flux of energy from the body takes place even at a constant temperature. But this loss is compensated by the absorption of radiant energy falling on the body, and its conversion into internal energy. We have thus a case of dynamic equilibrium. The decrease in internal energy during the cooling of a body by radiation can be used as a measure not of the total flow of energy out of the body, but of the difference between the outgoing and the absorbed fluxes.

Radiant energy propagates rectilinearly in a homogeneous medium with velocities which can be accurately computed from measurements, although they much exceed these to which we are generally used.

In thermal radiation the body emits or absorbs radiations of different wave length to a varying extent.

The emission of an absolutely black body, i.e., a body which completely absorbs incident radiation of any wave length, is of great importance from the theoretical point of view. Let the ratio of radiant energy absorbed by a body to the energy incident on it be called the absorption coefficient a of the body; then it can be said that an absolutely black body is a body whose absorption coefficient is equal to unity for all wave lengths and under all conditions. The absorption coefficients of other bodies depends on the temperature and on the wave length, i.e., their $a = a(\lambda, T)$. The body is called gray if a is equal for all wave lengths, though smaller than unity. The absorption of a body is selective if a depends on λ It stands to reason that there are no absolutely black or gray bodies in nature.

Of all existing bodies, carbon black and platinum black have the highest absorption coefficients, amounting to about 0.98. Absolutely black and gray bodies are, however, useful auxiliary concepts, facilitating the study of various radiation problems.

Absorption and emission of energy are both characteristic of physical objects. An adopted measure of the emission capacity is the quantity of energy emitted by the body per unit time, through a unit surface element and within a unit solid angle (for an isotropic body). This emission coefficient depends only on the properties and on the state of the emitting body, and not on the properties and the state of the surrounding medium. For thermal emission, the state of a body is determined by the absolute temperature T. The emission coefficient is, besides, a function of λ and T, i.e., $e = e(\lambda, T)$, since it is different for different wave lengths. The emission coefficient of a black body is greater than that of any other body at the same temperature.

Kirchhoff's law (1882) is the fundamental law of thermal radiation; the

ratio of the emission coefficient of a body e (λ , T) to its absorption coefficient a(λ , T) according to this law does not depend on the properties of the body, but only on the wave length λ , and on the temperature T;

$$\frac{e(\lambda,T)}{a(\lambda,T)} = E(\lambda,T) \tag{1}$$

a = 1 for an absolutely black body; therefore, E (λ , T) is the emission coefficient of an absolutely black body. Kirchhoff's law is correct only for monochromatic radiation, i.e., for radiation having a constant wave length. The equality e (λ , T) = a (λ , T) = 0 may hold for an exceptional case. In such a case the body is transparent to radiation of a wave length λ. Many bodies, particularly gases and vapors, have selective absorption properties. Therefore they emit radiation of only definite wave lengths.

According to Kirchhoff's law, bodies which absorb well emit well, and vice-versa. Thus, for example, rough surfaces emit heat well but smooth, polished surfaces emit badly.

Dry air is nearly transparent for all the wave lengths, and its emission coefficient is practically zero. Water vapor is almost transparent to solar short-wave radiations but absorbs long waves, which are emitted at terrestrial and atmospheric temperatures.

The energy distribution in the emission spectrum of an absolutely black body is given by Planck's law, deduced by him in 1900 on the basis of the assumption of quantum emission*:

$$E(\lambda, T)\, d\lambda = \frac{2c^2 h \lambda^{-5} d\lambda}{e^{\frac{hc}{\lambda kT}} - 1}, \tag{2}$$

where E (λ, T) d λ is the radiant energy of wave length in a spectral interval d λ emitted in vacuum by a square centimeter of the surface of an absolutely black body at a temperature T, into a unit solid angle, and where

c = 3×10^{10} cm/sec - the velocity of light

k = 1.38×10^{-16} erg/deg - Boltzmann's constant, which is the gas constant per molecule.

h = 6.61×10^{-27} erg·sec - Planck's constant ("the elementary quantum of action").

Expressing the radiation in cal/min· cm^2, the wave length in microns, and putting

$$c_1 = 2c^2 h = 1.7 \times 10^{-11} \text{ cal cm}^2 \text{min}^{-1}$$

$$c_2 = \frac{hc}{k} = 1.43 \text{ cm deg},$$

we obtain Planck's law in the form:

$$E(\lambda, T)\, d\lambda = \frac{c_1 \lambda^{-5} d\lambda}{e^{\frac{c_2}{\lambda T}} - 1}. \tag{3}$$

The function E (λ , T) vanishes at λ = 0 and $\lambda = \infty$ and has at some λ a maximum for any constant value of the product λ . T. The energy emitted by a black body at a fixed temperature attains its maximum at a clearly defined wave length. We thus obtain Wien's law of displacement:

$$\lambda_{max} = \frac{b}{T}, \tag{4}$$

where $b = 0.2884$ см deg $= 2884$ μ deg.

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* [Actually Planck advanced the quantum hypothesis as an explanation of his previously obtained semiempirical law. — Translator's note.]

The radiation intensity at λ_{max} will be

$$E_{max} = 5.98 \times 10^{-15} \times T^5 \text{ cal/min cm}^2$$

Thus, the higher the temperature of the black body, the smaller the wave length corresponding to the maximum intensity of radiation, and the greater the maximum intensity itself. For the visible part of the spectrum this law corresponds to the well known phenomenon of a hot body whose luminosity increases with temperature.

It was shown by Stefan's experiments (1879) that the total emission of a black body is proportional to the fourth power of the absolute temperature. Stefan's assumption that his law was accurate for all bodies has proved to be wrong. Boltzmann (1884) derived this law theroretically and proved that it is accurate only for absolutely black bodies. Stefan-Boltzmann's law can be obtained as a result of Planck's law, by integrating Planck's equation for all the wave lengths from 0 to ∞:

Hence

$$\int_0^\infty E(\lambda, T)\, d\lambda = c_1 T^4 \int_0^\infty \frac{(\lambda T)^{-5} d(\lambda T)}{e^{\frac{c_2}{\lambda T}} - 1} = c_1 T^4 \int_0^\infty \frac{u^{-5} du}{e^{\frac{c_2}{u}} - 1} = \sigma T^4.$$

$$E = \sigma T^4, \tag{5}$$

where σ is the Stefan-Boltzmann constant.

$$\sigma = 5.75 \times 10^{-5} \frac{\text{erg}}{\text{cm}^2 \text{ sec deg}^4} = 0.825 \times 10^{-10} \frac{\text{cal}}{\text{cm}^2 \text{ min deg}^4}$$

(see Appendix 15).

Stefan-Boltzmann's law is a precise physical law for an absolutely black body. This law can be applied to real physical bodies only with certain restrictions. Since all physical bodies absorb radiation to a lesser extent than the perfect black body, they also emit less than the latter at the same temperatures. The spectra of gases are not continuous but are line spectra, because of the selectivity of absorption and emission of radiant energy. Strictly speaking, Stefan-Boltzmann's law, derived theoretically for an absolutely black body, cannot be applied to gases. This law will nevertheless be applied subsequently to atmospheric radiation, by introducing the factor f , which takes this atmospheric selectivity into account and which depends on the temperature of the atmosphere. Thus, Stefan-Boltzmann's law will be written for the atmosphere in the following form:

$$E = f \cdot \sigma \cdot T^4. \tag{6}$$

Furthermore, the radiation of the earth's surface must be taken into account in some of the problems. If we consider this to be a gray-body radiation, we can express Stefan-Boltzmann's law for gray bodies by the following equation:

$$E = q\sigma T^4, \tag{7}$$

where q designates the "factor of grayness of the body".
It is obvious that

$$0 < f < 1,$$
$$0 < q < 1. \tag{8}$$

Figure 65 gives the distribution of energy in the spectrum of an absolutely black body at a temperature range which corresponds to conditions on the earth's surface and in the atmosphere. All radiation intensities are computed for wave lengths lying in intervals 1 micron wide. It is easy to see that the wave length of the radiation having a maximum energy is displaced toward the short-wave region when the temperature increases, and that the total radiation increases with the increase of the temperature (see Appendix 16).

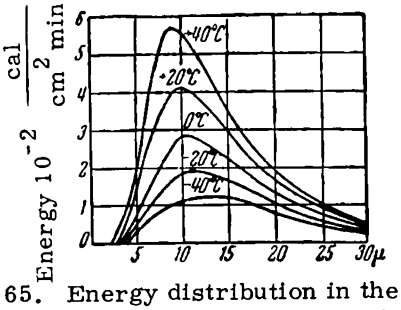

Figure 65. Energy distribution in the emission spectrum of an absolutely black body at different temperatures

§ 2. Solar Radiation

The Sun is the sole source of radiation which maintains the temperature both of the earth's surface and of the atmosphere, and which primarily causes atmospheric motions. The energy received from the stars and from the moon is negligible.

The sun is a yellow fixed star, situated at an average distance of 1.4910×10^{11} m from the earth's center. The sun's radius is 695, 300 km. The earth's orbit is an ellipse. The minimum distance from the earth to the sun, observed on 2 January, is 1.4701×10^{11} m, whereas the maximum distance, observed on 4 January, is 1.5200×10^{11} m.

The quantity of solar radiation energy, normally incident on a unit area of the upper boundary of the atmosphere, is equal to 1.88 cal/min·cm^2 (at the mean distance of the earth from the sun). This value of solar radiation is called the "solar constant". The word "constant" is not meant in the sense that the solar radiation is really constant; it only indicates the fact that great variations of solar radiation, caused by the changes in atmospheric transmittance, as well as by the variation of the distance between earth and sun, are excluded from the calculation of this quantity. The problem whether the small changes of the solar constant are caused by effective changes in the solar radiation, or are related to the partial elimination of the atmospheric influence, has not been elucidated until now. The variation in the value of the solar constant due to the change of the distance between Earth and sun amounts to 7 % at the utmost.

The effective temperature of the sun i.e., the temperature which an absolutely black body of the same size as the sun, situated at the same distance from the earth, should have in order to emit the same quantity of radiation, can be determined from equation (5) § 1. The effective temperature of the sun equals 5,700° K.

On the other hand, the so-called "chromatic" temperature of the sun can also be determined. The maximum energy in the solar spectrum outside the limits of earth's atmosphere corresponds to a wave length of 0.475 microns. It follows from Wien's law that the temperature of the sun is 6,090°. The difference between the effective and the chromatic temperature is explained by the fact that the continuous spectrum, emitted by the incandescent mass of the sun—the photosphere, is interrupted by a great number of dark lines, the so-called Fraunhofer's lines, which are a result of the absorption of radiation of the corresponding wave lengths

in the cooler sun envelope, the chromosphere. The total emission therefore decreases and the effective temperature is smaller than the chromatic temperature of the sun.

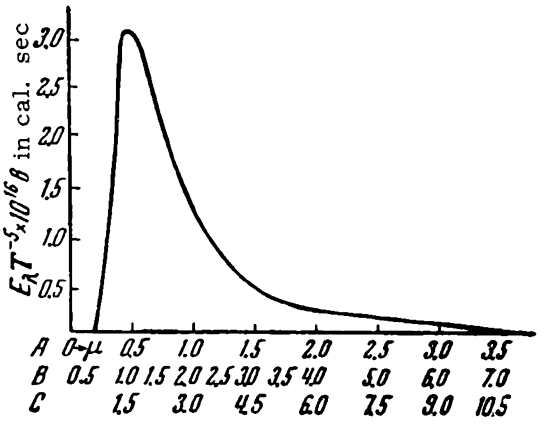

Figure 66. Theoretical emission curve of an absolutely black body. Scale A: T = = 6,000°K (the solar temperature); scale B: T = 300°K (~temperature of the earth's surface); scale C: T = = 200°K (~temperature of the stratosphere)

In the study of the absorption of solar radiation in the atmosphere and the emission from the earth and the atmosphere, it must be kept in mind that the solar radiation is effectively emitted by a body of surface temperature of about 6,000°K, while the earth and the atmosphere emit at temperatures of about 200°-300°K. Ninety-nine per cent of the energy emitted by a black body at the solar temperature belongs to wave lengths between the limits 0.17 μ -4 μ, with a maximum in the green-blue part of the spectrum at λ = 0.5μ; 56% of the energy corresponds to the visible part of the spectrum (0.4μ-0.8μ); 36%—to the infra-red (0.8μ-4.0μ) and 8%—to the ultra-violet (λ < 0.4) (see Appendix 17). Ninety-nine per cent of the energy emitted by a black body at temperatures observed at the earth's surface (T ≈ 300°K) belongs to wave lengths within the limits from 3 to 80 μ . The maximum energy corresponds to λ = 10μ . Ninety-nine per cent of the radiation emitted by a black body at temperatures of the stratosphere T ≈ 200°K, belong to wave lengths within the limits from 4 μ to 120 μ, with a maximum at λ = 15μ (Figure 66). Thus, the radiation of the earth surface and of the earth's atmosphere falls mainly in the infra-red part of the spectrum. Solar radiation is therefore usually called short-wave radiation, and that of the earth and the atmosphere long-wave radiation.

Solar radiation is attenuated in passing through the earth's atmosphere, because part of the radiation is scattered by the gas molecules of the atmosphere, and by the solid and liquid particles suspended in it, while another part is absorbed by the atmospheric gases and converted into their internal energy. Only the remnant reaches the earth's surface in the form of direct solar radiation. As a result of these phenomena of scattering and absorption, each of the molecules becomes a source of two sorts of radiation, namely: a short-wave scattered radiation, and a long-wave thermal emission of the atmosphere.

The two kinds of radiation are emitted in all directions. These fluxes of energy ultimately reach the earth's surface, or return into outer space or else are scattered and absorbed in the same way as the direct solar radiation.

Thus, three different fluxes of radiant energy are incident on the earth's

surface: direct solar radiation, scattered radiation, and atmospheric radiation. A part of the incident radiation is reflected by the earth's surface and is absorbed or scattered again by the atmosphere, or returns into outer space. The remaining energy is used up in heating the dry soil or the water surface, or in vaporization. The earth's surface in its turn radiates energy which is absorbed and scattered on its way in the atmosphere, and finally gets lost in outer space.

The clouds which are present in the atmosphere partly reflect the incident radiation and partly absorb it, and consequently they, too, become thermal emitters.

Short-wave solar radiation is absorbed in the atmosphere mostly by water vapor, and to a lesser extent by oxygen and by ozone in the upper layers of the atmosphere. The absorption lines and bands in the solar spectrum which are of terrestrial origin can be discerned from the other lines and bands due to absorption in the chromosphere. When a spectroscope is directed at the disc of the sun, the lines of solar origin evince the Doppler effect, caused by the rotation of the sun (about its axis) and which manifests itself by a displacement of these lines towards one or the other side of the spectrum, the lines of terrestrial origin showing no such displacement.

In addition, the intensity of the lines of terrestrial origin increases with the increase of the zenith distance of the sun, whereas the intensity of the lines of solar origin remains unchanged.

The most intense absorption of solar radiation is observed in the red and in the infra-red part of the spectrum. An intense absorption of ultra-violet radiation by ozone may also be noted. The short-wave end of the spectrum terminates suddenly at $\lambda = 0.3\,\mu$. This happens even in the case when the spectrograms are taken at great heights. It has been established that the absorption of ultra-violet radiation takes place at very great heights and is caused by the atmospheric ozone which is mainly concentrated in the layer from 20 to 30 km, although the maximum ratio of the density of ozone to the air density is observed at a height of 40 km. It may also be noted that numerous measurements of the ozone content and of the correlation between it and the meteorological elements revealed a strikingly close relationship between the amount of ozone present at these heights on one hand and the pressure distribution at the earth's surface, and also the potential temperature of the stratosphere, on the other.

The main water vapor absorption lines, located near the infra-red part of the spectrum, have wave lengths of 0.72μ; 0.81μ; 0.93μ; 1.13μ; 1.42μ; $1.89\,\mu$ (Figure 67). In addition, two broad bands belonging to water vapor are centered at about 2.01μ-2.05μ. Two narrow absorption lines in the spectrum belong to oxygen (0.69μ-0.76μ). The other gases (even carbon dioxide) do not absorb radiation at all. Any heating of the atmosphere, due to the absorption of direct solar radiation, is therefore negligible. The attenuation of solar radiation due to its absorption by the atmosphere is about 6-8 per cent of the total radiation, initially incident on a clear day when the sun is at the zenith. The major part of solar radiation hence passes right through the atmosphere on a clear day, and reaches the earth's surface where it is absorbed or reflected.

The solar radiant energy is also weakened as a result of scattering. The energy of radiation is decreased by its scattering in all directions if the rays pass through a transparent medium containing suspended fine particles with a different index of refraction from that of the medium. It has been shown by Rayleigh that the loss of energy caused by the scattering of light on molecules of air is inversely proportional to the fourth power of the wave length. Thus the

short-wave radiation is more intensely scattered than the long-wave radiation. Short waves therefore preponderate in diffuse scattered light, and this causes the blue coloration of the sky. On the other hand, the short-wave component of direct sunlight becomes less intense as the sun approaches the horizon. The wave length of maximum intensity is thus shifted towards the red part of the spectrum. Tobacco smoke, which is composed of particles whose dimensions do not exceed the molecular dimensions [sic] (0.2μ) scatters light of all wave lengths in the same way, the blue light being scattered more than the red. For this reason, smoke appears blue in diffuse light. The sun appears red when seen through tobacco smoke or through a fog composed of drops which are of the same size as smoke particles, because solar radiation loses its blue component as a result of scattering, and the maximum intensity is shifted towards the red part of the spectrum.

The coefficient of scattering is inversely proportional not to the fourth but to a lesser power of the wave length, if the diameter of the particles is greater than the wave length. The energy loss becomes independent of λ, and the radiation scattering becomes diffuse reflection, if the particles are large enough. This is confirmed by the fact that the light reflected from a cloud appears quite white when the sun is behind the observer, if the cloud is composed of drops with a diameter $d = 10\mu$. Exactly in the same way the sun appears to be white when observed through a cloud or fog composed of drops with a diameter of about 10μ. Since diffuse reflection is the same for all wave lengths, the azure coloring of the sky becomes less pure the more abundant are large particles in the atmosphere. These large particles consist mainly of dust, drops of water, and ice crystals; the hue of the sky thus makes it possible to estimate the amounts of these impurities. They are usually found in a greater quantity in the lower layers of the atmosphere. In order to obviate their effect, measurements of solar radiation are carried out in high mountain stations, far from the sources of pollution of the atmosphere, and situated above the lower cloud layer. Only the scattering of light by air molecules and, to a lesser extent, absorption in the higher layers of the atmosphere, affect the intensity of the solar radiations as measured in such mountain stations.

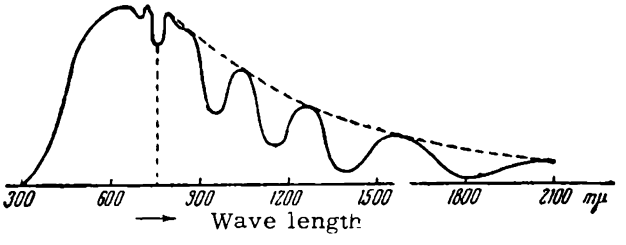

Figure 67. Energy distribution in the solar spectrum

The radiation scattered by gas molecules and diffusely reflected by the particles in suspension, is partly reflected back into outer space (possible after a second reflection which takes place in the atmosphere). Some amount of radiation is in addition reflected both from the earth's surface and from the clouds. The ratio of the radiant energy returned to space to the total incident radiant energy is called the albedo of the earth (including the atmosphere).

The value of the albedo depends on the physical properties of the reflecting surfaces. So, for example, the albedo is equal to 0.78 for a continuous covering of clouds; 0.81-0.85 for a fresh fall of snow. The albedo increases for a water surface with the zenith distance of the sun, being 0.29 at $\zeta = 78$ degrees and 0.62 at $\zeta = 86°$degrees. For grass it equals 0.10 to 0.33; for a rocky surface 0.12 to 0.15; for dry soil—0.14; for moist soil—0.08 to 0.09.

It follows from the cited values that the differences in the albedo are not significant if the earth's surface is not covered with snow. Cloudiness is of greatest importance for the earth's albedo. Thus, the albedo is about 0.10 when the sky is not covered with clouds, but equals 0.78 for a continuous overcast. The average value of the albedo for the whole surface of the earth can be accepted as equal to 0.42, assuming the mean cloudiness to be 52 %. The earth's albedo is greater for short-wave than for long-wave radiation.

§ 3. Absorption of Long-Wave Radiation in the Atmosphere

We have seen above that the absorption of the short-wave solar radiation in the atmosphere is not great, and that it is restricted only to some narrow bands belonging mainly to water vapor, and partly also to ozone and oxygen. The emission of radiant energy by bodies having the temperature of the earth and the atmosphere, gives rise to an entirely different extension of the spectral range, form 3 to 100 . Within these limits there is no appreciable absorption of radiant energy by the gases of which dry and pure air is composed. Only ozone has an intense absorption band at 9-10μ.

Carbon dioxide in this spectral range has one broad absorption band from 12μ to 16.3μ.

Long-wave radiation is absorbed mainly by water vapor which gives a series of broad absorption bands for these wave lengths. Numerous attempts to consider water vapor as a "gray body" which would absorb all the wave lengths, although to a lesser extent than an absolutely black body (in the same proportion for all the wave lengths), were only the first step in the study of the part played by water vapor in atmospheric radiation processes. Water vapor is actually not a gray body since its spectrum has strongly prominent broad absorption bands, together with bands of nearly complete transmission. The assumption of the "grayness" of absorption leads to the conclusion that the radiation from the earth and the atmosphere into outer space does not depend on the emission temperature, which contradicts the real situation.

It is therefore necessary to take into account the variation of the vapor's absorption coefficient with the wave length. The absorption coefficients obtained by Gettner (Hettner) for water vapor at T = 400°K and for a pressure considerably exceeding the actual pressure of atmospheric water vapor were used in meteorology until recently.

Gettner's experiments indicated an intense absorption band, whose center is situated at about 6.26μ, and a broad band, which begins at about 10μ, the absorption increasing towards the long-wave region up to 34μ, after which point Gettner carried out no further measurements. Subsequent research carried out for wave lengths greater than 10μ with moist air at room temperature, led to the same qualitative results as those obtained by Gettner, though the numerical values of the absorption coefficient were lower than Gettner's.

Simpson schematically divided the atmosphere into layers containing water vapor in an amount equivalent to 0.3 mm of precipitation, i.e., 0.03 g in an air column having a height h and a base of 1 cm^2, in order to simplify the investigations of absorption of long-wave radiations. If h is the height of an air column with a cross section equal to 1 cm^2, containing a quantity of water vapor equivalent to a mm of precipitation, i.e., to 0.1 a g of water vapor, then $0.1\ a = h\rho_w$, where ρ_w is the water vapor's density. Applying equation (1)

§2 Chapter IV, we obtain h = 4.62 $a \cdot \frac{T}{e}$, where h is expressed in meters, and e in mb.

Such a quantity of vapor (0.03 g), when encountered in the tropical zones characterized by their great humidity, can be contained in an air column 10 meters high; in the driest regions of the globe it is contained in an air column 1,000 m high, whereas in the upper layers of the atmosphere—in the whole stratosphere. Simpson also assumes that this layer contains 0.06 g of carbon dioxide. These data were chosen by Simpson on the assumption that they represent the quantities of water vapor and of carbon dioxide contained in the stratosphere.

Simpson, by using Gettner's data, with the exception of the region from 9 to 12 μ, divided the whole spectrum into three regions with respect to the absorption of radiation in each layer:

1. total transmission from 8.5 μ to 11 μ and not less than 4 μ;
2. total absorption from 5.3 μ to 7 μ and not more than 14 μ;
3. partial absorption from 4 μ to 5.5 μ, from 7 μ to 8.5 μ and from 11 μ to 14 μ.

The absorption by carbon dioxide takes place in the spectral region around 15 μ.

Simpson assumes that in the region of absorption, water vapor emits as an absolutely black body would at the same temperature. Let us examine Figure 68, borrowed from Simpson. The earth's surface emits practically as a black body at the same temperature. Let us assume T = 280° K. Thus, the energy distribution in the emission spectrum of the black body will be represented according to Planck's radiation law, by the curve ABCDEF, whereas the total energy emitted in unit time through a unit surface area will be expressed by an area bounded by this curve, and by the λ axis. The area NCDP will then give the amount of energy which is not absorbed by the atmosphere. This energy leaves the atmosphere, and is completely lost from the earth; it departs into outer space if there are no clouds in the atmosphere, or other suspended admixtures which are able to reflect this energy back to the earth, or to absorb it. This fact is of considerable importance, since the maximum intensity of the earth's radiation falls exactly within this spectral region.

The areas ABM and QEF represent the quantity of energy absorbed by a concentration of water vapor equivalent to 0.3 mm of precipitation. Partial absorption will take place in the spectral regions which correspond to the energy represented by the areas MBCN and DEPQ. It can be seen from the comparison of these areas that the greater part of the long-wave radiation is intercepted by the atmosphere, the smaller part leaving the atmosphere and departing into outer space, never to return.

The radiant energy absorbed by water vapor will of course be used up to heat not only this vapor, but also the dry air with which it is mixed; although dry air in itself does not absorb radiant energy (apart from the small amounts absorbed by carbon dioxide), it nevertheless effectively increases the heat capacity of the water vapor by being subjected in common with the vapor to influx and loss of heat.

The curve GHIJKF in Figure 68 also represents the distribution of energy in the emission spectrum of an absolutely black body at a stratospheric temperature T = 218° K.

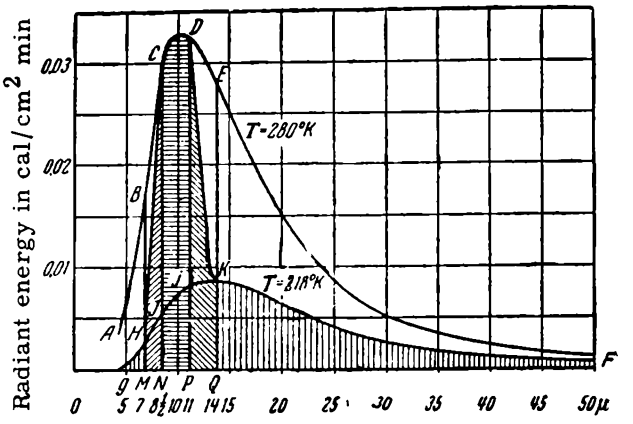

Figure 68. The balance of long-wave radiant energy in the atmosphere

The maximum intensity at this temperature corresponds to a wave length of 12.5 μ . The stratosphere will emit the energy corresponding to the areas GHM and QKF, and to the partial areas MHIN and PIKQ. The total energy loss by emission into outer space will therefore be given by the area GHCDKF.

Accepting an approximate evaluation, we find that the air layer which contains a quantity of water vapor equivalent to 0.3 mm of precipitation absorbs the total radiation in the 5.5-7 μ region, and even wave lengths exceeding 14 μ , if the partial absorption in the regions (3) is not taken into account. More exact computations show that water vapor absorbs 80 % of the long-wave radiation in this region of the spectrum. This percentage is greater if the absorption by carbon dioxide, whose maximum absorption corresponds to 15 μ , is taken into consideration. Since the lower layers of the atmosphere contain about 0.4 % of carbon dioxide, a unit air column containing 0.06 g of carbon dioxide absorbs 88 % of the long-wave radiation. This amount of carbon dioxide is contained in the lowest 1 km layer of the atmosphere. In Appendix 18 mean absorption coefficients are given for narrow spectral intervals from 5 to 35 μ ; these results were obtained in recent measurements.

The absorption coefficient of liquid water is so great that drops comparable in size to those found in clouds and fog absorb and emit radiation practically like an absolutely black body.

§ 4. The Heat Balance of the Earth

The mean annual average temperature both of the earth's surface and of the atmosphere can be regarded as constant (if we pay no attention to the variations over a long period). This can only occur providing that a radiative equilibrium is reached at the atmosphere's boundary which manifests itself by the fact that the total amount of energy transferred to the upper limit of the atmosphere equals the amount of energy emitted into outer space. We shall first deal with balance of radiant energy averaged over the whole globe. The geographic distribution of radiant energy will be examined afterwards.

The sun is practically the only supply of energy to the atmosphere and the earth. The other sources of energy (the radiations of the moon and stars, as well as the internal heat of the earth) have no importance for the energy balance of the atmosphere.

The total amount of energy transmitted from the sun to the earth is equal, as at the upper boundary of the atmosphere, to the solar constant I_0 multiplied by the area of the earth's great circle $I_0 R^2$, i.e., $I_0 \pi R^2$. This value must be divided by

the area of the globe $4\pi R^2$ and multiplied by 1,440 mins in order to obtain the mean value I_m of solar radiation, falling on $1 cm^2$ per 24 hours:

$$I_m = \frac{I_0}{4} \times 1{,}440 \approx 700 \frac{cal}{24\ hr \cdot cm^2}$$

Only 27 % of this energy reaches the earth's surface directly; 16 % reaches the earth in the form of radiation scattered by the atmosphere and clouds, while 15 % is absorbed both by the atmosphere and the clouds. The remaining 42 % is reflected back into outer space, 33 % of it by a direct reflection from the earth's surface and from clouds, and 9 % as a result of scattering by the air molecules.

In order to maintain a radiative equilibrium at the upper limits of the atmosphere it is necessary that, apart from the 42 % of short-wave radiation, 58 % of the long-wave radiation should be emitted into outer space, 50 % by the atmosphere and 8 % by the earth's surface. The earth's surface at a mean temperature of 17°C emits on the average 0.58 cal/cm^2 min, or 840 cal per 24 h. This is 120 % of the total incident solar radiation. However, the atmosphere, in its turn, transmits to the earth's surface 670 calories, i.e., 96 % of the total solar radiant energy. This is called counter-radiation. As a result, the radiation of the earth effectively amounts to 24 % only; 8 % is lost, as said, in outer space, and 16 % is absorbed by the atmosphere. The existence of the counter-radiation very substantially increases the temperature of earth's surface. The earth would have to emit 43 % of the energy (about 300 calories per 24 h), in order to maintain the radiative equilibrium, if the counter-radiation of the atmosphere did not exist.

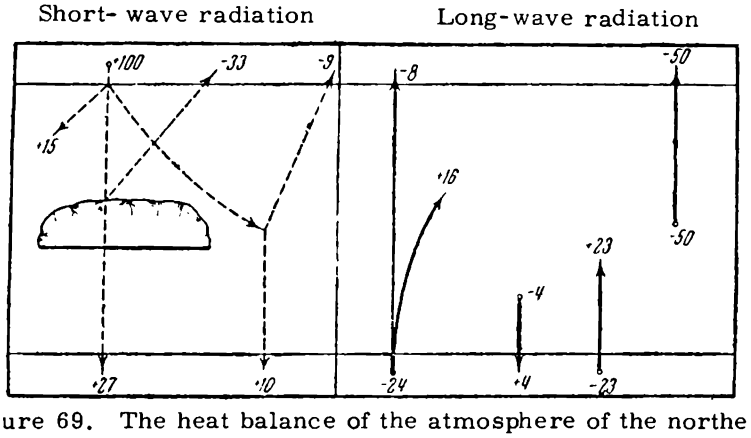

Figure 69. The heat balance of the atmosphere of the northern hemisphere

The mean temperature of the earth would then be t = −19°C. The mean temperature of the earth's surface actually equals 17°C, thanks to the counter-radiation. Thus, the counter-radiation of the atmosphere increases the earth's temperature by more than 30°. This is the so-called greenhouse effect of the atmosphere.

Other kinds of heat exchange, apart from radiative transfer, are of importance for the earth's heat balance: 23 % of the heat energy is transferred during condensation of water vapor in the upper layers of the atmosphere which has evaporated into the atmosphere from terrestrial sources, while 4 % of the energy is transferred from the atmosphere to the earth's surface by turbulent exchange. The given data are mean annual averages for the northern hemisphere; the conditions for the southern hemisphere have not been sufficiently studied.

A schematic distribution of the absorbed and emitted radiations is given in Figure 69. The left part of the figure represents the values of absorbed and emitted short-wave radiation, the radiation transferred both to the atmosphere and the earth being considered as positive, and the radiation lost by the atmosphere and the earth as negative. The right part of Figure 69 gives the values of heat transfer in the atmosphere by other modes, namely: long-wave radiation, latent heat, and the eddy transfer of heat.

Thus, a heat balance can be set up for the radiative equilibrium at the atmosphere's limit, and a separate one for the earth's surface and for the atmosphere, whereby both latent heat and eddy-transferred heat will be taken into account in the two last cases.

The earth-atmosphere system (at the upper boundary of the atmosphere)

Receives:		Loses:	
By short-wave solar radiation..	100 %	By reflection from the clouds and the earth's surface.....	33 %
		By scattering on air molecules	9 %
		By long-wave emission from the earth.....	8 %
		By long-wave emission from the atmosphere.....	50 %
	100 %		100 %

The earth's surface

Receives:		Loses:	
By direct solar radiation.....	27 %	By emission.....	120 %
By scattered radiation.....	16 %	By evaporation.....	23 %
By counter-radiation	96 %		
By eddy transfer.....	4 %		
	143 %		143 %

The atmosphere

Receives:		Loses:	
By absorption of solar radiation	15 %	By emission	146 %
By absorption of terrestrial radiation[intercepted radiation]	112 %	By eddy transfer	4 %
By condensation of water vapor	23 %		
	150 %		150 %

Chapter VIII

THE EQUATIONS OF HYDROMECHANICS, THERMODYNAMICS AND THE THEORY OF RADIATION

§ 1. The Methods of Lagrange and Euler

The motion of the atmosphere, as that of any fluid, can be studied by two fundamental methods. Both methods were developed by Euler. However, one of them is usually called Euler's method, and the other Lagrange's method. The object of study in Lagrange's method is the moving fluid, more precisely, the material particles which uniformly fill any moving volume element of the fluid. The variations of vector or scalar quantities which characterize the motion of any particle can be studied by following the motion of each particle separately. Variations which are spatially constant can be considered for the entire atmosphere at once, by passing from one moving air particle to another. The dynamic variables in Lagrange's method are regarded as functions of the time and of three other parameters, which serve as distinctive labels for the individual fluid particle. As such, parameters can serve, for example, the Cartesian coordinates a, b, c, which determine the position of the particle with respect to the fixed axes at an initial moment $t = t_0$. Thus, according to Lagrange, the variables a, b, c, t determine the values of vectors and scalars, characterizing the motion of the atmosphere. They are called the Lagrangean variables.

American research workers who have recently studied large-scale motions of the order of magnitude of the general circulation of the atmosphere, dealing with air volumes of the order of magnitude of tropospheric air masses, adopt the three mutually independent most conservative characteristics of air masses as Lagrangean variables, namely: the potential temperature θ, the specific humidity s and a third, new kinematic variable, the so-called potential vorticity ζ.

In Lagrangean variables the coordinates of the moving particle are determined by the equations

$$x = x(a, b, c, t); \quad y = y(a, b, c, t); \quad z = z(a, b, c, t); \tag{1}$$

and its radius vector by the equation

$$r = r(a, b, c, t). \tag{2}$$

The law of motion of a particular particle can be worked out from equations (1), when a, b, c, are fixed and t variable. From the same equations the spatial distribution of all particles at any fixed instant can be deduced, when a, b, c, are variable and t invariable.

The second method, developed by Euler, concerns, strictly speaking, not the fluid itself but a fixed space which is filled by the moving fluid. In this method of Euler attention is concentrated both on the secular changes of different dynamic variables at a fixed point in space, and on the changes of these variables at a fixed instant, when passing to adjacent points. Differently put, in Euler's method the different vector and scalar dynamic variables are regarded as functions of

the coordinates x, y, z and of the time t, i.e., as functions of four parameters, called the Eulerian variables. For example,

$$v = v(x, y, z, t); \quad j = \bar{j}(x, y, z, t); \quad p = p(x, y, z, t). \tag{3}$$

Thus, the different vector and scalar fields which characterize the motion of the fluid (the velocity field, the acceleration field, the field of pressure, etc), constitute the objects studied by Euler's method. The details of the fluid particle's motion prior to its arrival at some fixed point in space and of its subsequent motion are of secondary importance only in Euler's method.

Comparing the two methods, Langrange's method seems to have certain advantages from the physical point of view considering the motion as a whole. The mathematical difficulties arising in the application of Lagrange's method are greater than those encountered in Euler's method. Therefore we use Euler's method exclusively.

Let us consider any scalar function, given in Eulerian variables:

$$A = F(x, y, z, t). \tag{4}$$

By passing to the Lagrangean variables, we obtain

$$A = F[x(a, b, c, t), y(a, b, c, t), z(a, b, c, t), t] = \Phi(a, b, c, t). \tag{5}$$

It is obvious that the derivatives with respect to the Eulerian variables will be

$$\frac{\partial F}{\partial x}, \quad \frac{\partial F}{\partial y}, \quad \frac{\partial F}{\partial z}, \quad \frac{\partial F}{\partial t}.$$

The first three derivatives $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ determine the point-to-point variation of the magnitude A in a given spatial domain, and the last one $\frac{\partial F}{\partial t}$ the secular variation of this magnitude at a fixed spatial point.

The derivatives of this function with respect to the Lagrangean variables will be: $\frac{\partial \Phi}{\partial a}, \frac{\partial \Phi}{\partial b}, \frac{\partial \Phi}{\partial c}, \frac{\partial \Phi}{\partial t}$ The first three derivatives express the variation of the function A from one particle of air to another particle in its infinitesimal neighborhood, while the last derivative $\frac{\partial \Phi}{\partial t}$ expresses the variation of this function with time for an individual air particle.

The following relations between the Lagrangean and the Eulerian derivatives are evident, according to the rules of compound partial differentiation:

$$\left.\begin{aligned}
\frac{\partial \Phi}{\partial a} &= \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial a} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial a} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial a};\\
\frac{\partial \Phi}{\partial b} &= \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial b} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial b} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial b};\\
\frac{\partial \Phi}{\partial c} &= \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial c} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial c} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial c};\\
\frac{\partial \Phi}{\partial t} &= \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial t} + \frac{\partial F}{\partial t}.
\end{aligned}\right\} \tag{6}$$

§ 2. The Individual, Local and Convectional Derivatives

The last of equations (6) § 1 gives the so-called individual derivative $\frac{dF}{dt}$ of the function F in Eulerian variables.

Considering that $\frac{\partial \Phi}{\partial t} = \frac{dF}{dt}$, we obtain:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + v_x \frac{\partial F}{\partial x} + v_y \frac{\partial F}{\partial y} + v_z \frac{\partial F}{\partial z}. \tag{1}$$

The partial derivative $\frac{\partial F}{\partial t}$, which gives the temporal variation of the function F at a fixed spatial point, is called the local derivative. The variation

$$v_x \frac{\partial F}{\partial x} + v_y \frac{\partial F}{\partial y} + v_z \frac{\partial F}{\partial z} = (\boldsymbol{v}, \text{ grad } F), \tag{2}$$

of the function F as a result of spatial displacement of the individual particle is known in physics and hydromechanics as the convectional derivative. The last term of equation (1) $v_z \frac{\partial F}{\partial z}$, is often known as the convectional derivative in meteorology, whereas the sum of the first two terms $v_x \frac{\partial F}{\partial x} + v_y \frac{\partial F}{\partial y}$ is called the advectional derivative. The partial derivatives $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ are also called spatial derivatives.

Equation (1) is of importance in weather forecasting. By solving it with respect to $\frac{dF}{dt}$, we obtain

$$\frac{\partial F}{\partial t} = \frac{dF}{dt} - v_x \frac{\partial F}{\partial x} - v_y \frac{\partial F}{\partial y} - v_z \frac{\partial F}{\partial z}. \tag{3}$$

This expression shows that it is necessary to know the secular variations of the individual air particle, which are characterized by the derivative $\frac{dF}{dt}$ and by the spatial distribution of the values of the function F and of the velocity $\boldsymbol{v}$, in order to predict the variation of the function F at a given point.

The expression

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \tag{4}$$

is called the Eulerian operator*.

It should be noted that in practice one deals not with derivatives, but with finite difference quotients. But not all the quantities which appear in Euler's equation can be found with the same ease by observations.

The value of $\frac{\partial F}{\partial t}$ can be obtained at meteorological stations from the data of recording instruments. The values of $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$, and of v_x, v_y can be obtained from the observations of a network of stations, recorded on synoptic charts. Aerological observations are necessary in order to determine $\frac{dF}{dz}$, v_z ; these observations are also necessary in order to determine the values of the other variables which define the state of a free atmosphere. The determination of v_z and $\frac{dF}{dt}$ is the most difficult. No regular measurements of their values are carried out anywhere. Such methods for velocity measurements as have been tried until now (gliders, stabilized sounding balloons, etc) do not give reliable results. The approximate value of v_z can be obtained from the equation of continuity. The value of $\frac{dF}{dt}$ can be determined by observation on a stabilized free balloon.

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* [Also known as the mobile operator. Translator's note].

In spherical coordinates r, φ, λ Euler's operator has the following form:

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\dot{r}\frac{\partial}{\partial r}+\dot{\varphi}\frac{\partial}{\partial \varphi}+\dot{\lambda}\frac{\partial}{\partial \lambda}=\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+v_\varphi\frac{\partial}{r\partial \varphi}+v_\lambda\frac{\partial}{r\cos\varphi\partial\lambda}, \tag{5}$$

and in cylindrical coordinates r, θ, z – the form:

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\dot{r}\frac{\partial}{\partial r}+\dot{\theta}\frac{\partial}{\partial \theta}+\dot{z}\frac{\partial}{\partial z}=\frac{\partial}{\partial t}+v\frac{\partial}{r\partial r}+v_\theta\frac{\partial}{r\partial\theta}+v_z\frac{\partial}{\partial z}. \tag{6}$$

§ 3. The Deformation of an Air Body

It is known from theoretical mechanics that the instantaneous velocity $\boldsymbol{v}$ of any point of an absolutely rigid body, satisfies the equation

$$\boldsymbol{v}=\boldsymbol{v}_0+[\boldsymbol{\omega}, \boldsymbol{r}], \tag{1}$$

where $\boldsymbol{v}_0$ is the velocity of translation, common to all the points of the body, and $[\boldsymbol{\omega}, \boldsymbol{r}]$ is the linear velocity the point has in virtue of the rotation of the rigid body around its instantaneous axis; $\boldsymbol{\omega}$ of equation (1) designates the body's angular velocity, and $\boldsymbol{r}$ – the radius vector of the considered point of the body, which emanates from a point (pole) on the axis of rotation.

Each material particle of a moving air parcel participates in a more composite movement. Helmholtz, in his researches on fluid dynamics, developed a theory which is a generalization of the preceding description of rigid body motion. Helmholtz's theory can be formulated in the following way.

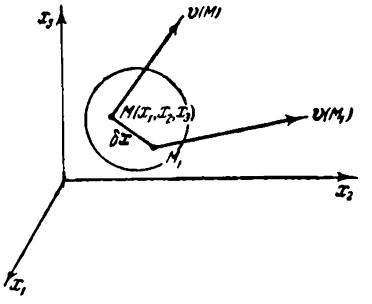

Figure 70. The motion of any point of a fluid body is composed of translation, rotation and deformation

The velocity $\boldsymbol{v}$ of any point M' of an infinitely small fluid particle can be regarded at any moment as a vector sum of the velocities of three components of the motion: 1) the translation of the particle as a whole, having the same velocity for every point M of the latter; 2) the rotational motion, with angular velocity equal to half the curl of the velocity $\boldsymbol{v}$ at the point M, around an instantaneous axis passing through the point M, and 3) the motion caused by the state of deformation at the point M, which is composed of dilatation and shear.

Let us consider a very small atmospheric air particle (Figure 70), and consider two adjacent points M and M', situated inside this particle. The coordinates of the point M will be denoted by (x_1, x_2, x_3) and those of the point M' by

$$(x_1+\delta x_1,\ x_2+\delta x_2,\ x_3+\delta x_3).$$

Let the velocity at the point M equal $\boldsymbol{v}(x_1, x_2, x_3)$; then the velocity at the point M' equals

$$\boldsymbol{v}'=\boldsymbol{v}(x_1+\delta x_1,\ x_2+\delta x_2,\ x_3+\delta x_3).$$

The velocities are assumed to be continuous functions of position.

Let each component of the velocity at the point M' be developed in a Taylor series around the point M; we will restrict these series to the terms, containing coordinate increments of the first order:

$$\begin{aligned} v_1' &= v_1 + \frac{\partial v_1}{\partial x_1}\delta x_1 + \frac{\partial v_1}{\partial x_2}\delta x_2 + \frac{\partial v_1}{\partial x_3}\delta x_3, \\ v_2' &= v_2 + \frac{\partial v_2}{\partial x_1}\delta x_1 + \frac{\partial v_2}{\partial x_2}\delta x_2 + \frac{\partial v_2}{\partial x_3}\delta x_3, \\ v_3' &= v_3 + \frac{\partial v_3}{\partial x_1}\delta x_1 + \frac{\partial v_3}{\partial x_2}\delta x_2 + \frac{\partial v_3}{\partial x_3}\delta x_3. \end{aligned} \tag{2}$$

The three equalities (2) can be written in a shorter vectorial form:

$$v' = v + \left(\frac{\partial v_j}{\partial x_k}\right)\delta x, \tag{3}$$

where $\left(\frac{\partial v_j}{\partial x_k}\right)$ is the velocity tensor of the motion, i.e., a physical quantity, which characterizes the velocity field of the fluid's motion at the point under consideration, and which is determined by nine components:

$$\left(\frac{\partial v_j}{\partial x_k}\right) = \begin{pmatrix} \frac{\partial v_1}{\partial x_1}, & \frac{\partial v_1}{\partial x_2}, & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1}, & \frac{\partial v_2}{\partial x_2}, & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1}, & \frac{\partial v_3}{\partial x_2}, & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \tag{4}$$

Let us decompose the tensor $\left(\frac{\partial v_j}{\partial x_k}\right)$ into two components--a symmetric and an antisymmetric tensor.

A tensor is called symmetric, if its components satisfy the condition

$$a_{jk} = a_{kj}, \tag{5}$$

i.e., those components symmetrically placed with respect to the main diagonal are equal. It is easy to see that the symmetric tensor is determined not by nine different values, but only by six.

The tensor whose components satisfy the condition

$$a_{jk} = -a_{kj}, \tag{6}$$

is called antisymmetric. Its components which are symmetrically placed with respect to the main diagonal are equal in absolute value, but of opposite sign. Equation (6) leads to the equality:

$$a_{jj} = 0, \tag{7}$$

i.e., all the main diagonal components of an antisymmetric tensor are equal to zero. The antisymmetric tensor is thus determined by three different components.

Each tensor can be resolved into the sum of two others: a symmetric tensor and an antisymmetric one. In order to show this we introduce the notion of a sum of two tensors.

The sum of two tensors (a_{jk}) and (b_{jk}) is the tensor whose components equal the sum of the corresponding components of the summands.

$$(a_{jk})+(b_{jk})=(a_{jk}+b_{jk}). \tag{8}$$

An arbitrary tensor (a_{jk}) can therefore be written in the form of a sum of two tensors:

$$(a_{jk})=\frac{1}{2}(a_{jk}+a_{kj})+\frac{1}{2}(a_{jk}-a_{kj}). \tag{9}$$

Or, fully:

$$(a_{jk})=\begin{pmatrix} a_{11}, & \frac{1}{2}(a_{12}+a_{21}), & \frac{1}{2}(a_{13}+a_{31}) \\ \frac{1}{2}(a_{21}+a_{12}), & a_{22}, & \frac{1}{2}(a_{23}+a_{32}) \\ \frac{1}{2}(a_{31}+a_{13}), & \frac{1}{2}(a_{32}+a_{23}), & a_{33} \end{pmatrix}+ \\ +\begin{pmatrix} 0, & \frac{1}{2}(a_{12}-a_{21}), & \frac{1}{2}(a_{13}-a_{31}) \\ \frac{1}{2}(a_{21}-a_{12}), & 0, & \frac{1}{2}(a_{23}-a_{32}) \\ \frac{1}{2}(a_{31}-a_{13}), & \frac{1}{2}(a_{32}-a_{23}), & 0 \end{pmatrix} \tag{10}$$

Thus:

$$\frac{\partial v_j}{\partial x_k}=\frac{1}{2}(S_{jk})+\frac{1}{2}(A_{jk}),$$

where

$$(S_{jk})=\begin{pmatrix} 2\frac{\partial v_1}{\partial x_1}, & \frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}, & \frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}, & 2\frac{\partial v_2}{\partial x_2}, & \frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1}, & \frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2}, & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}, \tag{11}$$

$$(A_{jk})=\begin{pmatrix} 0, & -\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right), & -\left(\frac{\partial v_3}{\partial x_1}-\frac{\partial v_1}{\partial x_3}\right) \\ \left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right), & 0, & -\left(\frac{\partial v_2}{\partial x_3}-\frac{\partial v_3}{\partial x_2}\right) \\ \left(\frac{\partial v_3}{\partial x_1}-\frac{\partial v_1}{\partial x_3}\right), & \left(\frac{\partial v_2}{\partial x_3}-\frac{\partial v_3}{\partial x_2}\right), & 0 \end{pmatrix} \tag{12}$$

Thus equation (3) may now be written in the following form:

$$\boldsymbol{v}'=\boldsymbol{v}+\frac{1}{2}(A_{jk})\,\delta\boldsymbol{x}+\frac{1}{2}(S_{jk})\,\delta\boldsymbol{x}. \tag{13}$$

It is proved in tensor analysis that a new vector is obtained as a result of the [inner] multiplication of a given vector by an antisymmetric tensor. In fact,the antisymmetric tensor is determined by three components:

$$\left.\begin{aligned} A_{12}&=-\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right), \\ A_{23}&=-\left(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}\right), \\ A_{31}&=-\left(\frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}\right). \end{aligned}\right\} \tag{14}$$

But the right-hand terms of equalities (14) are components of the vorticity;

$$\left.\begin{aligned} A_{23} &= -\operatorname{curl}_1 \boldsymbol{v}, \\ A_{31} &= -\operatorname{curl}_2 \boldsymbol{v}, \\ A_{12} &= -\operatorname{curl}_3 \boldsymbol{v}. \end{aligned}\right\} \tag{15}$$

Therefore

$$\frac{1}{2}(A_{jk})\cdot\delta\boldsymbol{x} = \frac{1}{2}[\operatorname{curl}\boldsymbol{v}\,\delta\boldsymbol{x}], \tag{16}$$

and the velocity of the infinitesimally removed point M' will be given in terms of the velocity of M by the equation

$$\boldsymbol{v}' = \boldsymbol{v} + \frac{1}{2}[\operatorname{curl}\boldsymbol{v}, \delta\boldsymbol{x}] + \frac{1}{2}(S_{jk})\cdot\delta\boldsymbol{x}. \tag{17}$$

Comparing equations (17) and (1) we see that the first terms in both equations represent the linear velocity of the body; the second terms are equal too and correct in their sign; they give the velocity which the point acquires through the rotation of the particle on any axis which passes in its interior, the angular velocity being determined by the vector curl $\boldsymbol{v}$, and equals half of it. It should however be kept in mind that equation (1) can be applied to any point of an absolutely rigid body, however far it may be removed from the pole, whereas equation (17) is correct only for an infinitesimal neighbourhood. The velocities of fluid particles which are far between are kinematically unrelated. Equation (1) does not contain a third term; this is clear, since the last term of equation (17) represents the velocity of deformation of the air body, and of course no rigid body undergoes such a deformation. The tensor

$$\frac{1}{2}(S_{jk}) = \begin{pmatrix} \frac{\partial v_1}{\partial x_1}, & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\right), & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1}\right), \\ \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\right), & \frac{\partial v_2}{\partial x_2}, & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2}\right), \\ \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1}\right), & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2}\right), & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \tag{18}$$

is correspondingly called the deformation velocity tensor.

It is easy to demonstrate that the diagonal components of the deformation velocity tensor give the relative velocities of dilatation or compression of the body in the directions of the corresponding coordinate axes. The off-diagonal terms give the shear velocities or, in other words, the velocities of change of the initially right angles.

§ 4. The Equation of Continuity

The effect of atmospheric molecular processes, like the diffusion of water vapor etc, is not taken into consideration in the kinematics of the atmosphere. In the same way the transition of water vapor from one state to another, the precipitations etc, are disregarded. These phenomena are of great importance for the thermodynamics of the atmosphere, but they are not essential from the point of view of kinematics, because the associated transport of mass is negligibly small in comparison with the mass transport brought about by atmospheric motion. Thus, a condition of mass conservation is imposed on each element of the atmosphere, i.e., we take each moving air particle to have a constant mass. An additional condition stipulates that the atmosphere should fill space continuously, i.e., there must be no vacuous regions in the atmosphere.

Two vector fields,the velocity field and the momentum density field can be used to describe the instantaneous state of motion. The atmosphere being a continuous material medium, these fields bear a fundamental relationship to the

density field, the nature of their correspondence remaining unspecified. Since the invariant mass elements filling space continuously are displaced among themselves, the knowledge of the instantaneous field of motion entails the knowledge of any future state of the density field. The preceding propositions thus lead to a prognostic relationship. The mathematical expression of this relationship is known as the equation of continuity. In some cases the time is eliminated from this equation; the equation then has only a diagnostic significance, because the motion appears to obey a certain restrictive condition.

Let us consider a separate elementary volume element in the moving fluid, having the form of a rectangular parallelepiped whose sides are parallel to the coordinate planes, and examine the mass balance in this volume during the time dt.

The following table can be drawn up without special explanations, denoting the components of the velocity by u, v, w.

The total mass gain in the volume under consideration will thus be equal to

$$-\left(\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}\right) dx\, dy\, dz\, dt.$$

This variation in the mass content will cause a change in the density from ρ to $\rho+\frac{\partial \rho}{\partial t} dt$, and the mass increment can also be put in the form:

$$\frac{\partial \rho}{\partial t} dt\, dx\, dy\, dz.$$

Through the side	having the coordinate	the following mass enters
$dy\, dz$	x	$\rho u\, dy\, dz\, dt$
$dy\, dz$	$x+dx$	$-\left(\rho u+\frac{\partial \rho u}{\partial x} dx\right) dy\, dz\, dt$
$dz\, dx$	y	$\rho v\, dz\, dx\, dt$
$dz\, dx$	$y+dy$	$-\left(\rho v+\frac{\partial \rho v}{\partial y} dy\right) dz\, dx\, dt$
$dx\, dy$	z	$\rho w\, dx\, dy\, dt$
$dx\, dy$	$z+dz$	$-\left(\rho w+\frac{\partial \rho w}{\partial z} dz\right) dx\, dy\, dt$

Equating these expressions and dividing them by the volume dx dy dz and by the time dt, we obtain the equation of continuity

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0 \qquad (1)$$

or

$$\frac{d\rho}{dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0. \qquad (2)$$

It can be seen from the equation of continuity that the velocity v of the air particle and its density ρ satisfy a definite relationship, and cannot be assigned independently of each other.

The equation of continuity is important for forecasts, since it enables us to compute the local change of the density, i.e., the redistribution of mass, on the basis of the velocity field or of the momentum density field.

In two exceptional cases the equation of continuity has merely a diagnostic value:

1) when the motion is stationary; then $\frac{\partial \rho}{\partial t}$ and

$$\operatorname{div} \rho \boldsymbol{v} = 0 \quad \text{or} \quad \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0; \tag{3}$$

2) in case the fluid is incompressible; then

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{4}$$

A vector field whose divergence identically vanishes is called a source-free field. The atmospheric momentum field is source-free in case the motion is stationary.

We write down the equation of continuity in curvilinear coordinates, but without explicit derivation:

In the spherical coordinates r, φ, λ:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial (\rho v_r \cdot r^2)}{\partial r} + \frac{1}{r \cos \varphi} \left\{ \frac{\partial (\cos \varphi \cdot \rho v_\varphi)}{\partial \varphi} + \frac{\partial (\rho v_\lambda)}{\partial \lambda} \right\} = 0. \tag{5}$$

In the cylindrical coordinates r, θ, z

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial (\rho v_r \cdot r)}{\partial r} + \frac{1}{r} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0. \tag{6}$$

§ 5. Viscosity (Internal Friction)

The motion of a fluid particle depends in a definite way on the forces exerted on it, and which are due to the presence of another substance, in particular of other fluid particles.

The forces acting on the fluid particle can be either volume forces or surface forces. The volume forces act on any point of the fluid particle, both on points situated inside the particle and those lying on its boundary. The volume force exerted on the fluid particle is proportional to its volume.

The surface forces are a result of interaction between the separate elements of the fluid particle. From the physical point of view these are short-range forces, depending only on the molecular structure of the fluid. The surface forces exerted on the fluid particle therefore act only on its boundary elements; the total surface force acting on the fluid particle is proportional to its surface area.

Gravity is the most important volume force. The forces of pressure and of viscosity are the most important surface forces.

Let us consider a given volume element of a fluid (Figure 71), making an imaginary partition dividing it into two sections, say an upper and a lower one. If the lower section is removed, then the remaining upper section must be considered to be not only under the action of the applied external forces, but also of these forces which have been exerted on it by the removed section. This force, called stress, is exerted only on those points of the remaining section of the fluid particle which are situated on the partition surface. Alternatively, the removal of the upper section of the particle gives rise to corresponding shear forces equal in magnitude to the former but of the opposite direction, acting on the remaining lower section of the partition surface. The sole internal force for a perfect fluid is the pressure—a well-defined scalar function $p(x,y,z,t)$.

This is true also for a viscous fluid in a state of rest.

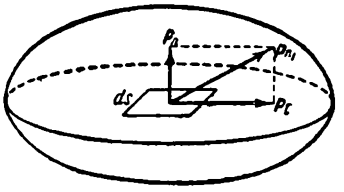

Figure 71. The internal force in a moving viscous fluid is not directed along the normal to a surface element, but at a certain angle to it

However, the moving viscous fluid differs from a perfect fluid in the tangential stresses developing in it, caused by the viscosity (internal friction).

When two adjacent fluid layers move with different velocities, tangential forces of internal friction, or viscosity forces, develop between these layers. It should be pointed out that viscosity is a fundamental property of fluids, both liquid and gaseous. This property is exhibited only during motion and results in the motion of the viscous fluid (gas) being transmitted from one layer to another, evolving a resistance to relative motion of the fluid particles. This phenomenon is outwardly similar to the traction resistance of two solid bodies in sliding contact, but the laws governing the action of internal friction have a quite different character.

In fluid flow, the irregular motion of the molecules causes a momentum exchange between adjacent layers moving with different velocities. The molecules in their ceaseless passage from one layer to another give rise to a mixing of the layers, and to the forces of cohesion.

The molecules which pass from a swift layer to a slower one cause an acceleration of the slowly flowing layer, and vice versa, the molecules entering from a slow into a more rapid layer retard the motion of the latter. The force $\boldsymbol{p}_n$, exerted by one layer of a viscous fluid on another is therefore not normal but inclined at a specific angle to the interface between these layers, i.e., $\boldsymbol{p}_n$ has both a normal and a tangential component. The tangential and normal stresses acting on differently inclined planes drawn through the same point of viscous fluid are generally different. An unlimited number of planes of different spatial orientation can be drawn through any point, and a correspondingly unlimited variety of normal and tangential stresses is thereby obtained. These stresses may, however, be expressed by just 9 [independent] magnitudes: 3 normal and 6 tangential stresses, which can be obtained in the following way. Let us draw planes through an arbitrary internal point of the fluid, perpendicular to the coordinate axes x, y, z (Figure 72), and denote by $\boldsymbol{p}_x$, $\boldsymbol{p}_y$, $\boldsymbol{p}_z$ the respective stresses acting on these planes. The stress $\boldsymbol{p}_n$ acting on an arbitrary plane determined by its normal n, will be expressed by the equation

$$\boldsymbol{p}_n = \boldsymbol{p}_x \cos(n, x) + \boldsymbol{p}_y \cdot \cos(n, y) + \boldsymbol{p}_z \cdot \cos(n, z) \qquad (1)$$

or, by projecting on the axes x, y, z:

$$\begin{aligned} p_{nx} &= p_{xx} \cos(n, x) + p_{yx} \cos(n, y) + p_{zx} \cos(n, z); \\ p_{ny} &= p_{xy} \cos(n, x) + p_{yy} \cos(n, y) + p_{zy} \cos(n, z); \\ p_{nz} &= p_{xz} \cos(n, x) + p_{yz} \cos(n, y) + p_{zz} \cos(n, z). \end{aligned} \qquad (2)$$

If n is the unit vector normal to the plane in question, and noting that the projections of this unit vector on the coordinate axes are respectively equal to cos(n, x); cos(n, y); cos(n, z), we can write equation (2) in a shorter form through the use of the tensor (p_{jk}):

$$p_n = (p_{jk})\, n. \tag{3}$$

It can be seen from equation (3) that the stress at any point of the moving viscous fluid is not determined by a scalar function, as would be the case for a perfect fluid, but by the tensor (p_{jk}). The tensor

$$(p_{jk}) = \begin{pmatrix} p_{xx} & p_{yx} & p_{zx} \\ p_{xy} & p_{yy} & p_{zy} \\ p_{xz} & p_{yz} & p_{zz} \end{pmatrix} \tag{4}$$

is called the stress tensor; each of its components represents an absolute value of the force-component referred to a unit surface element, the first subscript indicating the direction of the normal to the surface on which the force acts, and the second the direction of the force-component itself (Figure 73).

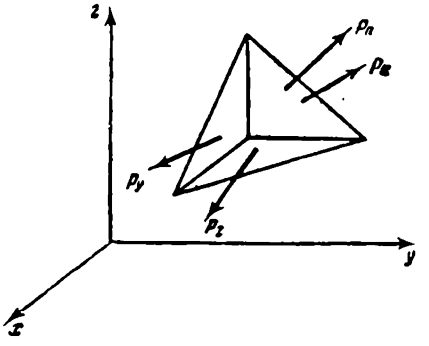

Figure 72. The stress acting on a plane with the normal n, can be expressed by stresses acting on the planes whose normals are parallel to the coordinate axes

It can be proved that the stress tensor is symmetric, i.e.,

$$p_{xz} = p_{zx}, \quad p_{xy} = p_{yx}, \quad p_{yz} = p_{zy}. \tag{5}$$

The equalities (5) can be explained in the following way (Figure 74). Let us consider a separate unit volume of the fluid in the form of a cube whose sides are orthogonal to the coordinate axes. The equalities (5) show that the moments of rotation exerted on this cube cancel out in pairs, and the stresses acting on the cube sides do not, therefore, cause it to rotate, but lead only to its deformation.

Viscosity plays a double role during the motion of every real fluid, including air.

It has been shown by observations that the velocity field in a real fluid is continuous. The formation of such a continuous field of velocities is warranted by the influence of the viscosity, causing a continuous transfer of momentum from one layer to another.

Newton related the stresses which develop in a viscous fluid to the velocity field. He laid down a simple hypothesis that a linear relation exists between the force of internal friction and the normal derivative of the velocity:

$$\tau = \mu \cdot \frac{dc}{dn}, \qquad [\mu] = [ML^{-1}T^{-1}], \tag{6}$$

where τ is the force of internal friction referred to a unit surface; μ is the coefficient of viscosity, or coefficient of internal friction; it is independent of the velocity field and determined in the case of a gas by its nature and temperature, whereas in the case of a drop-forming fluid, it also depends on the pressure. Equation (6) is an expression of the so-called Newton's law of viscosity.

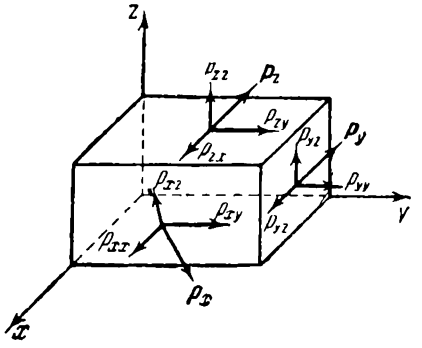

Figure 73. The state of stress in a viscous fluid is determined by nine magnitudes, constituting the stress tensor

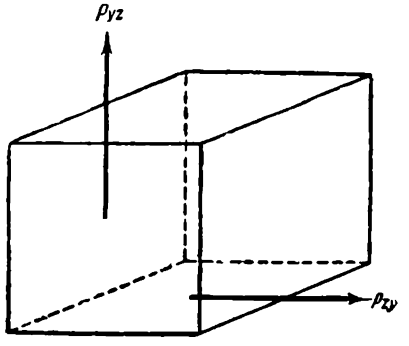

Figure 74. The moments of rotation in a viscous fluid are pairwise cancelled

The transfer of momentum from one layer to another results, apart from the levelling of the velocity field, in a dissipation of mechanical energy and its conversion into heat. In practice, any transformation of mechanical energy is considered a loss. A part of the mechanical energy is lost in this way, i.e., transformed into heat, during the motion of air. In such a case we deal with a process of dispersion or dissipation of mechanical energy, due to viscosity effects. The damping influence of viscosity forces on the motion may often be neglected, but its influence on the preservation of a continuous velocity field can never be ignored.

Nonviscous, i.e., "perfect", fluids do not exist in nature. One of the fundamental properties of viscous fluids—the continuity of the velocity field and the finiteness of normal derivatives of the velocity, are assumed to exist even in a "perfect" fluid, such as is studied in theoretical hydrodynamics. This property would be unnatural and incomprehensible if internal friction did not exist, since viscosity is the sole cause of the equalization of velocities. The velocity field can be discontinuous in a completely nonviscous fluid, because a mutual slipping of adjacent layers can take place at any point of the flowing fluid.

§ 6. The Equations of Motion of a Viscous Fluid

Considering the atmosphere as a viscous fluid it is natural to apply the equations of Navier to the study of its motion. Disregarding the mass forces for the time being, we can write these equations in the following form:

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&=\frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right),\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}&=\frac{1}{\rho}\left(\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zy}}{\partial z}\right),\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}&=\frac{1}{\rho}\left(\frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right),
\end{aligned}\qquad (1)$$

where u, v, w are the components of the velocity in an orthogonal coordinate system x, y, z; ρ —the fluid density, and p_{xx}, $p_{xy} = p_{yx}$, $p_{xz} = p_{zx}$, p_{yy},

$p_{yz} = p_{zy}$, p_{zz} are the components of the stress tensor, i.e., surface forces caused by viscosity, the first subscript indicating the direction of the normal to the surface on which this force acts, and the second indicating the direction of the force itself.

The system of these equations is incomplete, since the three equations of Navier contain ten unknown functions, u, v, w, p_{xx}, p_{yy}, p_{zz}, p_{xy}, p_{yz}, p_{zx}, ρ, It is necessary to add seven more equations, containing only the same unknown functions appearing in equation (1), in order to complete this system.

Six of the seven lacking functions can be given on the basis of the hypothesis formulated by Stokes in 1845, according to which a linear relationship exists between the stress components and the components of the velocity of deformation

$$\left.\begin{aligned} p_{xx} &= -p - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial u}{\partial x}, \\ p_{yy} &= -p - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial v}{\partial y}, \\ p_{zz} &= -p - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z}, \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} p_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \\ p_{yz} &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \\ p_{zx} &= \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right). \end{aligned}\right\} \quad (3)$$

The introduced function p is the hydrodynamic pressure that would nave been the fluid's had it been perfect. The minus sign indicates that the pressure is directed inwards, into the volume of the fluid under consideration.

μ —the coefficient of dynamic viscosity, is a parameter depending on the physical properties of the fluid. This quantity cannot be obtained from the preceding relationships and is experimentally determined; it depends on the molecular structure of the fluid.

Eliminating the stresses from equations (1) by means of equations (2) and (3), we obtain the equations of motion of a viscous fluid, called the Navier-Stokes equations

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \\ &\quad = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{3}\nu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \\ &\quad = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{3}\nu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \\ &\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \\ &\quad = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{3}\nu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right), \end{aligned}\right\} \quad (4)$$

where $\nu = \mu/\rho$ — coefficient of kinematic viscosity

$$[\nu] = [L^2T^{-1}]$$

The Navier-Stokes equations contain five unknown functions: u, v, w, p, ρ. This system of equations is therefore incomplete.

As a fourth equation we shall use the equation of continuity

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0, \tag{5}$$

expressing the principle of mass conservation.

§ 7. The System of Equations for an Incompressible Viscous Fluid

If the fluid is incompressible ρ = const., and the equation of continuity takes the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{1}$$

Joining equation (1) to the Navier-Stokes equations, which for ρ = const. assume the form

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right),\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}&=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right),
\end{aligned} \tag{2}$$

we obtain a complete system of four equations, containing four unknown functions: u, v, w, p, which do not include the temperature.

This means that the motion of the incompressible fluid does not depend on the thermodynamic processes occurring in it. However, the reverse conclusion should not be drawn, that the thermodynamic processes in incompressible fluids are independent of the motion of the fluid.

By introducing the mass forces F_x, F_y, F_z into the Navier-Stokes equations we arrive at the fundamental problem of classical hydromechanics, namely, the determination of the motion of the fluid and of its pressure from the given constant mass forces. From a mathematical point of view the solution of this problem reduces to the simultaneous solution of the following system of equations (3) and equation (1)

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&=F_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right),\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}&=F_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}&=F_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right),\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}&=0.
\end{aligned} \tag{3}$$

It is necessary to determine u, v, w, p as functions of the coordinates of the point x, y, z and of the time t, from these equations.

We have succeeded in completing the system of equations of motion for an incompressible fluid. Air, however, is a compressible fluid, and a special examination is required of the conditions under which it is possible to disregard its compressibility.

During the adiabatic change of pressure by Δp the fluid density changes by $\Delta\rho$, where

$$\Delta\rho=\left(\frac{\partial \rho}{\partial p}\right)_{ad}\Delta p. \tag{4}$$

It is known that $\left(\frac{\partial p}{\partial \rho}\right)_{ad}$ equals the square of the sound velocity c; therefore

$$\Delta\rho=\frac{1}{c^2}\cdot\Delta p. \tag{5}$$

However it follows from Bernoulli's equation that

$$\Delta p \sim \rho v^2;$$

therefore

$$\Delta\rho \sim \rho \cdot \frac{v^2}{c^2}. \tag{6}$$

The compressibility of the fluid can be disregarded if $\frac{\Delta\rho}{\rho} \ll 1$; the fluid velocity should be small compared with the velocity of sound in order to regard it as incompressible:

$$v \ll c. \tag{7}$$

This condition is not only necessary but also sufficient, if the fluid motion is steady. A supplementary condition must be satisfied in the case of an unsteady motion.

Let the velocities undergo an appreciable change in the time τ, over a distance l between the points under consideration. The separate terms of Euler's equation will then be of the following orders of magnitude

$$\frac{\partial v}{\partial t} \sim \frac{v}{\tau}, \qquad \frac{1}{\rho}\operatorname{grad} p \sim \frac{\Delta p}{l \cdot \rho}.$$

Therefore

$$\frac{v}{\tau} \sim \frac{\Delta p}{l\rho},$$

whence

$$\Delta p \sim \frac{l}{\tau} \cdot \rho \cdot v.$$

and

$$\Delta\rho \sim \frac{l\rho v}{\tau c^2}.$$

Let us now determine the orders of magnitude of the terms of the equation of continuity

$$\frac{d\rho}{dt} \sim \frac{\Delta\rho}{\tau} \sim \frac{l\rho v}{\tau^2 c^2},$$

$$\rho \operatorname{div} v \sim \frac{\rho \cdot v}{l}.$$

The compressibility of the fluid can be disregarded, if

$$\frac{d\rho}{dt} \ll \rho \operatorname{div} v,$$

i.e., if

$$\tau \gg \frac{l}{c}. \tag{8}$$

Thus, it is sufficient that conditions (7) and (8) be fulfilled for the fluid to be considered incompressible.

The accuracy of this result rests, however, only on a kinematic and dynamic basis, ignoring the energetic considerations. Notwithstanding, the compressibility of air cannot be disregarded if energetic considerations are also taken into account. Indeed, transformation of the internal energy of a gas into mechanical work and the reverse transformation of mechanical work into internal energy are related to the compressibility of air.

The system of four equations (4) and (5) § 6, containing five unknown functions u, v, w, p, ρ, is therefore incomplete in the case of a compressible fluid.

In searching for new equations for the laws which govern atmospheric processes we first turn our attention to the equation of state. Let us first assume that the air is dry. The fifth equation will then be Clapeyron's equation:

$$\rho = \frac{p}{RT}. \tag{9}$$

However, supplementing the system (4) and (5) § 6 by Clapeyron's equation does not complete this system, since Clapeyron's equation introduces a new variable T instead of the variable ρ; the advantage of the new variable over the former consists, however, in the fact that T can be directly measured in meteorology while the density can not.

Our system of equations is complete, if the temperature distribution is considered as given (for example through observations). However, only isolated, particular problems can be solved in this way. We are dealing here with a general problem, and will make use of such a method in order to complete the system of equations.

§ 8. Equation of Heat Balance

We can, in principle, obtain a new equation by using the equation of heat balance. Let us denote by q the influx of heat from the surroundings per unit time and unit mass; we write the equation of heat balance in the form

$$q = c_v \frac{dT}{dt} + p \frac{d}{dt}\left(\frac{1}{\rho}\right). \tag{1}$$

The equation of heat balance is taken from thermodynamics. The time is not taken into account in thermodynamics, but in dynamic meteorology it plays a very significant part. Therefore it is necessary to check if the equation of heat balance in the form (1) satisfies the requirements of dynamic meteorology. We note, first of all, that this equation cannot be applied to a viscous fluid, since it contains the gas pressure p, while in the case of a viscous fluid the shearing stress must also be taken into account.

We shall derive the equation of heat balance in a more general form, by proceeding from the law of the conservation of energy, which states that the change of the sum of the fluid's kinetic and internal energies is equal to the work done by the volume and surface forces on the fluid, plus the heat influx.

Let us consider some finite fluid volume τ. Referring all the changes to a unit of time, we can express the law of conservation of energy through the equation

$$\frac{d\mathcal{E}}{dt} + \frac{dU}{dt} = Q + \int_{\tau} \rho (\boldsymbol{F}, \boldsymbol{v})\, d\tau + \int_{\Sigma} (\boldsymbol{p}_n, \boldsymbol{v})\, d\sigma, \tag{2}$$

where Q is the influx of heat, U—the internal energy and $\mathcal{E}$ —the total kinetic energy of the volume τ;

$\int_{\tau} \rho (\boldsymbol{F}, \boldsymbol{v})\, d\tau$ —the work done by all the volume forces per unit time;

$\int_{\Sigma} (\boldsymbol{p}_n, \boldsymbol{v})\, d\sigma$ —the work done by all the surface forces per unit time.

Let us consider now the equations of motion of a viscous fluid in terms of the stresses

$$\frac{du}{dt} = F_x + \frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right),$$

$$\frac{dv}{dt} = F_y + \frac{1}{\rho}\left(\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}\right), \qquad (3)$$

$$\frac{dw}{dt} = F_z + \frac{1}{\rho}\left(\frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}\right).$$

Multiplying the equations (3) respectively by u, v, w, adding them together and integrating over the entire volume τ, we arrive at the theorem of kinetic energy

$$\frac{dE}{dt} = \int_\tau \rho (F, v)\, d\tau + \int \left\{ \left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right) u + \right.$$

$$\left. + \left(\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}\right) v + \left(\frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}\right) w \right\} d\tau.$$

Noting that

$$\frac{\partial}{\partial x}(p_{xx} \cdot u + p_{yx} \cdot v + p_{zx} \cdot w) =$$

$$= \frac{\partial p_{xx}}{\partial x} u + \frac{\partial p_{yx}}{\partial x} v + \frac{\partial p_{zx}}{\partial x} w + p_{xx} \frac{\partial u}{\partial x} + p_{yx} \frac{\partial v}{\partial x} + p_{zx} \frac{\partial w}{\partial x},$$

and forming the corresponding relations for the y and z axes, the equation of kinetic energy can be written in the form

$$\frac{dE}{dt} = \int_\tau \rho (F, v)\, d\tau + \int_\tau \left\{ \frac{\partial}{\partial x}(p_{xx}u + p_{yx}v + p_{zx}w) + \right.$$

$$\left. + \frac{\partial}{\partial y}(p_{xy}u + p_{yy}v + p_{zy}w) + \frac{\partial}{\partial z}(p_{xz}u + p_{yz}v + p_{zz}w) \right\} d\tau -$$

$$- \int_\tau \left\{ p_{xx}\frac{\partial u}{\partial x} + p_{yy}\frac{\partial v}{\partial y} + p_{zz}\frac{\partial w}{\partial z} + p_{yz}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + p_{zx}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \right.$$

$$\left. + p_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \right\} d\tau. \qquad (4)$$

The second integral on the right-hand side may be transformed into a surface integral, according to Gauss's theorem, stating that the flux of a vector A through a closed surface equals the volume integral of the divergence of this vector:

$$\int_\Sigma A_n\, d\sigma = \int_\tau \operatorname{div} A\, d\tau.$$

Let us denote by D' the integrand of the last integral of equation (4); the equation then assumes the form

$$\frac{dE}{dt} = \int_\tau \rho (F, v)\, d\tau + \int_\Sigma (p_n, v)\, d\sigma - \int_\tau D' d\tau. \qquad (5)$$

It should be noted that, in the case of viscous flow, not all the work done by forces exerted on the fluid goes to change the kinetic energy of the system; a part of the mechanical energy is evidently lost by conversion into heat.

The equation of heat balance will be obtained in the following form:

$$\frac{dU}{dt} = Q + \int_\tau D' d\tau. \qquad (6)$$

by subtracting equation (5) from equation (2).

It can be seen from this equation that the change of the internal energy of the fluid is composed firstly of the heat influx from the surroundings and secondly of a part of the mechanical energy converted into heat.

Let us now pass from the integral form of the equation of heat balance to its differential form. Denoting by u the internal energy, per unit mass, and by q the quantity of heat, transferred from without to a mass unit in a unit of time

$$U=\int \rho u\, d\tau; \quad Q=\int \rho q\, d\tau,$$

we obtain equation (6) in the form

$$\frac{d}{dt}\int_{\tau} \rho u\, d\tau=\int \rho q\, d\tau+\int D' d\tau. \tag{7}$$

If we reverse the order of the operations on the left-hand side (which is quite permissible, since the mass $\rho\, d\tau$ remains unchanged), collecting all the terms of the equation in the left side under a common integral sign, we obtain

$$\int_{\tau} \left\{ \rho \frac{du}{dt} - \rho q - D' \right\} d\tau = 0.$$

Since the volume τ is arbitrary, we have

$$\rho \frac{du}{dt}=\rho q+D' \tag{8}$$

or, fully

$$\rho q=\rho\frac{du}{dt}-\left\{ p_{xx}\frac{\partial u}{\partial x}+p_{yy}\frac{\partial v}{\partial y}+p_{zz}\frac{\partial w}{\partial z}+p_{yz}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+ \right.$$
$$\left. +p_{zx}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+p_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right. . \tag{9}$$

This is the most general form of the equation of heat balance.

The velocities of deformation, according to the equations (2) and (3) § 6, can be used in the equation of heat balance, instead of the stresses. After simple transformations we obtain

$$\rho q=\rho\frac{du}{dt}+p\,\mathrm{div}\,\boldsymbol{v}+\frac{2}{3}\mu(\mathrm{div}\,v)^2-\mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\right.$$
$$\left.+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right\}. \tag{10}$$

This is the definitive form of the equation of heat balance.

The physical meaning of the terms of equation (10) will not be explained. Taking the fluid to be perfect, $\mu=0$ and the equation of the heat balance assumes the form

$$\rho q=\rho\frac{du}{dt}+p\,\mathrm{div}\,\boldsymbol{v}. \tag{11}$$

However, according to the equation of continuity

$$\mathrm{div}\,\boldsymbol{v}=-\frac{1}{\rho}\frac{d\rho}{dt};$$

therefore

$$\rho q=\rho\frac{du}{dt}-p\cdot\frac{1}{\rho}\frac{d\rho}{dt}.$$

Or, taking into consideration that for an ideal gas $u = c_v T$

$$q=c_v\frac{dT}{dt}+p\frac{d}{dt}\left(\frac{1}{\rho}\right). \tag{12}$$

The second term of the equation of heat balance is not equal to zero in the case of an ideal compressible fluid. Thus the term p div $\boldsymbol{v}$ expresses the quantity of heat used up in the work of expansion or liberated as a result of

compression. The remaining terms of equation (10) are related to the viscosity. These terms indicate the quantity of heat liberated per unit time in a unit volume of the fluid, as a result of the conversion of mechanical energy into heat, caused by the viscosity.

The sum of these terms expresses the dissipation or dispersion of the mechanical energy:

$$D = -\frac{2}{3}\mu(\operatorname{div} v)^2 + \mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right\}. \quad (13)$$

The dissipated mechanical energy consists of two parts. The first is related to the compressibility of the viscous fluid, and the second to its deformation. The compressibility of air is usually ignored, and the dissipation of the mechanical energy is determined by the equation

$$D = \mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right\}. \quad (14)$$

Equation (14) shows that the dissipation of the mechanical energy is essentially a positive quantity. This means that what occurs as a result of dissipation cannot be a loss but is always a gain in heat, i.e., a conversion of the mechanical energy into heat.

By adding the equation of heat balance

$$\rho q + D = c_v \cdot \rho \cdot \frac{dT}{dt} + p \cdot \rho \cdot \frac{d}{dt}\left(\frac{1}{\rho}\right) \quad (15)$$

to the system of equations (4) and (5) § 6, six equations are obtained containing seven unknown functions: u, v, w, p, ρ, T, q, i.e., our system of equations still remains incomplete. Sometimes q is considered as a fixed function of the coordinates and the time, in order to complete the system. However, by proceeding in this manner we would be sidetracked from the problem in its general form, and consequently from its general solution.

It is necessary to take into account the following forms of energy transfer in fluids, in searching for a general solution of our problem:

1) dissipation of the mechanical into thermal energy, caused by the viscosity of the fluid;

2) heat transfer by advection (horizontal), and by convection (vertical);

3) thermal conduction;

4) absorption and emission of radiant energy;

5) transformation of energy related to phase changes.

The dissipation of the mechanical energy has been examined. It figures in the equation of heat balance as a separate explicit term, and is in no way related to the unknown function q.

Since equation (15) refers to a moving particle, we have

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}.$$

Since the advection of heat is related to the passive trasfer of masses having different temperatures, it is natural to study heat advection by means of the advective derivatives of the temperature. The advection of heat, like the dissipation of mechanical energy, forms a part of the equation of heat balance, and is independent of the function q.

The form of the dependence of the function q on the other forms of energy transfer will be determined below.

In turning to the study of the function q, let us first of all draw attention to the two following cases, which are very important for practical work.

1) Let us assume that the process is adiabatic; then q = 0.

Neglecting the dissipation of the mechanical energy in the equation of heat balance, the adiabatic equation

$$c_p\frac{dT}{dt} - \frac{RT}{p}\frac{dp}{dt} = 0, \tag{16}$$

is obtained, which completes our system of equations. Thus, the system of six equations (the three Navier-Stokes equations, the equation of continuity, the equation of state and the adiabatic equation) is now complete. It does not matter whether in the equations of motion we retain or reject the viscosity terms.

2) q can be considered as a fixed function of the position coordinates, the time, and the velocity components. This method of completing the system of equations has already been mentioned.

Heat transfer is realized not only by advection and dissipation, but also by means of thermal conduction, if the temperature of the fluid changes from point to point.

By thermal conduction a direct transfer of energy by moving molecules is meant. This transfer is by no means related to the macroscopic motion of the fluid. Thermal conduction therefore occurs also in an immobile fluid.

The heat flux S caused by the thermal conduction is determined by Newton's equation. It is directly proportional to the decrease of temperature $-\frac{\partial T}{\partial n}$:

$$S = -k\,\mathrm{grad}\,T = -k\frac{\partial T}{\partial n}, \tag{17}$$

where k is the coefficient of thermal conductivity:

$$[k] = [LT^{-3}\theta^{-1}].$$

Let us isolate an elementary parallepiped in a fluid, with sides parallel to the coordinate planes. By considering the balance of heat which flows in and out, it is easy to find that the quantity of heat q which is conducted to a unit mass per unit time, is equal to

$$q = \frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]. \tag{18}$$

The coefficient of thermal conductivity k generally changes little with temperature and pressure. It can therefore be considered as a constant, a physical property characteristic of the fluid.

The equation of heat balance now assumes the form

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+D=c_v\rho\frac{dT}{dt}+p\,\mathrm{div}\,\boldsymbol{v}. \tag{19}$$

This is the so-called Kirchhoff's equation.

Let us consider some particular cases.

a) If the fluid is perfect D = 0 and consequently

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=c_v\rho\frac{dT}{dt}+p\,\mathrm{div}\,\boldsymbol{v}. \tag{20}$$

b) If, in addition, the fluid is incompressible, then div $\boldsymbol{v}$ = 0 and consequently

$$\frac{dT}{dt}=\frac{k}{c_v\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right) \tag{21}$$

or

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\frac{k}{c_v\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right). \tag{22}$$

c) Finally, if the fluid is immobile we have

$$\frac{\partial T}{\partial t}=\frac{k}{c_v\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right). \tag{23}$$

In this case the equation of heat balance is transformed into the equation of heat conduction, generally called Fourier's equation.

d) For a compressible perfect fluid the equation of heat balance can be written in a precise form as follows:

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\frac{k}{c_p\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{1}{\rho c_p}\frac{dp}{dt}. \tag{24}$$

If the fluctuations of pressure are very small, $\frac{dp}{dt}\approx 0$, and the equation of heat conduction assumes the form (23), with the sole difference that the coefficient of thermometric conductivity (diffusivity) contains c_p instead of c_v:

$$\frac{\partial T}{\partial t}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right). \tag{25}$$

The quantity $\lambda=\frac{k}{c_p\rho}$ is the coefficient of thermometric conductivity $[\lambda]=[L^2T^{-1}]$.

Equation (25) is often called Kirchhoff's equation. It should be kept in mind that this equation is only approximate, and that it is not always possible to neglect the term $\frac{1}{\rho c_p}\frac{dp}{dt}$.

§ 9. The Heat Influx due to the Conversion of Radiant Energy

Radiant and heat energy are of two completely different kinds, and they should by no means be confused. Heat energy, for example, may be transferred from point to point by the process of thermal conduction, but only in presence of a material medium. The process of thermal conduction is impossible in a vacuum.

In contrast, radiant energy is emitted freely in vacuum, the material medium only obstructing it. The emission and absorption of radiant energy are related to the disturbance of the heat balance of the radiating medium.

One of the essential peculiarities of atmospheric processes is the extraordinary complexity of the radiant heat exchange in the atmosphere. This complexity is so great that until recently no exact equations of the radiant energy transfer in the atmosphere could be worked out. Only in 1941 was a complete system of equations for hydromechanics, incorporating radiative exchange, formulated by the Soviet scientist E.S. Kuznetsov.

Kuznetsov did not study the atomic mechanism of emission and absorption. In deriving the equations, he took the phenomenological approach, generally used in the study of the motion of a fluid as a continuous medium.

Kuznetsov introduces new functions to characterize the radiation, in order to compute the heat influx q caused by it.

The intensity of radiation I is, according to E.S. Kuznetsov, the fundamental function of the radiation field; the intensity is defined as that quantity of radiant energy per unit time belonging to a unit spectral interval and propagating within a unit solid angle, which is normally incident per unit time on a unit surface.

The intensity of radiation depends upon its frequency ν, upon the time t, upon the position coordinates x, y, z, and on the direction of the ray r, which is well defined by two angles, such as the two angular spherical coordinates θ and α (Figure 75).

Thus, the intensity of radiation is a function of seven variables

$$\nu,\ t,\ x,\ y,\ z,\ \theta,\ \alpha: \qquad I_\nu = I_\nu(t, x, y, z, \theta, \alpha) \tag{1}$$

or

$$I_\nu = I_\nu(t, P, r), \tag{2}$$

where P designates the point having the coordinates x, y, z, while r is the direction of the ray.

Such form of expression of the intensity of radiation creates great mathematical difficulties, aggravated by the fact that atmospheric radiation cannot be generally considered as isotropic.

Kuznetsov afterwards introduced a second function of radiation, namely the flux of radiant energy, defined by the equation

$$F_{\nu, n} = \int I_\nu(t, P, r) \cos(n, r)\, d\omega, \tag{3}$$

in which the integration extends over all the directions of r.

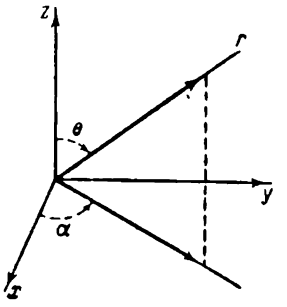

Figure 75. The angular spherical co-ordinates θ and α, which define the direction of the ray

Thus, the flux of radiant energy is defined as the quantity of radiant energy with a frequency ν, which passes through a surface unit from every possible direction per unit time; the orientation of the surface is determined by the direction of the normal n. The flux of radiant energy therefore depends on the frequency and on the orientation of the surface.

The following identity is well known

$$\cos(n, r) = \cos(n, x)\cos(x, r) + \cos(n, y)\cos(y, r) + \cos(n, z)\cdot\cos(z, r).$$

Substituting the expression in equation (3) we obtain

$$F_{\nu, n} = \cos(n, x)\int I_\nu(P, r)\cos(r, x)\,d\omega + \cos(n, y)\int I_\nu(P, r)\cos(r, y)\,d\omega + \cos(n, z)\int I_\nu(P, r)\cos(r, z)\,d\omega$$

or

$$F_{\nu, n} = F_{\nu, x}\cos(n, x) + F_{\nu, y}\cos(n, y) + F_{\nu, z}\cos(n, z), \tag{4}$$

where $F_{\nu, x}$, $F_{\nu, y}$, $F_{\nu, z}$ are the fluxes of radiant energy through unit surfaces whose normals parallel the coordinate axes x, y, z.

Thus, equation (4) shows that $F_{\nu, n}$ is the projection of the vector F_ν, on the vector n, if the values $F_{\nu, x}$, $F_{\nu, y}$, $F_{\nu, z}$ are regarded as components of some vector F_ν. Therefore Kuznetsov called the vector F_ν also a flux of radiant energy.

Kuznetsov formed the total quantities of the radiation field by integrating the intensity of radiation and the flux of radiant energy over all the frequencies from 0 to ∞:

$$I = \int_0^\infty I_\nu d\nu, \qquad F = \int_0^\infty F_\nu d\nu. \tag{5}$$

The characteristic quantities introduced above, namely the intensity of radiation and the flux of radiant energy, are typical of the radiating medium.

Kuznetsov introduced a series of new characteristic quantities, apart from those previously mentioned, which express the interaction between the radiation field and the material medium.

The mass coefficient η_ν of the radiation is defined as the quantity of radiant energy which belongs to the spectral interval from ν to $\nu + 1$ and appears in a

unit of mass per unit time, emitted into a unit solid angle. Kuznetsov assumed that η_ν is a function of position, but does not depend on direction, i.e., that

$$\eta_\nu = \eta_\nu (t, P). \tag{6}$$

This means that a fluid particle emits the same quantity of radiant energy in all directions. The total coefficient of radiation for all frequencies is determined by the formula

$$\eta = \int_0^\infty \eta_\nu \, d\nu. \tag{7}$$

The concept of a mass coefficient of radiation is related to the existence of the radiant energy in a material medium. The disappearance of radiant energy is due to its absorption.

The coefficient of absorption of radiant energy A_ν is defined in meteorology as the ratio of the radiant energy absorbed by the medium per unit length of the ray path to the total intensity of radiant energy. The coefficient of absorption α_ν is defined in astrophysics as the ratio of the radiant energy absorbed by the medium, when a ray travels through it a distance numerically equal to the specific volume $1/\rho$ of the medium, to the total radiant energy.

The first coefficient may be called the volume coefficient of absorption, whereas the second one, the mass coefficient of absorption. It is obvious that the following relationship holds between them.

$$A_\nu = \rho \alpha_\nu. \tag{8}$$

E.S. Kuznetsov used the mass coefficient of absorption α_ν, assuming it to be independent of direction, i.e.,

$$\alpha_\nu = \alpha_\nu (t, P). \tag{9}$$

It is obvious that the coefficient of absorption is equal to zero for a vacuum.

In order to characterize the attenuation of radiant energy as a result of scattering, Kuznetsov introduced the mass coefficient of scattering σ_ν, defining it as the ratio of the energy scattered by the medium when a ray travels through it a distance numerically equal to the specific volume of the medium, to the total energy.

Kuznetsov assumed that the coefficient of scattering does not depend on the direction of the incident ray r, i.e.,

$$\sigma_\nu = \sigma_\nu (t, P). \tag{10}$$

While the absorbed radiant energy is converted into other forms of energy, primarily thermal, the scattered energy remains radiant. Scattering only redistributes this energy among the different directions of r, issuing from the point P.

In order to characterize the distribution of the scattered radiant energy, Kuznetsov introduced the law of scattering determined by the function γ_ν:

$$\frac{1}{4\pi} \gamma_\nu = \frac{1}{4\pi} \gamma_\nu (t, P, r, r').$$

This scattering function gives the part of the energy scattered in the direction r' per unit solid angle. It is obvious that the scattering function must satisfy the condition:

$$\frac{1}{4\pi}\int \gamma_\nu(t, P, r, r')\,d\omega' = 1. \tag{11}$$

The scattering is uniform in all directions, if $\gamma_\nu = 1$. The intensity of the scattered radiation depends on the direction of the ray if $\gamma_\nu \neq 1$.

Kuznetsov thus introduced six characteristic quantities to describe the radiation field

$$I_\nu(t, P, r),\ F_\nu(t, P),\ \eta_\nu(t, P),\ \alpha_\nu(t, P),\ \sigma_\nu(t, P),\ \gamma_\nu(t, P, r, r').$$

However F_ν is not independent, but is related to I_ν by equation (3). Consequently, there are only five independent functions

$$I_\nu,\ \eta_\nu,\ \alpha_\nu,\ \sigma_\nu,\ \gamma_\nu.$$

Kuznetsov assumed that the last three functions can be considered as constants, and there remain only two unknown functions: I_ν and η_ν.

Let us now derive the expressions for the heat influx caused by radiant energy. Since the scattering of radiant energy does not cause its conversion into thermal energy, it is natural that scattered energy will not enter into the expression for the heat influx. Consequently, the heat influx q will be determined by only two processes: the absorption of radiant energy, and its emission.

Let us imagine a fluid particle exposed to radiation from the direction r (Figure 76). Within this particle let us consider a separate cylinder, whose axis is parallel to r, of height ds and cross section $d\sigma$.

The quantity of energy absorbed per unit time by this elementary cylinder is equal to:

$$\rho\alpha_\nu I_\nu\,ds\,d\sigma = \rho\alpha_\nu I_\nu\,d\tau = \alpha_\nu I_\nu\,dm,$$

where $d\tau = ds \cdot d\sigma$ is the cylinder's volume, and dm—the mass of the fluid contained in it. It is obvious that the quantity of energy absorbed by the whole mass contained in a volume having an arbitrary form, is determined by the same expression, where dm stands for the elementary fluid mass contained in this volume.

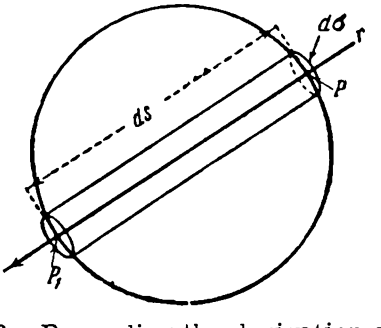

Figure 76. Regarding the derivation of the equation of radiative heat influx

The total energy, absorbed by a unit mass over all frequencies and from all directions, is given by the integral

$$\int_0^\infty \alpha_\nu d\nu \int I_\nu(t, P, r)\, d\omega.$$

This is the "income" of energy per unit time, referred to unit mass.

However, along with the absorption of energy the fluid mass also emits energy in all directions simultaneously. The total energy, emitted per unit time by a unit mass, is given by the integral

$$\int d\omega \int_0^\infty \eta_\nu d\nu = 4\pi \int_0^\infty \eta_\nu d\nu = 4\pi\eta.$$

This is the "expenditure" of energy, referred to a unit mass.

Consequently, the quantity of energy q, transferred per unit time to a mass unit of the fluid, as a result of radiation processes, equals

$$q = \int_0^\infty \alpha_\nu d\nu \int I_\nu(t, P, r)\, d\omega - 4\pi\eta. \tag{12}$$

Finally, let us introduce the heat transferred by radiation to a mass unit of the fluid into the equation of heat balance (§ 8):

$$q = \int_0^\infty \alpha_\nu d\nu \int I_\nu(t, P, r)\, d\omega - 4\pi \int_0^\infty \eta_\nu d\nu.$$

Taking into consideration the processes of dissipation, heat conduction and radiation, the equation of heat balance takes the form

$$k\Delta T + \rho \int_0^\infty \alpha_\nu d\nu \int I_\nu d\omega - \frac{2}{3}\mu (\operatorname{div} v)^2 + \mu \left\{ 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right\} = \rho c_v \frac{dT}{dt} + p \operatorname{div} v + 4\pi\rho \int_0^\infty \eta_\nu d\nu,$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2}$ — is the Laplacian operator. Thus, we obtain a system of six equations, containing 8 unknown functions: u, v, w, p, ρ, T, I_ν, η_ν. This system still remains incomplete.

§ 10. The Equation of Radiative Transfer

In his search for new equations, E. S. Kuznetsov drew on the results obtained by astrophysicists, and derived the equations for transfer of radiant energy in the following way.

Let us consider a fluid volume element of mass $dm = \rho\, d\tau$, which absorbs, emits and scatters radiant energy. Let us keep track of the radiant energy, which propagates in a given direction inside this fluid particle, having the form of a cylinder of unit cross section (Figure 76). For simplicity we assume that the radiation field is stationary.

The following variation in the radiant energy will take place, if it is transferred from point P to an infinitesimally neighboring point P':

1) The intensity will vary as a result of absorption along the path ds by the magnitude

$$-\rho(P)\,\alpha_\nu(P)\, I_\nu(P, r)\, ds.$$

2) The intensity will vary as a result of scattering by the magnitude

$$-\rho(P)\,\sigma_\nu(P)\,I_\nu(P,r)\,ds.$$

3) The intensity will vary as a result of emission by the magnitude

$$\rho\eta_\nu\,ds,$$

if this emission is referred to a unit solid angle, and to a unit spectral interval.

4) Finally, radiation scattered in the direction r is added to the radiation emitted by the cylinder.

The total attenuation of energy radiated along r' equals

$$\rho(P)\,\sigma_\nu(P)\,I_\nu(P,r')\,ds'\,d\sigma'.$$

Use can be made of the law of scattering in order to separate the part of the energy scattered in the direction r. i.e., it is sufficient to multiply the preceding expression by $\frac{1}{4\pi}\gamma_\nu(P,r,r')$.

Consequently the following quantity of energy is scattered in the direction r

$$\frac{1}{4\pi}\sigma_\nu(P)\,I_\nu(P,r')\,\gamma_\nu(P,r',r)\,dm.$$

Thus expression must be integrated over all possible directions, in order to account for all the rays scattered in the direction r'. We then obtain

$$dm\,\frac{\sigma_\nu(P)}{4\pi}\int I_\nu(P,r')\cdot\gamma_\nu(P,r',r)\,d\omega'$$

or, by referring this quantity to a cylinder of the cross section

$$\rho\,ds\,\frac{\sigma_\nu(P)}{4\pi}\int I_\nu(P,r')\,\gamma_\nu(P,r',r)\,d\omega'.$$

The intensities of the ray at the points P and P' differ infinitesimally; therefore

$$I_\nu(P',r)-I_\nu(P,r)=\frac{\partial I_\nu}{\partial s}\,ds.$$

Consequently, the variation of the radiation intensity is expressed by the equation

$$\frac{\partial I_\nu}{\partial s}=-\rho(\alpha_\nu+\sigma_\nu)\,I_\nu+\rho\eta_\nu+\rho\,\frac{\sigma_\nu}{4\pi}\int I_\nu(P,r')\,\gamma_\nu(P,r',r)\,d\omega' \qquad (1)$$

or

$$\frac{1}{\rho}\frac{\partial I_\nu}{\partial s}=\eta_\nu+\frac{\sigma_\nu}{4\pi}\int I_\nu(P,r')\,\gamma_\nu(P,r',r)\,d\omega'-(\alpha_\nu+\sigma_\nu)\,I_\nu. \qquad (2)$$

Thus, we obtain a new relation between the unknown functions, i.e., between the characteristic quantities I_ν and η_ν of the radiation field. Consequently, only one of these functions can be arbitrarily assigned.

Kuznetsov gives another, more concise form to the equation of the radiative heat influx, Since

$$\frac{\partial I_\nu}{\partial s}=\frac{\partial I_\nu}{\partial x}\cos(r,x)+\frac{\partial I_\nu}{\partial y}\cos(r,y)+\frac{\partial I_\nu}{\partial z}\cos(r,z),$$

the equation of radiative transfer can be written in the following form:

$$\frac{1}{\rho}\left[\frac{\partial I_\nu}{\partial x}\cos(r,x)+\frac{\partial I_\nu}{\partial y}\cos(r,y)+\frac{\partial I_\nu}{\partial z}\cos(r,z)\right]= \\ =\eta_{l\nu}+\frac{\sigma_\nu}{4\pi}\int I_\nu(P,r')\,\gamma_\nu(P,r',r)\,d\omega'-(\alpha_\nu+\sigma_\nu)\,I_\nu. \tag{3}$$

This equation is derived for a definite direction r of the ray. Let us multiply it by a corresponding element of solid angle $d\omega$ and integrate it over all possible directions, assuming that

$$\int\frac{\partial I_\nu}{\partial s}d\omega=\frac{\partial}{\partial x}\int I_{\nu,x}\cos(r,x)\,d\omega+\frac{\partial}{\partial y}\int I_{\nu,y}\cos(r,y)\,d\omega+ \\ +\frac{\partial}{\partial z}\int I_{\nu,z}\cos(r,z)\,d\omega. \tag{4}$$

But the integrals of equation (4) are, according to the relations (3) § 9, components of the energy flux vector F_ν.

Therefore

$$\int\frac{\partial I_\nu}{\partial s}\,d\omega=\frac{\partial F_{\nu,x}}{\partial x}+\frac{\partial F_{\nu,y}}{\partial y}+\frac{\partial F_{\nu,z}}{\partial z}=\operatorname{div}F_\nu. \tag{5}$$

The following integral can be transformed in this way:

$$\int\left\{\int I_\nu(P,r')\,\gamma_\nu(P,r',r)\,d\omega'\right\}d\omega=\int d\omega' I_\nu(P,r')\int\gamma_\nu(P,r',r)\,d\omega= \\ =4\pi\int I_\nu(P,r')\,d\omega'=4\pi\int I_\nu(P,r)\,d\omega. \tag{6}$$

Thus, the equation of radiative transfer will assume the form

$$\frac{1}{\rho}\operatorname{div}F_\nu=4\pi\eta_\nu-\alpha_\nu\int I_\nu(P,r)\,d\omega. \tag{7}$$

By integrating over all frequencies, we obtain

$$\frac{1}{\rho}\operatorname{div}F=4\pi\eta-\int_0^\infty\alpha_\nu d\nu\int J_\nu(P,r)\,d\omega. \tag{8}$$

It has, however, been shown above that

$$q=\int_0^\infty\alpha_\nu d\nu_\nu\int I_\nu d\omega-4\pi\eta.$$

Consequently, the heat influx can be expressed by the equation

$$q=-\frac{1}{\rho}\operatorname{div}F. \tag{9}$$

Thus, the heat influx at a given point is determined without ambiguity by the divergence of the radiant energy flux. The particle obtains more energy than it absorbs, if $\operatorname{div}F>0$. The particle absorbs more that it emits if $\operatorname{div}F<0$. Finally, radiative equilibrium prevails for $\operatorname{div}F=0$, i.e., the particle emits the same quantity of heat as it absorbs.

By integrating the equation of heat influx over the volume, we obtain

$$-\int_\tau q\rho\,d\tau=\int_\tau\operatorname{div}F\,d\tau, \tag{10}$$

or, by transforming the volume integral on the right-hand side into a surface integral by an application of Gauss' theorem, we obtain

$$-\int_\tau q\rho\,d\tau=\int_\Sigma F_n\,d\sigma, \tag{11}$$

§ 11. A Complete System of Equations for Hydromechanics, Thermodynamics, and Radiation Theory. Considerations of Humidity

Let us now collect the equations for hydromechanics, thermodynamics and radation theory:

$$\frac{du}{dt} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{3}\frac{\mu}{\rho}\frac{\partial}{\partial x}\operatorname{div} v + \frac{\mu}{\rho}\Delta u, \tag{1}$$

$$\frac{dv}{dt} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{3}\frac{\mu}{\rho}\frac{\partial}{\partial y}\operatorname{div} v + \frac{\mu}{\rho}\Delta v, \tag{2}$$

$$\frac{dw}{dt} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{3}\frac{\mu}{\rho}\frac{\partial}{\partial z}\operatorname{div} v + \frac{\mu}{\rho}\Delta w, \tag{3}$$

$$\frac{d\rho}{dt} + \rho \operatorname{div} v = 0, \tag{4}$$

$$\frac{p}{\rho} = RT, \tag{5}$$

$$k\Delta T + \rho\int_0^\infty a_\nu\, d\nu \int I_\nu\, d\omega - \frac{2}{3}\mu(\operatorname{div} v)^2 + \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right\} = \rho c_v \frac{dT}{dt} + p \operatorname{div} v + 4\pi\rho\int_0^\infty \eta_\nu\, d\nu, \tag{6}$$

$$\frac{1}{\rho}\frac{\partial I_\nu}{\partial s} = \eta_\nu + \frac{\sigma_\nu}{4\pi}\int I_\nu(P, r')\cdot\gamma_\nu(t, P, r', r)\, d\omega' - (a_\nu + \sigma_\nu) I_\nu, \tag{7}$$

$$\eta_\nu = a_\nu E_\nu, \tag{8}$$

$$E_\nu = \frac{c_1}{c^5}\cdot\frac{\nu^5}{e^{\frac{c_2\cdot\nu}{c\cdot T}} - 1}. \tag{9}$$

Thus nine equations, determining nine unknown functions: u, v, w, p, ρ, T, η_ν, E_ν, I_ν, are obtained; the first six unknown functions depend on the coordinates and on the time; the seventh and the eighth depend in addition on the frequency of the radiation, and the last on the ray direction r. Consequently there are seven independent variables: $x, y, z, t, \nu, \theta, \alpha$.

In addition, the equations (1) - (9) also contain, together with the constants μ, k, c, c_1, c_2, the functions

$$a_\nu(P, t),\ \sigma_\nu(P, t),\ \gamma_\nu(P, t, r, r'),$$

which can be regarded as given. Instead of assigning the functions a_ν, σ_ν, γ_ν we may specify their connection with some of the unknown functions, for example, with the temperature T and the pressure p.

The system of equations for hydromechanics, thermodynamics and radiation theory was worked out on the assumption that the moving fluid is at the same time the material medium which absorbs, emits and scatters radiant energy.

These equations cannot, however, be directly applied to the atmosphere. This can be explained first of all by the fact that dry air plays an insignificant role in the processes of absorption and radiation. The role of the gas molecules constituting dry and pure air is important only for the processes of scattering.

Water vapor plays a decisive role in atmospheric radiation processes. A much slighter role, though quite important, is played by carbon dioxide (in the lower layers of the atmosphere), and by ozone (in the stratosphere). Finally, during these processes a very important role is played by water in the form of liquid drops, by ice suspended in the atmosphere in the form of clouds, and also by solid impurities.

It is as yet impossible, from the theoretical point of view, to take into account all the essentially significant factors, which govern the processes of radiation in the atmosphere. We will be content here with a demonstration of the way in which the influence of water vapor can be taken into account, on the basis of the equations obtained above, and shall do so by following E.S. Kuznetsov's reasoning.

The values of R, c_p, c_v for moist air differ slightly from the corresponding constants for dry air. This difference may therefore be disregarded and should not be taken into account in the equations.

The coefficient of absorption is the only magnitude entering in our equations which essentially depends on the content of water vapor. Disregarding the absorption by dry air and assuming that water vapor is the only substance which is of importance for atmospheric radiation processes, we can substitute the product $\rho \cdot \alpha$ of the preceding equations by the product $\rho_w \alpha'$, where ρ_w is the water vapor density, and α' — the coefficient of absorption of water vapor. These products are only a part of the equations of heat balance and of the equations of radiative transfer.

The equations (6) and (7) will then be written as follows:

$$k\Delta T + \rho_w \int_0^\infty \alpha'_\nu \, d\nu \int I_\nu \, d\omega - \frac{2}{3}\mu (\operatorname{div} v)^2 + \mu \left\{ 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 = \rho c_v \frac{dT}{dt} + p \operatorname{div} v + 4\pi \rho_w \int_0^\infty \alpha'_\nu E_\nu \, d\nu, \tag{6'}$$

$$\frac{\partial I_\nu}{\partial s} = \rho_w \alpha'_\nu E_\nu + \rho \frac{\sigma_\nu}{4\pi} \int I_\nu (P, r') \gamma_\nu (t, P, r', r) \, d\omega' - (\rho_w \alpha'_\nu + \rho \sigma_\nu) I_\nu. \tag{7'}$$

By assuming

$$\alpha_\nu = \alpha'_\nu \frac{\rho_w}{\rho} = \alpha'_\nu s, \tag{10}$$

where s is the specific humidity, it can be seen that for moist air one can use the same system of equations (6), (7), assuming that its absorption coefficient is determined by equation (10).

However, such a determination of the absorption coefficient greatly complicates the problem. The absorption coefficient is still not a physical constant in this case, but depends on a new function — the specific humidity s.

The systems of equations (1) - (9) will thus still be incomplete, since the nine equations have ten unknown functions: u, v, w, p, ρ, T, γ_ν, E_ν, I_ν, s.

In the problems of dynamic meteorology s is often considered as a given function of the coordinates and of the time.

In general it is necessary, however, to work out an equation of the transport of water vapor in the atmosphere.

The equation of water vapor transport has the form of a diffusion equation, if neither condensation nor evaporation take place

$$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}+w\frac{\partial s}{\partial z}=D\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}+\frac{\partial^2 s}{\partial z^2}\right), \qquad (11)$$

$$[D]=[L^2T^{-1}],$$

where D is the coefficient of diffusion.

The system of equations (1) – (9) is closed by the equation of diffusion (11), if there is no evaporation or condensation. The problem of working out a complete system of equations becomes more complicated if evaporation or condensation do take place. In this case not only the equation of water vapor transport changes, but also the equation of heat balance.

The equation of water vapor transport takes the form:

$$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}+w\frac{\partial s}{\partial z}=D\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}+\frac{\partial^2 s}{\partial z^2}\right)+W, \qquad (12)$$

where W is the rate of evaporation, i. e., the quantity of water evaporated in a unit mass of moist air per unit time.

The term:

$$\varepsilon_3 = -L_w, \qquad (13)$$

appears in the equation of heat balance, where L is the latent heat of evaporation, and ε_3 – the quantity of heat used up per unit time for the vaporization of w g of water.

In order to determine the value of w it is necessary to derive a supplementary equation, describing the processes of evaporation of water drops or ice crystals.

As regards the equation of radiative transfer, it must be derived anew, and with it the expression which determines the heat influx caused by radiant energy, because it is obvious that the radiation processes taking place in a medium containing drops and crystals of different sizes must be determined by quite different equations.

Thus, it can be seen that a complete system of equations simultaneously describing the totality of atmospheric processes has not yet been obtained. Some sort of schematization of the process is inevitable, not only for the solution of these equations, but also in their derivation.

§ 12. The Different Types of Fluid Motion

A complete system of equations for hydromechanics, thermodynamics and radiation theory was worked out in the previous section. Some boundary and initial conditions must be furnished to these equations, in order to make the problem well defined.

The obtained system of equations can be solved in principle, since it is complete. It is only necessary to overcome the mathematical difficulties encountered in the integration of this complicated system. We still do not know, however, whether the eventually obtained solution for given initial and boundary conditions will be unique. No uniqueness theorem for the solution of the equations of hydromechanics for certain given boundary and initial conditions has yet been proved.

The sought solution of the system of equations must satisfy one more condition, namely, the motion determined by this solution must be stable. This means that small disturbances of the motion must gradually die out. The motion becomes turbulent, if the small disturbances are not extinguished, but amplified.

It has been shown by observations that the motion of fluids is turbulent as a rule. Only in exceptional cases, when special conditions are provided, can laminar motions be obtained. Natural flows are always turbulent. Any attempt to solve completely the problem of the turbulent motion of a fluid is quite futile, if by this is understood a determination of the laws of motion for all fluid particles. The determination of the trajectory of a separate fluid particle in turbulent flow is quite impossible. The equations of motion for a viscous fluid have not a great practical value for the study of natural flows.

It is natural to consider a field of turbulent motion as a random distribution of the field vector. A random field can be only studied by statistical methods. In such a case some "mean" dynamical variables are used – the "mean" velocity, "mean" pressure, etc. Osborne Reynolds, the founder of the theory of turbulence, was the first to indicate such a possibility.

Reynolds classifies all fluid motions into molecular (or thermal), and macroscopic (or mechanical).

The molecular or thermal motion, some idea of which is given by the Brownian movement, is defined as the irregular incessant movement of molecules in apparently motionless fluids or gases. The molecular or thermal motion, of course, never ceases also in moving fluids (or gases).

Molecular motion does not change the macroscopic state of the fluid, if it is in statistical equilibrium, since the processes proceeding in any direction are completely compensated by processes in the opposite direction. A systematic preponderance of processes in one direction over the reverse processes is observed when the fluid is not in a statistical equilibrium, as a result of which the macroscopic state of the system changes, gradually tending toward a state of statistical equilibrium.

Processes which bring a fluid nearer to a state of equilibrium are called relaxation processes. This concerns: 1) internal friction or viscosity, which becomes apparent in a gradual stopping of the fluid motion; 2) thermal conduction, which becomes apparent in equalization of temperatures; 3) diffusion, i. e., a mutual penetration of two gases initially separated from each other.

The periods of molecular motions are infinitely small as compared with the periods of macroscopic motions. Therefore, the molecular and macroscopic motions can always be differentiated. The molecular motion can be considered as relative, and the macroscopic as transport motion, considering the actual motion of a fluid particle to be absolute. In this case the macroscopic motion appears as a sort of average over a huge number of molecules.

The macroscopic motion of an actual fluid can be either laminar or turbulent.

Laminar motion is characterized by the fact that all the particles in a sufficiently thin layer of the fluid move with the same velocity, as a result of which the currents in the fluid would appear, if they were visible, as solid shafts. When passing from one layer to another in laminar motion the velocity changes gradually and continuously as a consequence of internal friction, and as a result of adhesion to the bounding walls, if such walls exist.

The laminar nature of the fluid flow lasts as long as the quantity called the Reynolds number (see Chapter IX § 4) does not reach a definite critical value. The motion ceases to be laminar and becomes turbulent when the Reynolds number exceeds the critical value.

Turbulent motion is characterized by the fact that, apart from the average flow velocity, every particle of the fluid has its own additional velocity. Therefore during turbulence, while the particle moves with the mean flow velocity, its motion will appear to be very irregular, recalling by its chaotic character the thermal motion of molecules, with the only difference that in molecular motion the molecules themselves are the moving bodies, but the aggregates of molecules which participate in turbulent motion change both their form and their mass during the motion.

Thus, macroscopic turbulent motion can be considered as an absolute motion of fluid particles, consisting of an average fundamental motion, common to the whole flow, and of relative additional motions which differ for different particles. The periods of the relative additional motions are considerably smaller than the period of the fundamental average motion.

The first experiments carried out under laboratory conditions in order to compare laminar and turbulent flow have already shown that the difference between them is not only of an external kinematic kind, but a more profound dynamic difference. It is established that the drag exerted by the streaming fluid during laminar flow is proportional to the first power of the velocity, but that of turbulent flow is proportional to the square of the velocity.

This is bound up with the fact that during the laminar motion the kinetic energy of the flow is converted directly into heat, i. e., into kinetic energy of the molecules, but during turbulent motion the kinetic energy of the fundamental averaged flow is not converted directly into heat, but first into kinetic energy of a relative additional macroscopic motion, and only subsequently into thermal energy.

Just as the molecular motion in the absence of equilibrium leads to processes of internal friction, of heat conduction and diffusion, so the additional macroscopic motion leads to turbulent friction, to turbulent thermal conduction and turbulent diffusion.

Laminar motion is rarely observed in nature – more often in the so–called laminar boundary layers, adjoining the boundaries of a fluid.

§ 13. The Equations of Average Motion of a Fluid

The equations of motion of a viscous fluid appear to be completely inapplicable to the study of turbulent motion.

The structure of turbulent flow is so complicated, and all the magnitudes which enter in the equations of motion (velocity, temperature, pressure) are subjected to such spatial and temporal variations during turbulent motion, that even if the equations of the instantaneous velocities could be applied to an isolated particle, the integration of these equations would have proved to be quite hopeless.

On the other hand, none of our instruments are suitable for the measurement of instantaneous hydrodynamic magnitudes; measurements yield only averages of these magnitudes over some range. Therefore, it is natural to derive average equations of motion for the average variables, refraining for the time being from

any attempt to reproduce an exact picture of the turbulent flow at an arbitrary point in space, and at any instant of time.

We start the study of atmospheric turbulence with the elaboration of the operation of averaging.

We will mostly average the values of the function over some period of time Δt, which must be of a long enough duration to yield representative average values which will change not too quickly. On the other hand this period should not be excessive in order to avoid undue suppression of the essentially important changes in the averaged function.

The average value of the function will be represented by a horizontal bar over the symbol which designates this function.

Thus, the time average of a function will be defined by the following equation

$$\bar{\varphi} = \frac{1}{\Delta t} \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \varphi \cdot dt. \tag{1}$$

Since the order of the operations of averaging and spatial differentiation (in an arbitrary direction) may be inverted, the following relation holds:

$$\overline{\frac{\partial \varphi}{\partial l}} = \frac{\partial \bar{\varphi}}{\partial l}; \quad \overline{\frac{\partial^2 \varphi}{\partial l^2}} = \frac{\partial^2 \bar{\varphi}}{\partial l^2}. \tag{2}$$

In addition we postulate the following equality:

$$\overline{\frac{\partial \varphi}{\partial t}} = \frac{\partial \bar{\varphi}}{\partial t}. \tag{3}$$

For two functions φ and ψ we have

$$\overline{\varphi + \psi} = \bar{\varphi} + \bar{\psi}, \tag{4}$$

and in general

$$\overline{\sum_i \varphi_i} = \sum_i \bar{\varphi}_i. \tag{5}$$

The average value of a vector (or tensor) will be obtained by averaging each of its components separately; thus

$$\overline{\operatorname{grad} \varphi} = \operatorname{grad} \bar{\varphi}; \quad \overline{\operatorname{div} \varphi} = \operatorname{div} \bar{\varphi}; \quad \overline{\Delta \varphi} = \Delta \bar{\varphi}. \tag{6}$$

Let us consider now not the function φ, but its deviation φ' from the mean value

$$\varphi' = \varphi - \bar{\varphi}. \tag{7}$$

The following identities are obvious

$$\bar{\varphi}' = \overline{\varphi - \bar{\varphi}} = \bar{\varphi} - \bar{\bar{\varphi}} = \bar{\varphi} - \bar{\varphi} = 0. \tag{8}$$

Let us evaluate the average product of two functions φ and ψ

$$\overline{\varphi \cdot \psi} = \overline{(\bar{\varphi} + \varphi')(\bar{\psi} + \psi')} = \bar{\varphi} \cdot \bar{\psi} + \overline{\bar{\varphi} \cdot \psi'} + \overline{\bar{\psi} \cdot \varphi'} + \overline{\varphi' \cdot \psi'}.$$

Since the second and the third terms vanish when averaged, we finally obtain:

$$\overline{\varphi \cdot \psi} = \bar{\varphi} \cdot \bar{\psi} + \overline{\varphi' \psi'}. \tag{9}$$

We assume for the sake of simplicity that the fluid is incompressible. Let us average any of the Navier–Stokes equations such as the equation for the x axis (the two others are similarly treated), not taking as yet the external forces into consideration. Let us write the equation of motion in terms of the stresses:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z} \tag{10}$$

and add to it the equation of continuity for an incompressible fluid after first multiplying it by

$$\rho u \frac{\partial u}{\partial x} + \rho u \frac{\partial v}{\partial y} + \rho u \frac{\partial w}{\partial z} = 0. \tag{11}$$

By adding equations (10) and (11) we obtain

$$\rho \frac{\partial u}{\partial t} + \frac{\partial \rho u u}{\partial x} + \frac{\partial \rho u v}{\partial y} + \frac{\partial \rho u w}{\partial z} = \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z} \tag{12}$$

or

$$\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(p_{xx} - \rho u u) + \frac{\partial}{\partial y}(p_{xy} - \rho u v) + \frac{\partial}{\partial z}(p_{xz} - \rho u w). \tag{13}$$

Let us integrate each term of equation (13) with respect to the time from $t - (\Delta t/2)$ to $t + (\Delta t/2)$ in order to obtain the equation of the averaged motion. Taking into consideration the equalities for the operation of averaging derived above, we obtain:

$$\rho \frac{\partial \bar{u}}{\partial t} = \frac{\partial}{\partial x}(\bar{p}_{xx} - \rho \bar{u}\bar{u} - \rho \overline{u'u'}) + \frac{\partial}{\partial y}(\bar{p}_{xy} - \rho \bar{u}\bar{v} - \rho \overline{u'v'}) + \frac{\partial}{\partial z}(\bar{p}_{xz} - \rho \bar{u}\bar{w} - \rho \overline{u'w'}). \tag{14}$$

By transferring the terms containing the mean velocities to the left-hand side of the equation, and taking into consideration that the equation of continuity for the average motion takes the form of

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0, \tag{15}$$

we obtain the Reynolds equation for the average motion:

$$\left.\begin{aligned}
&\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}\right) = \\
&\quad = \frac{\partial}{\partial x}(\bar{p}_{xx} - \rho \overline{u'u'}) + \frac{\partial}{\partial y}(\bar{p}_{xy} - \rho \overline{u'v'}) + \frac{\partial}{\partial z}(\bar{p}_{xz} - \rho \overline{u'w'}), \\
&\rho\left(\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \bar{w}\frac{\partial \bar{v}}{\partial z}\right) = \\
&\quad = \frac{\partial}{\partial x}(\bar{p}_{xy} - \rho \overline{u'v'}) + \frac{\partial}{\partial y}(\bar{p}_{yy} - \rho \overline{v'v'}) + \frac{\partial}{\partial z}(\bar{p}_{yz} - \rho \overline{v'w'}), \\
&\rho\left(\frac{\partial \bar{w}}{\partial t} + \bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{v}\frac{\partial \bar{w}}{\partial y} + \bar{w}\frac{\partial \bar{w}}{\partial z}\right) = \\
&\quad = \frac{\partial}{\partial x}(\bar{p}_{xz} - \rho \overline{u'w'}) + \frac{\partial}{\partial y}(\bar{p}_{yz} - \rho \overline{v'w'}) + \frac{\partial}{\partial z}(\bar{p}_{zz} - \rho \overline{w'w'}),
\end{aligned}\right\} \tag{16}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are the components of the mean velocity; $u' = u - \bar{u}$, $v' = v - \bar{v}$, $w' = w - \bar{w}$ the components of the pulsation velocity [eddy velocity]. It should be noted that the eddy velocities generally are not at all small in comparison with the mean velocities, but are quantities of the same order of magnitude. The additional

stresses which evolve in the fluid as a result of turbulence are, on the contrary, significantly greater than the stresses of viscosity. The terms of the Reynolds equations expressing the influence of the viscosity can therefore be omitted, and these equations for an incompressible fluid assume the form

$$\left.\begin{aligned}
&\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}-\left(\frac{\partial \overline{u'u'}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right),\\
&\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}-\left(\frac{\partial \overline{u'v'}}{\partial x}+\frac{\partial \overline{v'v'}}{\partial y}+\frac{\partial \overline{v'w'}}{\partial z}\right),\\
&\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z}-\left(\frac{\partial \overline{u'w'}}{\partial x}+\frac{\partial \overline{v'w'}}{\partial y}+\frac{\partial \overline{w'w'}}{\partial z}\right).
\end{aligned}\right\} \qquad (17)$$

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0.$$

The equations which we derived for the average motion of an incompressible fluid can be generalized to include the case of a compressible fluid, if the momentum density $\boldsymbol{m}=\rho\boldsymbol{v}$. is introduced instead of the velocity $\boldsymbol{v}$. We will now write the averaged equation of motion for a compressible fluid in a definitive form, omitting its derivation:

$$\left.\begin{aligned}
&\frac{\partial \bar{m}_x}{\partial t}+\frac{\partial \overline{um_x}}{\partial x}+\frac{\partial \overline{vm_x}}{\partial y}+\frac{\partial \overline{wm_x}}{\partial z}=\\
&\quad=-\frac{\partial \bar{p}}{\partial x}+\frac{\partial}{\partial x}\overline{u'm'_x}+\frac{\partial}{\partial y}\overline{v'm'_x}+\frac{\partial}{\partial z}\overline{w'm'_x},\\
&\frac{\partial \bar{m}_y}{\partial t}+\frac{\partial \overline{um_y}}{\partial x}+\frac{\partial \overline{vm_y}}{\partial y}+\frac{\partial \overline{wm_y}}{\partial z}=\\
&\quad=-\frac{\partial \bar{p}}{\partial y}+\frac{\partial}{\partial x}\overline{u'm'_y}+\frac{\partial}{\partial y}\overline{v'm'_y}+\frac{\partial}{\partial z}\overline{w'm'_y},\\
&\frac{\partial \bar{m}_z}{\partial t}+\frac{\partial \overline{um_z}}{\partial x}+\frac{\partial \overline{vm_z}}{\partial y}+\frac{\partial \overline{wm_z}}{\partial z}=\\
&\quad=-\frac{\partial \bar{p}}{\partial z}+\frac{\partial}{\partial x}\overline{u'm'_z}+\frac{\partial}{\partial y}\overline{v'm'_z}+\frac{\partial}{\partial z}\overline{w'm'_z},
\end{aligned}\right\} \qquad (18)$$

where m_x, m_y, m_z are the components of the momentum density $\boldsymbol{m}$, while m'_x, m'_y, m'_z are their pulsations [fluctuations].

It is obvious that the system of the Reynolds equations is incomplete, since we have four equations, containing ten unknown functions $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{p}$, and six Reynolds parameters of turbulence $\overline{u'u'}$, $\overline{u'v'}$, $\overline{u'w'}$, $\overline{v'v'}$, $\overline{v'w'}$, $\overline{w'w'}$.

In the search for additional relationships between the velocity fluctuations, the theory of turbulence has developed two approaches, the semi-empirical and the statistical theory of turbulence.

The fundamental considerations of the semi-empirical theory of turbulence are the following. After Stokes the stress components due to molecular viscosity are linearly expressed in terms of the components of the velocity of deformation. Is it impossible to use the relations

$$\left.\begin{aligned}
&-\rho\overline{u'u'}=2\mu_T\frac{\partial \bar{u}}{\partial x};\quad -\rho\overline{v'v'}=2\mu_T\frac{\partial \bar{v}}{\partial y};\quad -\rho\overline{w'w'}=2\mu_T\frac{\partial \bar{w}}{\partial z},\\
&-\rho\overline{u'v'}=\mu_T\left(\frac{\partial \bar{u}}{\partial y}+\frac{\partial \bar{v}}{\partial x}\right);\qquad -\rho\overline{u'w'}=\mu_T\left(\frac{\partial \bar{u}}{\partial z}+\frac{\partial \bar{w}}{\partial x}\right);\\
&-\rho\overline{w'w'}=\mu_T\left(\frac{\partial \bar{v}}{\partial z}+\frac{\partial \bar{w}}{\partial y}\right).
\end{aligned}\right\} \qquad (19)$$

for the average motion, in analogy with the expressions for the stresses?

The system of equations for the average motion of an incompressible fluid will then be complete, and will assume the form

$$\left.\begin{aligned}
&\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}= \\
&\quad=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(\mu_T\frac{\partial \bar{u}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu_T\frac{\partial \bar{u}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu_T\frac{\partial \bar{u}}{\partial z}\right)\right], \\
&\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}= \\
&\quad=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}+\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(\mu_T\frac{\partial \bar{v}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu_T\frac{\partial \bar{v}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu_T\frac{\partial \bar{v}}{\partial z}\right)\right], \\
&\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}= \\
&\quad=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z}+\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(\mu_T\frac{\partial \bar{w}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu_T\frac{\partial \bar{w}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu_T\frac{\partial \bar{w}}{\partial z}\right)\right].
\end{aligned}\right\} \qquad (20)$$

While this may be possible, it should not, however, be forgotten that the coefficient of eddy viscosity μ_T has nothing to do with the coefficient of ordinary viscosity μ. The coefficient of viscosity μ depends on the physical properties of the fluid. The coefficient of eddy viscosity μ_T does not depend on them at all. It is entirely determined by the character of the flow.

In contrast to the coefficient of viscosity, which is a scalar magnitude, the coefficient of eddy viscosity is generally a tensor.

Thus, even though an analogy does exist between the irregular turbulent motion and the irregular molecular motion, this analogy is restricted.

Similarly, though we can draw an analogy between the averaged equations of Stokes and the Navier-Stokes equations, this analogy is also restricted. It can, however, be helpful in some cases.

We should note that the viscosity plays no role in the derivation of the averaged equation of Reynolds. This equation could be obtained in the same way by averaging the equations of motion for perfect fluids.

Thus, the semi-empirical theory of turbulence makes it possible to express only very roughly the eddy stresses by the components of the mean velocity, $\bar{u}$, $\bar{v}$, $\bar{w}$. In the equations of average motion, generally speaking, other different coefficients of turbulence appear together with the unknown functions $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{p}$, which must also be determined.

A vertical coefficient of eddy viscosity μ, and a horizontal coefficient μ' are necessary for the atmosphere. The equations of average motion can be then written in the following form:

$$\left.\begin{aligned}
&\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z}=\\
&\quad=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}+\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(\mu'\frac{\partial\bar{u}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu'\frac{\partial\bar{u}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial\bar{u}}{\partial z}\right)\right],\\
&\frac{\partial\bar{v}}{\partial t}+\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}+\bar{w}\frac{\partial\bar{v}}{\partial z}=\\
&\quad=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial y}+\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(\mu'\frac{\partial\bar{v}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu'\frac{\partial\bar{v}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial\bar{v}}{\partial z}\right)\right],\\
&\frac{\partial\bar{w}}{\partial t}+\bar{u}\frac{\partial\bar{w}}{\partial x}+\bar{v}\frac{\partial\bar{w}}{\partial y}+\bar{w}\frac{\partial\bar{w}}{\partial z}=\\
&\quad=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial z}+\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(\mu'\frac{\partial\bar{w}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu'\frac{\partial\bar{w}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial\bar{w}}{\partial z}\right)\right],\\
&\qquad\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}=0.
\end{aligned}\right\}\qquad(21)$$

§ 14. Transformation of Kinetic Energy of the Average Motion into Kinetic Energy of Eddies

Let us consider the transformation of the kinetic energy of average motion into kinetic energy of eddy pulsation. We will use the equations of motion of an incompressible fluid in the following form, neglecting the volume forces for the sake of simplicity:

$$\left.\begin{aligned}
\rho\frac{\partial u}{\partial t}&=-\left\{\frac{\partial}{\partial x}(p_{xx}+\rho uu)+\frac{\partial}{\partial y}(p_{yx}+\rho uv)+\frac{\partial}{\partial z}(p_{zx}+\rho uw)\right\},\\
\rho\frac{\partial v}{\partial t}&=-\left\{\frac{\partial}{\partial x}(p_{xy}+\rho vu)+\frac{\partial}{\partial y}(p_{yy}+\rho vv)+\frac{\partial}{\partial z}(p_{zy}+\rho vw)\right\},\\
\rho\frac{\partial w}{\partial t}&=-\left\{\frac{\partial}{\partial x}(p_{xz}+\rho wu)+\frac{\partial}{\partial y}(p_{yz}+\rho wv)+\frac{\partial}{\partial z}(p_{zz}+\rho ww)\right\}.
\end{aligned}\right\}\qquad(1)$$

By assuming

$$u=\bar{u}+u';\ v=\bar{v}+v';\ w=\bar{w}+w',$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are the components of the velocity, averaged over the volume and referred to the center of mass of this volume, we obtain for the x axis the equation of averaged motion in the form

$$\rho\frac{\partial\bar{u}}{\partial t}=-\left\{\frac{\partial}{\partial x}(\bar{p}_{xx}+\rho\bar{u}\cdot\bar{u}+\rho\overline{u'u'})+\frac{\partial}{\partial y}(\bar{p}_{yx}+\rho\bar{u}\cdot\bar{v}+\rho\overline{u'v'})+\right.$$
$$\left.+\frac{\partial}{\partial z}(\bar{p}_{zx}+\rho\bar{u}\cdot\bar{w}+\rho\overline{u'w'})\right\}\qquad(2)$$

and two analogous equations for the y and z axes. Since

$$\rho uu-(\rho\bar{u}\,\bar{u}+\rho\overline{u'u'})=\rho(\bar{u}u'+u'\bar{u})+\rho(u'u'-\overline{u'u'}),$$

we obtain by subtracting equation (2) term by term from equation (1) the following equation for the x axis:

$$\rho\frac{\partial u'}{\partial t}=-\left\{\frac{\partial}{\partial x}\left[p'_{xx}+\rho\left(\bar{u}u'+u'\bar{u}\right)+\rho\left(u'u'-\overline{u'u'}\right)\right]+\frac{\partial}{\partial y}\left[p'_{yx}+\right.\right.$$
$$+\rho\left(\bar{u}v'+u'\bar{v}\right)+\rho\left(u'v'-\overline{u'v'}\right)\Big]+\frac{\partial}{\partial z}\left[p'_{zx}+\rho\left(\bar{u}w'+u'\bar{w}\right)+\right. \tag{3}$$
$$\left.\left.+\rho\left(u'w'-\overline{u'w'}\right)\right]\right\}$$

and two analogous equations for the y and z axes.

By multiplying equation (2) for the average motion by $\bar{u}$, $\bar{v}$, $\bar{w}$, respectively, adding the results, and setting

$$2\bar{E}=\rho\left(\bar{u}^2+\bar{v}^2+\bar{w}^2\right),$$

we obtain

$$\left.\begin{aligned}
&\frac{\partial\bar{E}}{\partial t}=\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}+\bar{v}\frac{\partial}{\partial y}+\bar{w}\frac{\partial}{\partial z}\right)\bar{E}=\\
&=-\left\{\frac{\partial}{\partial x}\left[\bar{u}\left(\bar{p}_{xx}+\rho\overline{u'u'}\right)\right]+\frac{\partial}{\partial y}\left[\bar{u}\left(\bar{p}_{yx}+\rho\overline{u'v'}\right)\right]+\right.\\
&\quad+\frac{\partial}{\partial z}\left[\bar{u}\left(\bar{p}_{zx}+\rho\overline{u'w'}\right)\right]+\frac{\partial}{\partial x}\left[\bar{v}\left(\bar{p}_{xy}+\rho\overline{v'u'}\right)\right]+\\
&\quad+\frac{\partial}{\partial y}\left[\bar{v}\left(\bar{p}_{yy}+\rho\overline{v'v'}\right)\right]+\frac{\partial}{\partial z}\left[\bar{v}\left(\bar{p}_{zy}+\rho\overline{v'w'}\right)\right]+\\
&+\frac{\partial}{\partial x}\left[\bar{w}\left(\bar{p}_{xz}+\rho\overline{w'u'}\right)\right]+\frac{\partial}{\partial y}\left[\bar{w}\left(\bar{p}_{yz}+\rho\overline{w'v'}\right)\right]+\frac{\partial}{\partial z}\left[\bar{w}\left(\bar{p}_{zz}+\rho\overline{w'w'}\right)\right]+\\
&+\left[\bar{p}_{xx}\frac{\partial\bar{u}}{\partial x}+\bar{p}_{yx}\frac{\partial\bar{u}}{\partial y}+\bar{p}_{zx}\frac{\partial\bar{u}}{\partial z}+\bar{p}_{xy}\frac{\partial\bar{v}}{\partial x}+\bar{p}_{yy}\frac{\partial\bar{v}}{\partial y}+\bar{p}_{zy}\frac{\partial\bar{v}}{\partial z}+\right.\\
&+\left.\bar{p}_{xz}\frac{\partial\bar{w}}{\partial x}+\bar{p}_{yz}\frac{\partial\bar{w}}{\partial y}+\bar{p}_{zz}\frac{\partial\bar{w}}{\partial z}\right]+\rho\left[\overline{u'u'}\frac{\partial\bar{u}}{\partial x}+\overline{u'v'}\frac{\partial\bar{u}}{\partial y}+\overline{u'w'}\frac{\partial\bar{u}}{\partial z}+\right.\\
&\left.\left.+\overline{v'u'}\frac{\partial\bar{v}}{\partial x}+\overline{v'v'}\frac{\partial\bar{v}}{\partial y}+\overline{v'w'}\frac{\partial\bar{v}}{\partial z}+\overline{w'u'}\frac{\partial\bar{w}}{\partial x}+\overline{w'v'}\frac{\partial\bar{w}}{\partial y}+\overline{w'w'}\frac{\partial\bar{w}}{\partial z}\right]\right\}.
\end{aligned}\right\} \tag{4}$$

We thus obtained the energy equation for the average motion.

By multiplying the equations (3) for pulsating eddy motion by u', v', w' respectively, and by adding the results of multiplication, we obtain the equation of energy for eddy motion at each point of the flow. In order to be able to compare this equation to equation (4), let us refer them to the same volume, which moves with the average velocity of the flow. Averaging the equation over this volume, and setting

$$2E'=\rho\left(u'^2+v'^2+w'^2\right),$$

where E' stands for the kinetic energy of eddy pulsation, we obtain the following equation of kinetic energy for pulsating eddy motion:

$$\frac{d\bar{E}'}{dt}=\frac{\partial}{\partial t}\left(\bar{u}\frac{\partial}{\partial x}+\bar{v}\frac{\partial}{\partial y}+\bar{w}\frac{\partial}{\partial z}\right)E'=$$
$$=-\left\{\frac{\partial}{\partial x}[u'(p'_{xx}+\rho u'u')]+\frac{\partial}{\partial y}[u'(p'_{yx}+\rho u'v')]+\right.$$
$$+\frac{\partial}{\partial z}[u'(p'_{zx}+\rho u'w')]+\frac{\partial}{\partial x}[v'(p'_{yx}+\rho v'u')]+$$
$$+\frac{\partial}{\partial y}[v'(p'_{yy}+\rho v'v')]+\frac{\partial}{\partial z}[v'(p'_{zy}+\rho v'w')]+$$
$$+\frac{\partial}{\partial x}[w'(p'_{zx}+\rho w'u')]+\frac{\partial}{\partial y}[w'(p'_{zy}+\rho w'v')]+$$
$$+\frac{\partial}{\partial z}[w'(p'_{zz}+\rho w'w')]+\left[p'_{xx}\frac{\partial u'}{\partial x}+p'_{yx}\frac{\partial u'}{\partial y}+\right. \quad (5)$$
$$+p'_{zx}\frac{\partial u'}{\partial z}+p'_{xy}\frac{\partial v'}{\partial x}+p'_{yy}\frac{\partial v'}{\partial y}+p'_{zy}\frac{\partial v'}{\partial z}+p'_{xz}\frac{\partial w'}{\partial x}+$$
$$\left.+p'_{yz}\frac{\partial w'}{\partial y}+p'_{zz}\frac{\partial w'}{\partial z}\right]-\rho\left[\overline{u'u'}\frac{\partial\bar{u}}{\partial x}+\overline{u'v'}\frac{\partial\bar{u}}{\partial y}+\right.$$
$$+\overline{u'w'}\frac{\partial\bar{u}}{\partial z}+\overline{v'u'}\frac{\partial\bar{v}}{\partial x}+\overline{v'v'}\frac{\partial\bar{v}}{\partial y}+\overline{v'w'}\frac{\partial\bar{v}}{\partial z}+$$
$$\left.\left.+\overline{w'u'}\frac{\partial\bar{w}}{\partial x}+\overline{w'v'}\frac{\partial\bar{w}}{\partial y}+\overline{w'w'}\frac{\partial\bar{w}}{\partial z}\right]\right\}$$

From this equation are omitted the terms containing the products of the three fluctuating magnitudes, because these terms disappear after integration of equation (5) over most regions of the flow pattern.

The rate of change of the total kinetic energy of the flow will be obtained by adding the rate of change of the kinetic energy of the average motion to the rate of change of the kinetic energy of the eddy pulsations, i.e.,

$$\frac{dE}{dt}=\frac{d\bar{E}}{dt}+\frac{d\bar{E}'}{dt}.$$

It is easy to see that by adding equations (4) and (5) the same expression is obtained, as by direct derivation of the expression for $\frac{dE}{dt}$, by means of a corresponding multiplication of the equations (1) by u, v, w, and their summation.

When adding the equations of energy (4) and (5), the terms in the last square brackets will not enter in the equation of the total energy. It is obvious that these terms, the only ones which do not influence the total kinetic energy of the flow, express the conversion of the kinetic energy of the average motion into kinetic energy of eddy pulsation. Attention should be drawn to the fact that the transformation of kinetic energy of the average flow into eddy kinetic energy does not depend on the physical properties of the fluid, but is determined only by the kinematic character of the flow, i.e., by the distribution of the average and eddy motions. If we denote by T the rate of conversion of kinetic energy of the average flow into kinetic energy of pulsation, referred to a unit volume, we obtain

$$T=-\rho\left[\overline{u'u'}\frac{\partial\bar{u}}{\partial x}+\overline{u'v'}\frac{\partial\bar{u}}{\partial y}+\overline{u'w'}\frac{\partial\bar{u}}{\partial z}+\overline{v'u'}\frac{\partial\bar{v}}{\partial x}+\overline{v'v'}\frac{\partial\bar{v}}{\partial y}+\overline{v'w'}\frac{\partial\bar{v}}{\partial z}+\right.$$
$$\left.+\overline{w'u'}\frac{\partial\bar{w}}{\partial x}+\overline{w'v'}\frac{\partial\bar{w}}{\partial y}+\overline{w'w'}\frac{\partial\bar{w}}{\partial z}\right]. \quad (6)$$

If we express by mean velocities all the terms appearing in the equation describing the transformation of kinetic energy, and assume

$$\left.\begin{aligned} -\rho\,\overline{u'u'} &= 2\mu_T\frac{\partial\bar{u}}{\partial x}, & -\rho\,\overline{u'v'} &= \mu_T\left(\frac{\partial\bar{v}}{\partial x}+\frac{\partial\bar{u}}{\partial y}\right), \\ -\rho\,\overline{v'v'} &= 2\mu_T\frac{\partial\bar{v}}{\partial y}, & -\rho\,\overline{u'w'} &= \mu_T\left(\frac{\partial\bar{w}}{\partial x}+\frac{\partial\bar{u}}{\partial z}\right), \\ -\rho\,\overline{w'w'} &= 2\mu_T\frac{\partial\bar{w}}{\partial z}, & -\rho\,\overline{v'w'} &= \mu_T\left(\frac{\partial\bar{w}}{\partial y}+\frac{\partial\bar{v}}{\partial z}\right), \end{aligned}\right\} \tag{7}$$

we shall obtain

$$T=\mu_T\left\{2\left[\left(\frac{\partial\bar{u}}{\partial x}\right)^2+\left(\frac{\partial\bar{v}}{\partial y}\right)^2+\left(\frac{\partial\bar{w}}{\partial z}\right)^2\right]+\left(\frac{\partial\bar{v}}{\partial x}+\frac{\partial\bar{u}}{\partial y}\right)^2+\left(\frac{\partial\bar{w}}{\partial x}+\frac{\partial\bar{u}}{\partial z}\right)^2+\left(\frac{\partial\bar{w}}{\partial y}+\frac{\partial\bar{v}}{\partial z}\right)^2\right\} \tag{8}$$

Thus, the expression for the converted kinetic energy of the average motion is quite similar to the expression for the dissipation of mechanical energy into heat with the sole difference that the coefficient of eddy viscosity μ_T, enters in the expression for the converted kinetic energy, while the coefficient of viscosity μ enters in the expression for the dissipation.

We have still to indicate the physical meaning of the other terms appearing on the right side of the energy equations (4) and (5). The first term of equation (4) expresses the rate of change of the kinetic energy of the average motion, as a result of the work performed by the stresses, and of the transfer of the average momentum by eddy pulsations. The first term of equation (5) expresses the rate of change of the kinetic energy of pulsation as a result of the work done by the stresses, and due to transfer of the momentum of the eddies.

The second term of equation (4) expresses the dissipation of the kinetic energy of the average motion, which is completely independent of the pulsation. Finally, the second term of equation (5) expresses the dissipation of the kinetic energy of eddy pulsations, and clearly this dissipation is independent of the average velocities.

Let us now integrate the equation of energy (5) over a volume moving with the mean velocity of the flow. The first term will then be expressed by two surface integrals, the first of which stands for the work done by the stresses on the boundary of the volume, while the second term is a measure of the rate of change of the kinetic energy of eddy pulsation, as a result of its transfer through the boundary surface. The volume under consideration can be chosen in such a way that the total energy transferred by pulsations through the boundary is equal to zero. The change of the kinetic energy of eddy pulsation will then be expressed by the difference between the quantity of energy, which is obtained by conversion of the kinetic energy of the average motion, and the quantity of energy dissipated into heat by viscosity.

Thus, we finally obtain the so-called resolving equation of Reynolds, in the form

$$\int_\tau \frac{dE'}{dt}\,d\tau=-\int_\tau D\,d\tau+\int_\tau T\,d\tau. \tag{9}$$

Equation (9) shows that the eddy kinetic energy increases if the part of the kinetic energy of the average motion which is converted into it exceeds the dissipation of mechanical into thermal energy. Otherwise the eddy kinetic energy decreases.

The cardinal point of Reynolds' theory, the division of motions into three forms (the average macroscopic, the relative macroscopic, and the thermal) by means of equation (9), can be put in the form of the following scheme: (Energy of the average motion) $\underrightarrow{T}$ (Energy of the relative motion) $\underrightarrow{D}$ (Energy of the thermal motion).

If T>D, the newly developed kinetic energy of eddy pulsation has no time to be converted into thermal energy. In this case the laminar motion is unstable, and turbulence increases. If, on the contrary, T<D, the developed pulsations are damped by the forces of viscosity. The laminar motion is in this case stable, and turbulence is suppressed.

§ 15. The Fridman–Keller Equations

The questions, whether the phenomena occurring in turbulent flow do or do not conform with some general law, and what is the inner structure of turbulent flow, remained quite untouched until lately. Only in the last years did a number of Soviet scientists – Fridman, Keller, Kolmogorov, Obukhov, Loistyanskii, Millionshchikov and others – make considerable advances in this problem and founded an entirely new statistical theory of turbulence. Fridman and Keller indicated in 1925 a system of parameters for the structure of the turbulent flow; they also worked out a mathematical method for an approximate solution of this problem.

In this section we will give the derivation of the Fridman–Keller equations, restricting ourselves to the examination of the incompressible case.

Let us consider the equations of Navier-Stokes for a viscous incompressible fluid in the form

$$\nu \Delta u_l - \frac{1}{\rho}\frac{\partial p}{\partial x_l} - \frac{\partial u_l}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_l}{\partial x_j} = 0; \quad (l = 1, 2, 3) \qquad (1)$$

and the equation of continuity

$$\sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} = 0. \qquad (2)$$

With the help of these equations for a laminar flow, it is possible in principle, as we have already indicated, to determine the unknown functions u_j (M) and p(M) at any point M of the flow pattern [at any future time] from their initial values.

However, this cannot be applied to the turbulent flow, since for it the initial values of the unknown functions are random.

The statistical school of Academician Kolmogorov attributes some rather common properties to the distributions of the hydrodynamic variables u_1, u_2, u_3, p in the turbulent flow, which assume there a great number of random values. The problem of turbulent motion is approached from this standpoint, by Millionshchikov for instance, in the following way.

The probability distribution of the initial conditions is given. The probability distribution of the "random functions" u_j(M) and p(M) of the point M at any subsequent moment t > 0 is to be determined.

Millionshchikov does not indicate the method for calculating the expectation values of the hydrodynamic variables. However, in order to pass from theory to experiment, it is necessary to introduce the ergodic hypothesis, extensively applied in statistical mechanics. This hypothesis states that averages over a sufficiently long period of time come arbitrarily close to the spatial expectation values.

The definition of averaging, used by Millionshchikov as an operation of obtaining the expectation values, has essential advantages as compared with the methods of averaging by means of various smoothing functions. This definition dispenses with the necessity to stipulate relationships concerning repeated application of averaging since repeated formation of the mathematical expectation only yields again the value obtained on the first time.

The deviations of the unknown functions u_1, u_2, u_3, p from their expectation values are denoted, following Millionshchikov, by the following symbols

$$\left.\begin{aligned} v_i &= u_i - \bar{u}_i, \\ v_0 &= p - \bar{p}, \end{aligned}\right\} \quad (i = 1, 2, 3). \tag{3}$$

Let us also assume for the sake of brevity that:

$$b_{\alpha\beta} = \overline{v_\alpha \cdot v_\beta}. \tag{4}$$

We obtain the Reynolds equations for the average motion of the point M' by forming the expectation values for equations (1) and (2)

$$\nu\Delta'\bar{u}_i' - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i'} - \frac{\partial \bar{u}_i'}{\partial t} - \sum_{j=1}^{3} \bar{u}_j' \cdot \frac{\partial \bar{u}_i'}{\partial x_j} - \sum_{j=1}^{3} \frac{\partial b_{ij}}{\partial x_j} = 0 \tag{5}$$

and the equation of continuity

$$\sum \frac{\partial \bar{u}_j}{\partial x_j} = 0. \tag{6}$$

As mentioned above, this system of equations is incomplete, since the four equations include ten unknown functions

$$\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{p}, b_{11}, b_{12}, b_{13}, b_{22}, b_{23}, b_{33}.$$

Fridman and Keller proposed to work out a complete system of equations, by correlating the fluctuations of velocity and pressure at two points in the flow pattern, and at two different instants. They proposed to take into consideration the so-called moments of correlation, i.e., the expectation values of the generation of pulsations of the hydrodynamic variables at different points of the flow, and at different instants.

We are interested in the moments of correlation since they complete the system of our equations and allow to solve them with respect to the unknown functions. The notion of "moment of correlation" makes it possible to introduce a new important parameter of turbulence - the scale of turbulence - which is no less important than the previous parameters, namely, the frequency and the amplitude of eddy pulsations. In addition, the moments of correlation can be experimentally measured. This opens the possibility of comparing theory with experiment.

Thus, the Fridman–Keller theory makes it possible to elucidate the physical structure of turbulence, and furnishes the basis for investigating the flows artificially created in wind tunnels.

For the second and third moments of correlation we use the notation

$$\left.\begin{aligned} B_{\alpha\beta} &= \overline{v_\alpha(M')\, v_\beta(M'')}, \\ B_{\alpha\beta\gamma} &= \overline{v_\alpha(M') \cdot v_\beta(M')\, v_\gamma(M''')}, \end{aligned}\right\} \tag{7}$$

where M', M'', M''' designate generally different points of four-dimensional space

$$M' \equiv (x_1', x_2', x_3', t');$$
$$M'' \equiv (x_1'', x_2'', x_3'', t'');$$
$$M''' \equiv (x_1''', x_2''', x_3''', t''').$$

By multiplying the systems of equations (1) and (2) respectively by the fluctuations v_i and v_0, and averaging the resulting products, we obtain 16 additional equations, interrelating the same unknown functions and some new functions as well, including the 10 moments of correlation of the second order: B_{11}, B_{12}, B_{13}, B_{22}, B_{23}, B_{33}, B_{01}, B_{02}, B_{03}, B_{00}. In addition, moments of correlation of the third order, which were rejected by Fridman and Keller because they considered them negligible in comparison with the second order moments, will enter in the equations. We thus obtain twelve additional equations of the type

$$\nu\Delta' B_{ik}(M', M'') - \frac{1}{\rho}\frac{\partial B_{0k}(M', M'')}{\partial x_i'} - \frac{\partial B_{ik}(M', M'')}{\partial t} - \sum_{j=1}^{3} \overline{u}_j' \frac{\partial B_{ik}(M', M'')}{\partial x_j} - \sum_{j=1}^{3} \frac{\partial \overline{u}_i}{\partial x_j} B_{jk}(M', M'') = 0 \qquad (8)$$

and four equations of the type

$$\sum_{j=1}^{3} \frac{\partial B_{jk}(M', M'')}{\partial x_j'} = 0 \qquad (9)$$

forming together with equations (5) and (6), a complete system of twenty equations in twenty unknown functions. A similar system of equations can also be worked out for the point M''.

We should note that more recent studies by Soviet scientists have shown that the moments of correlation of the third order can not always be neglected. However, many methods of completing the system of the equations of motion by means of new hypotheses, making it possible to express the third and the fourth moments by lower-order moments, have been indicated in these studies. Therefore these latest studies not only do not belittle the importance of the remarkable work of Fridman and Keller, but put an even greater emphasis on its immense importance for the study of the structure of turbulent flow.

Let us note, however, that the Fridman-Keller equations are so complicated that their integration in the general case has not yet been accomplished.

The system of Fridman-Keller's equations is usually treated in exceptional cases of turbulence in a quiescent or in a uniformly moving fluid when for the whole space $\overline{u}_i = 0$ (i = 1, 2, 3) can be assumed. In this case the Fridman-Keller equations assume the form

$$\nu\Delta' B_{ik}(M', M'') - \frac{1}{\rho}\frac{\partial B_{0k}(M', M'')}{\partial x_i'} - \frac{\partial B_{ik}(M', M'')}{\partial t'} = 0,$$
$$\sum_{j=1}^{3} \frac{\partial B_{jk}(M', M'')}{\partial x_j'} = 0,$$
$$\nu\Delta'' B_{ik}(M', M'') - \frac{1}{\rho}\frac{\partial B_{0k}(M'', M')}{\partial x_i''} - \frac{\partial B_{ik}(M'', M')}{\partial t''} = 0, \qquad (10)$$
$$\sum_{j=1}^{3} \frac{\partial B_{jk}(M'', M')}{\partial x_j''} = 0.$$

Chapter IX

THE EQUATIONS OF MOTION OF THE ATMOSPHERE AND THEIR SIMPLIFICATION

§ 1. The Equations of Motion of the Atmosphere Regarded as a Perfect Fluid

In the present section we shall study the equations of motion of the atmosphere for the simplest case, when the air is considered to be a perfect fluid.

Placing the origin of coordinates somewhere on the earth's surface, let us consider the equation of the relative motion of a perfect fluid in the form given by Euler:

$$\frac{dv}{dt} = -\frac{1}{\rho}\operatorname{grad} p - \operatorname{grad}\Phi - 2[\omega, v] \tag{1}$$

or

$$\frac{\partial v}{\partial t} + (v, \operatorname{grad})\, v = -\frac{1}{\rho}\operatorname{grad} p - \operatorname{grad}\Phi - 2[\omega, v], \tag{2}$$

where $-2[\omega, v]$ is the Coriolis force, caused by the earth's rotation, and Φ is the potential of gravity.

The vectorial equation (2) is equivalent to the three component equations:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - 2(\omega_y w - \omega_z v),\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2(\omega_z u - \omega_x w),\\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g - 2(\omega_x v - \omega_y u).
\end{aligned}\right\} \tag{3}$$

If we add the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0, \tag{4}$$

to the system of equations (3) we do not obtain a complete system of equations, because the four equations (3) and (4) contain five unknown functions u, v, w, p, ρ.

In the study of the motion of a perfect fluid, we completely exclude from our considerations both the process of energy dissipation occuring in a real fluid, and the heat exchange between its different parts. This means that the motion of a perfect fluid must be regarded as adiabatic. Consequently the adiabatic equation in the form

$$\frac{c_p}{T}\frac{dT}{dt} - \frac{R}{p}\frac{dp}{dt} = 0. \tag{5}$$

can be taken as the fifth equation.

The new unknown function—the temperature T—is related to p and ρ by the equation of state:

$$p = \rho R T. \tag{6}$$

Thus, the system of six equations (3), (4), (5), (6) contains six unknown functions u, v, w, ρ, T, and is complete.

Let us now consider the conditions which must be fulfilled at the boundary surfaces of the fluid.

The boundary condition, in the case of a perfect fluid, must express the fact that the fluid cannot penetrate through the solid surface, or break away from it; consequently, the normal component of the velocity at the solid surface must be zero

$$v_n = 0. \tag{7}$$

The normal components of the velocities of any two contiguous particles belonging to two different immiscible fluids must be equal at the interface.

$$v_{n1} = v_{n2}, \tag{8}$$

The pressures in the two fluids must also equal

$$p_1 = p_2. \tag{9}$$

Finally, the adiabatic condition must hold on the boundary as well as in the interior of the perfect fluid under consideration.

It is sometimes more convenient to use another form of the equation of motion for perfect fluids. If the entropy in a perfect fluid is initially equal everywhere, then it will not change as a result of the fluid motion; consequently, the adiabatic condition is simplified and becomes:

$$d\varphi = c_p \frac{dT}{T} - R \frac{dp}{p} = 0. \tag{10}$$

Let us now introduce the enthalpy* into the equations of motion. According to equation (10) we obtain

$$d\vartheta = c_p dT = \frac{RT}{p} dp = \frac{1}{\rho} dp \tag{11}$$

or

$$\frac{1}{\rho} \operatorname{grad} p = \operatorname{grad} \vartheta. \tag{12}$$

The equation of motion can therefore be written as follows

$$\frac{dv}{dt} = -\operatorname{grad}(\Phi + \vartheta) - 2[\omega, v]. \tag{13}$$

We shall now indicate some forms of the equation of motion for the atmosphere, regarded as a perfect fluid.

It is known from vector analysis that:

$$(v, \operatorname{grad}) v = \operatorname{grad}\left(\frac{v^2}{2}\right) - [v, \operatorname{curl} v]. \tag{14}$$

By combining equations (2) and (14) we obtain the equation of motion of the atmosphere in the following form

$$\frac{\partial v}{\partial t} = -\operatorname{grad}\frac{v^2}{2} - \frac{1}{\rho}\operatorname{grad} p - \operatorname{grad}\Phi - 2[\omega, v] - [\operatorname{curl} v, v]. \tag{15}$$

Let us apply the curl operator to both sides of equation (2). Writing $\Omega = \operatorname{curl} v$, we obtain the equation of atmospheric motion in the form given

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* [Translator's note: The enthalpy is denoted everywhere by the Russian letter ϑ].

by A. A. Fridman:

$$\frac{d\Omega}{dt} - (\Omega, \text{grad})\, v + \Omega \operatorname{div} v = -2\operatorname{curl}[\omega, v] - \frac{1}{\rho T}[\operatorname{grad} T, \operatorname{grad} p]. \qquad (16)$$

Fridman's equation determines the temporal variation of the vorticity vector in the air parcel under consideration.

Let us now consider the momentum density of the flow

$$m = \rho v; \quad m_x = \rho u; \quad m_y = \rho v; \quad m_z = \rho w.$$

We will transform the left side of one of the equations (3), by the equation of continuity (4) written in the following form:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} =$$
$$= \frac{\partial \rho u}{\partial t} + \frac{\partial \rho u u}{\partial x} + \frac{\partial \rho v u}{\partial y} + \frac{\partial \rho w u}{\partial z} - u\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) =$$
$$= \frac{\partial m_x}{ot} + \frac{\partial u m_x}{\partial x} + \frac{\partial v m_x}{\partial y} + \frac{\partial w m_x}{\partial z}.$$

The equations of motion for the atmosphere as a perfect fluid will then assume the following form:

$$\left.\begin{aligned}
&\frac{\partial m_x}{\partial t} + \frac{\partial u m_x}{\partial x} + \frac{\partial v m_x}{\partial y} + \frac{\partial w m_x}{\partial z} = -\frac{\partial p}{\partial x} - 2\rho(\omega_y w - \omega_z v),\\
&\frac{\partial m_y}{\partial t} + \frac{\partial u m_y}{\partial x} + \frac{\partial v m_y}{\partial y} + \frac{\partial w m_y}{\partial z} = -\frac{\partial p}{\partial y} - 2\rho(\omega_z u - \omega_x w),\\
&\frac{\partial m_z}{\partial t} + \frac{\partial u m_z}{\partial x} + \frac{\partial v m_z}{\partial y} + \frac{\partial w m_z}{\partial z} = -\frac{\partial p}{\partial z} - g - 2\rho(\omega_x v - \omega_y u).
\end{aligned}\right\} \qquad (17)$$

We now write down the equations of motion in spherical coordinates r, φ, λ without deriving them explicitly.

The components of the velocity in spherical coordinates are given by the equations

$$v_\lambda = r\cos\varphi\dot{\lambda},$$
$$v_\varphi = r\dot{\varphi},$$
$$v_r = \dot{r},$$

and the equations of motion for the atmosphere as a perfect fluid assume the form:

$$\left.\begin{aligned}
&\frac{dv_\lambda}{dt} - 2\omega\sin\varphi\cdot v_\varphi + 2\omega\cos\varphi\cdot v_r + \frac{1}{r}(v_\lambda\cdot v_r - \operatorname{tg}\varphi\cdot v_\lambda\cdot v_\varphi) =\\
&\qquad = -\frac{1}{r\cos\varphi}\frac{\partial\Phi}{\partial\lambda} - \frac{1}{\rho r\cos\varphi}\frac{\partial p}{\partial\lambda},\\
&\frac{dv_\varphi}{dt} + 2\omega\sin\varphi\cdot v_\lambda + \frac{1}{r}(v_\varphi\cdot v_r + \operatorname{tg}\varphi\cdot v_\lambda^2) =\\
&\qquad = -\frac{\partial\Phi}{r\partial\varphi} - \frac{1}{\rho r}\frac{\partial p}{\partial\varphi},\\
&\frac{dv_r}{dt} - 2\omega\cos\varphi v_\lambda - \frac{1}{r}(v_\lambda^2 + v_\varphi^2) =\\
&\qquad = -\frac{\partial\Phi}{\partial r} - \frac{1}{\rho}\frac{\partial p}{\partial r}.
\end{aligned}\right\} \qquad (18)$$

On the left-hand sides of the equations we have:
1) the components of the Euler acceleration J:

$$\left.\begin{aligned}
J_\lambda &= \frac{dv_\lambda}{dt};\\
j_\varphi &= \frac{dv_\varphi}{dt};\\
j_r &= \frac{dr}{dt}.
\end{aligned}\right\} \qquad (19)$$

2) the components of the Coriolis acceleration **C**:

$$\left.\begin{aligned} C_\lambda &= -2\omega \sin\varphi \cdot v_\varphi + 2\omega \cos\varphi \cdot v_r; \\ C_\varphi &= +2\omega \sin\varphi \cdot v_\lambda; \\ C_r &= -2\omega \cos\varphi \cdot v_\lambda. \end{aligned}\right\} \tag{20}$$

3) the components of the metric acceleration **M**:

$$\begin{aligned} M_\lambda &= \frac{1}{r}(v_\lambda v_r - \operatorname{tg}\varphi \cdot v_\lambda v_\varphi), \\ M_\varphi &= \frac{1}{r}(v_\varphi v_r + \operatorname{tg}\varphi \cdot v_\varphi^2), \\ M_r &= -\frac{1}{r}(v_\lambda^2 + v_\varphi^2). \end{aligned} \tag{21}$$

The metric acceleration is not a result of the earth's rotation, since it does not depend on the angular velocity ω of the earth. It is caused by the fact that the metric of the system λ, φ, r does not conform with the principle of inertial motion.

The centrifugal acceleration, due to the curvature of the trajectory, which differs from the curvature in the system λ, φ, r, is contained in the Euler acceleration. The metric acceleration and the Coriolis acceleration are orthogonal to the velocity vector, since

$$M_\lambda \cdot v_\lambda + M_\varphi \cdot v_\varphi + M_r \cdot v_r = 0 \tag{22}$$

and

$$C_\lambda \cdot v_\lambda + C_\varphi \cdot v_\varphi + C_r \cdot v_r = 0. \tag{23}$$

§ 2. The Equations of Motion for a Turbulent Atmosphere

An irregular turbulent air motion is usual in the atmosphere. We can convince ourselves of that literally at every step. The clouds of smoke issuing from chimneys consist of puffs of soot particles whose visible trajectories are completely intermingled. The same can be observed in cloudy air masses.

The gustiness of the earth's surface wind, both directly perceptible and as recorded by anemographs in meteorological stations, airplanes going over air-bumps, irregularities in the vertical speeds of pilot balloons – are all different manifestations of atmospheric turbulence.

The vertical transport of water vapor and of other material impurities (dust, condensation nuclei, bacteria, etc) is also related to the turbulent character of atmospheric motion. The turbulence of the atmosphere plays a decisive role in processes of heat exchange between different layers. Even such processes as the variation of wind with height, the periodical variations of atmospheric temperature, the periodical changes of humidity, and others – are closely related to the turbulent character of atmospheric motions.

The causes of atmospheric turbulence have not yet been completely elucidated just as there is no clear notion of the nature of the elements of turbulence.

On account of the irregular character of turbulent motion, it is not possible to use instantaneous values of the actual air velocities. Averaged equations of motion are therefore used in the investigation of turbulent motions in the atmosphere.

In order to work out equations of motion for the turbulent atmosphere, it is sufficient to add to the right side of equations (3) § 1, terms expressing the forces caused by the turbulence:

$$\left.\begin{aligned}
\frac{du}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} - 2(\omega_y w - \omega_z v) + \frac{\partial}{\partial x}\left(\nu'\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu'\frac{\partial u}{\partial y}\right) + \\
&\qquad + \frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right), \\
\frac{dv}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2(\omega_z u - \omega_x w) + \frac{\partial}{\partial x}\left(\nu'\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu'\frac{\partial v}{\partial y}\right) + \\
&\qquad + \frac{\partial}{\partial z}\left(\nu\frac{\partial v}{\partial z}\right), \\
\frac{dw}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g - 2(\omega_x v - \omega_y u) + \frac{\partial}{\partial x}\left(\nu'\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu'\frac{\partial w}{\partial y}\right) + \\
&\qquad + \frac{\partial}{\partial z}\left(\nu\frac{\partial w}{\partial z}\right), \\
\frac{\partial \rho}{\partial t} &+ \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0,
\end{aligned}\right\} \qquad (1)$$

where ν and ν' are respectively the vertical and horizontal coefficients of kinematic viscosity.

Turbulence is not a physical property of air; it is only a special, prevalent form of the air motion. One can imagine the atmosphere without turbulence, under the same external conditions. In order to point out the special importance of turbulence in meteorology, let us examine some examples of the conditions which would obtain in a hypothetical atmosphere in which all motion is laminar.

In a laminar atmosphere we would first notice, even with the naked eye, that only layer clouds exist. Cumuli would completely disappear. We would further find by means of temperature soundings that the lapse rate in this hypothetic atmosphere reaches exceedingly high values. In fact, turbulence moderates the altitude variation of the temperature.

The chemical analysis of the composition of air at different altitudes would indicate that each gaseous constituent of the atmospheric air forms an independent atmosphere, obeying its own barometric formula. The amount of heavier gases in the atmosphere would therefore decrease with altitude more rapidly than that of lighter gases, and the percentual composition of air at different altitudes would be different. This would lead to the result that the total pressure at a given altitude of a free atmosphere would be greater than it really is; nevertheless, this difference would reach significant value only in very high layers. Turbulence ensures that there be no altitude-dependent diffusive equilibrium of separate atmospheric gases, and equalizes the composition of the air at all altitudes.

Life in big towns would have been impossible in a laminar atmosphere, because the quantity of oxygen supplied by molecular diffusion would evidently be insufficient. We might add that the contamination of air by combustion products would have led to an intolerable situation near mills and factories.

The absence of turbulence would have had a pernicious effect not only on human beings, but also on the entire organic world.

§ 3. Helmholtz's Principle of Similarity

The integration of system (1) § 2 is impossible, as we have already mentioned. We will, therefore, restrict ourselves to the consideration of simplified equations of atmospheric motions, which occur in particular problems. However, before passing on to these particular problems, we must find the criteria, showing exactly what kind of simplifications of the problems are admissible in principle.

Much headway in this direction was made by Helmholtz's principle of geometric similarity, by the principle of dynamic similitude, and finally by the very important assumptions of Hesselberg and Fridman, concerning the order of magnitude of meteorological elements and their derivatives.

Let us first consider Helmholtz's principle of geometric similarity.

The motions of two viscous fluids, determined each by the Navier-Stokes equations, can be compared. Let us explain under what conditions will these motions be geometrically similar. We will denote the quantities which refer to the first fluid by the subscript 1, while these referring to the second fluid will be left without a subscript. We will assume that also the gravity and the angular velocity of rotation might generally have different values for the fluids under consideration.

Helmoholtz used the equation of state in the form

$$p + c = a\rho \tag{1}$$

(c is some constant and a depends on the temperature) – in order to take into consideration the compressibility of the fluid. The density remains finite in this form of the equation, even when the pressure vanishes, because c is, generally speaking, different from zero. c = 0 only in the case of an ideal gas. The coefficient a, which accounts for the influence of the temperature, is essentially positive. Let the quantities, referring respectively to the first and second fluids, bear the following mutual relationships

$$u_1 = Uu;\ v_1 = Uv;\quad w_1 = Uw;\ p_1 = Pp;\quad \rho_1 = R\rho;\ x_1 = Lx;$$
$$y_1 = Ly;\ z_1 = Lz;$$
$$t_1 = Tt;\quad \mu_1 = M\mu;\quad g_1 = Gg;\quad \omega_1 = \Omega\omega;\quad c_1 = Cc;\quad a_1 = Aa, \tag{2}$$

where U, P, R, L, T, M, G, Ω, C, A are evidently constant dimensionless quantities. Since the two fluids satisfy both the Navier-Stokes equations and equation (1), quite definite relationships exist between these quantities. By inserting the values characterizing the first flow in the first of the Navier-Stokes equations, we obtain:

$$\frac{U}{T}\frac{du}{dt} = -\frac{P}{RL}\cdot\frac{1}{\rho}\frac{\partial p}{\partial x} + \Omega U(2w\omega_y - 2v\omega_z) + \frac{MU}{RL^2}\cdot\nu\nabla U. \tag{3}$$

Since the quantities referring to the second flow satisfy the first of the Navier-Stokes equations, all the dimensionless coefficients of equation (3) must fulfil

$$\frac{U}{T} = \frac{P}{RL} = \Omega U = \frac{MU}{RL^2}.$$

Nothing new is given by the second of the Navier-Stokes equations. From the third equation we obtain an additional relation containing G in a similar way. Thus

$$\frac{U}{T} = \frac{P}{RL} = \Omega U = \frac{MU}{RL^2} = G. \tag{4}$$

In addition we obtain from the equation of continuity:

$$\frac{R}{T} = \frac{RU}{L},$$

and from the equation of state:

$$P = C = AR.$$

Thus, 10 dimensionless quantities, characterizing the scale of motion, are linked together other by seven independent relations. Consequently, seven quantities can be expressed by the three remaining ones, for which we will use the ratios of lengths L, velocities U, and densities R:

$$T=\frac{L}{U};\quad P=U^2R;\quad M=RLU;$$

$$G=\frac{U^2}{L};\quad \Omega=\frac{U}{L};\quad C=U^2R;\quad A=U^2. \tag{5}$$

We are considering two flow patterns of the same fluid (air); therefore R = 1 and

$$T=\frac{L}{U};\quad P=U^2;\quad M=LU^2;\quad G=\frac{U^2}{L};$$

$$\Omega=\frac{U}{L};\quad C=U^2;\quad A=U^2. \tag{6}$$

These relations allow to obtain an answer to a number of important questions.

Let us first explain the possibility of existence of geometrically similar motions in the atmosphere, if their velocity fields are equal, i. e., if U = 1. These geometrically similar motions differ only in the length-scale L of the phenomenon. Thus

$$T=L;\ P=1;\ M=L;\ G=\frac{1}{L};\quad \Omega=\frac{1}{L};\quad C=A=1. \tag{7}$$

The analysis of relations (7) shows that motions geometrically similar to those actually occuring in the atmosphere are impossible, because in that case some magnitudes, certainly constant, would have to be varied. Thus, for example, the gravity appears to be inversely proportional to the length-scale of the flow, whereas the viscosity would be directly proportional to this magnitude.

It further follows from relations (7) that the greater the length-scale of the air flow under consideration, the greater must be the scale of time for the same velocity of the flow. Time, therefore, plays a greater role in small-scale than in large-scale motions.

The larger the flow extent, the greater must be the force of viscosity in order to produce the same effect. Since the viscosity of air has a fixed, nearly constant value, it plays an essential role only in small-scale motions. The greater the scale of the motion, the smaller is the role played by viscosity. Hence we may completely disregard the viscosity during large scale motions.

In the same way we can convince ourselves of the fact that with the increase of the scale of motion gravity and the Coriolis force play an increasingly important role. Let us put a second question, concerning the possibility of occurrence in the atmosphere of kinematically similar motions having the same linear extent.

L = 1 in this case, and relations (6) give:

$$T=\frac{1}{U};\ P=U^2;\ M=U;\ G=U^2;\ \Omega=U;\ C=U^2;\ A=U^2. \tag{8}$$

It appears that motions having the same extent and similar from the kinematic point of view are impossible in the atmosphere, because in that case some evidently constant magnitudes would have to be variable (gravity, viscosity).

Relations (8) enable to draw a number of very important qualitative conclusions.

The forces of viscosity must attain a high value at great velocities in order to make the motion kinematically similar to that at small velocities. The viscosity, however, remains constant at any velocity. Consequently, the influence of the viscosity is smaller, the greater the velocity of the motion.

The same can also be said of the gravity and the pressure: the greater the velocity, the slighter the effect of both these factors.

Finally, we should explain the role of the compressibility, which is expressed by the coefficient $C = U^2$.

In order to warrant a similarity of motions, it is necessary that the air should be less compressible ($C = U^2 > 1$) at high velocities than at low ones The compressibility therefore plays a lesser role at low air speeds than at high ones.

Thus, the following propositions can be formulated on the strength of Helmholtz's principle of similarity.

1) The following motions can not occur in the atmosphere: a) having different linear extension from the actual motion but the same velocity; b) having the same extension, but different velocity.

2) The role of the viscosity is essential in a slow small-scale motion, and inessential in large-scale rapid motions.

3) The role of the compressibility is essential only at very great velocities, the fluid being considered as incompressible at small velocities.

4) The influence of gravity and of the Coriolis force increases with the scale of the motion.

5) The influence of pressure decreases with increasing velocity.

These propositions make it possible to simplify the equations of motion, and hence to obtain an approximate solution of some problems concerning the motion of the atmosphere.

§ 4. Dynamic Similitude. The Reynolds Number

We have seen above that no currents can occur in the atmosphere which are geometrically similar to these actually existing there, because the mass forces must be different for them, whereas, for example, gravity is a fixed quantity which is not at our disposal to be altered.

We will put the problem of similarity in another form Let us assume that two fluids are examined: the atmosphere, whose motion is uncontrollable, and a laboratory fluid, on whose motion we can impose the required conditions. In other words, we may wish to reproduce atmospheric motions in the laboratory, and we should take care that the laboratory-created motion is similar to the atmospheric one. We will determine whether such simulation is feasible.

In the attempt to simulate the motion of material particles by a motion, similar not only geometrically and kinematically but also from the dynamic point of view, it is necessary and sufficient that the force-scale f should bear the following relationship to the length-scale l, mass-scale m, and time-scale τ:

$$f = m\frac{l}{\tau^2}.$$

This condition, called Newton's principle of dynamic similitude, is the only one necessary, and is sufficient for the motion of two systems of material particles to be similar geometrically (the geometrical proportions of the systems are similar), kinematically (the velocities of the corresponding points are similar), and dynamically (the forces acting at corresponding points are similar).

In the attempt to recreate atmospheric motions we encounter enormous difficulties which are inherent in the system and hence insurmountable. Some of these difficulties are common problems of reproducing a flow bounded by solid surfaces and by free boundaries, but to these difficulties new ones are added, by the compressibility of the atmosphere, to its thermal stratification, its unboundedness, and finally by the rotation of the earth. These difficulties are so great that not one successful attempt to reproduce atmospheric processes can be named.

Nevertheless, the principle of dynamic similitude facilitates (though to a limited extent) the simplification of the general equations of motion; however, in applications to atmospheric motion the principle of dynamic similitude is less fruitful than in aerodynamics.

We will restrict ourselves to the examination of horizontal motions, and assume that there are no external forces, and that the motion therefore takes place only under the action of internal forces. If, in addition, the compressibility of the atmosphere is ignored, the Navier–Stokes equation assumes the form

$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + \mu \Delta u. \tag{1}$$

Accepting these assumptions, what is the condition for the dynamic similitude of flows, if geometric and kinematic similarity is to be maintained? This condition can be determined with the help of the preceding section. Of the seven relations (5) § 3, only three are retained in the case under consideration:

$$T = \frac{L}{U}; \quad P = U^2 R; \quad M = RLU. \tag{2}$$

We are now interested only in the last of these relations, which connects the scales of the viscosity forces and of the inertial forces. Passing from the scales to the functions themselves, and taking into consideration that

$$M = \frac{\mu_1}{\mu}; \quad R = \frac{\rho_1}{\rho}; \quad L = \frac{x_1}{x} = \ldots = \frac{l_1}{l}; \quad U = \frac{u_1}{u} = \ldots \frac{v_1}{v},$$

we obtain the relation

$$\frac{\mu_1}{\mu} = \frac{\rho_1}{\rho} \frac{l_1 \cdot v_1}{l \cdot v}$$

or, introducing the coefficient of kinematic viscosity

$$\frac{v_1 l_1}{\nu_1} = \frac{vl}{\nu} = \text{const}. \tag{3}$$

This is the condition of dynamic similitude for flows moving only under the action of inertial forces and of the internal forces of viscosity.

The dimensionless quantity

$$\mathrm{Re} = \frac{vl}{\nu} \tag{4}$$

is called the Reynolds number

Thus the constancy of the Reynolds number is, providing geometric and kinematic similarity exists, the necessary and sufficient condition for the dynamic similitude of two flows of an incompressible viscous fluid, on which external forces do not act (it is obvious that the values of the velocity v, of the length l, and of the coefficient of kinematic viscosity can be altered thereby).

It can be seen from the structure of equation (4) that small Reynolds numbers

correspond to small inertial forces and a high viscosity, while high Reynolds numbers correspond to great inertial forces and to a small viscosity.

Under atmospheric conditions we constantly encounter motions taking place at high Reynolds numbers. Let us evaluate this number for three cases of atmospheric motion: 1) breeze blowing on the shore of a great lake (l_1 = 1,000 m, v_1 = 1.5 m/sec), 2) the motion of air in a monsoon (l_1 = 500,000 m, v_2 = 10 m/sec), and 3) the motion of the atmosphere in its general circulation (l_3 = 10,000,000 m, v_3 = 10 m/sec). Let us assume, in addition, that the coefficient of kinematic viscosity is $\nu = 0.13 \cdot 10^{-4}$ m^2/sec under normal conditions (t = 0°C, p = 760 mm). The Reynolds number for these motions will have the following values respectively:

$$Re_1 = 1 \cdot 10^8; \quad Re_2 = 4 \cdot 10^{11}; \quad Re_3 = 8 \cdot 10^{12}.$$

It can be seen from the cited examples that the Reynolds numbers of atmospheric motions are enormous. This fact shows that the influence of the viscosity can always be neglected in the study of atmospheric motions, and that air can be regarded as a perfect fluid.

Numerous measurements carried out in laboratories have shown that the Reynolds number is also a criterion for the transition from laminar to turbulent motion; the flow is laminar for small Reynolds numbers; it is turbulent for high ones. The flow in cylindrical tubes becomes turbulent already at values of Re_{cr} = 1,160, if no special precautions are taken. The flow can maintain its laminar character even at Re_{cr} = 20,000 and more if special precautionary measures are applied. No turbulence can occur at $Re_{cr} < 1160$.

It follows from the cited values of Re_{cr} that atmospheric currents are always completely turbulent. The equations for viscous fluids are therefore not needed for the study of atmospheric motions; it is, however, always necessary to proceed from Reynolds' averaged equations of motion, in which the forces of viscosity can be ignored but the additional stresses must generally be retained. The possibility of rejecting the additional stresses of turbulence must be carefully checked.

§ 5. The Laminar Boundary Layer

We have already mentioned several times that the influence of viscosity on atmospheric motions can be disregarded; we have at the same time pointed out the necessity of carefully allowing for the influence of turbulence. But before entirely dispensing with considerations of viscosity, we shall in the present section examine a motion for which the effects of viscosity are decisive, and cannot be disregarded. The problem concerns the laminar boundary layer in the field of flow, and also in the fields of temperature and humidity.

The results obtained in the present section will be compared afterwards with the corresponding results for the turbulent boundary layer. However, we must keep in mind that the results of the study of the laminar boundary layer also have an independent significance in the theory of some meteorological instruments.

In the previous section we pointed out that, due to the huge value of Re for the atmosphere, the influence of viscosity on atmospheric motions can be disregarded everywhere, except for the layer directly adjacent to the solid boundary to which the viscous fluid adheres without slipping. This layer is called the boundary layer.

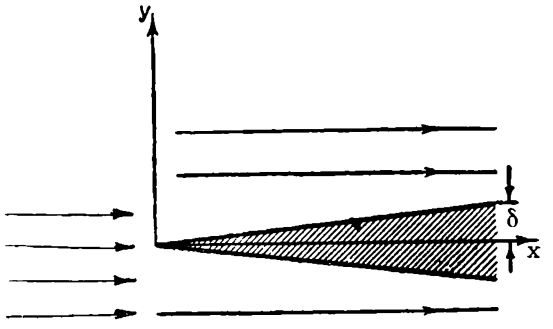

Figure 77. The formation of a laminar boundary layer in flow past a plate

Let us imagine an undisturbed flow of air moving with velocity u (Figure 77). Let us introduce into this flow a polished flat plate, and orient it so that its surface is parallel to the direction of the undisturbed flow. It stands to reason that the flow will be disturbed in the immediate proximity of the plate.

The layer in which the disturbing influence of the surface on the flow will be observed, will be the boundary layer. It has been ascertained by observations that the thickness of the boundary layer δ increases the farther we move from the leading edge of the plate, where $\delta = 0$.

Within this layer the flow velocity rises sharply from zero at the surface of the plate, to some finite value on the outer surface of the layer, equal to the velocity $\bar{u}$ of the undistrubed flow. The stress tensor therefore appears to be of essential importance for this layer, in spite of the small value of μ.

The integration of the equation of motion for the laminar flow of a viscous fluid is in principle always possible, just as the integration of equations of motion for the potential flow outside the boundary layer is possible.

We shall restrict ourselves to the examination of the plane-parallel motion of a fluid.

It is expedient to use the Reynolds number in the study of the boundary layer, because the unknown function will then be dimensionless. We write

$$u_1 = Uu; \quad v_1 = Uv; \quad w_1 = Uw; \quad x_1 = Lx; \quad y_1 = Ly; \quad z_1 = Lz; \quad t_1 = Tt; \quad p_1 = Pp, \tag{1}$$

where the actual physical variables are denoted by the subscript 1, capitals stand for the "characteristic" variables of the flow, T = L/U being used as characteristic time, and $P = \rho U^2$ as characteristic pressure (according to Bernoulli's equation); small letters without subscripts stand for dimensionless magnitudes, which are the ratios of the physical variables to the characteristic ones (these dimensionless variables may assume every value from 0 to 1).

After substitution of (1) in the equations of motion (2) § 7 Chapter VIII and some elementary transformations, we obtain the following dimensionless equations

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \end{aligned} \tag{2}$$

As characteristic velocity we shall use the velocity of the undisturbed flow $\bar{u}$, and as characteristic length – the length l of the plate.

Let us assume in addition that $\delta' = \delta/l$ – the dimensionless thickness of the boundary layer, is a small magnitude of the first order. We shall estimate the order

of magnitude of each term which enters in the equation (2), designating the order of magnitude by the symbol Π.

Since $\Pi u = \Pi x = 1$, we have $\Pi \frac{\partial^2 u}{\partial x^2} = \Pi \frac{\partial u}{\partial x} = 1$ and $\Pi \frac{\partial u}{\partial t} = 1$.

Thus $\Pi u = \Pi \int_0^{\delta'} \frac{\partial u}{\partial y} dy = 1$; the derivative of the horizontal velocity in the direction of the normal to the plate, is therefore very large: $\Pi \frac{\partial u}{\partial y} = \frac{1}{\delta'}$. In the same way $\Pi \frac{\partial^2 u}{\partial y^2} = \frac{1}{\delta'^2}$.

From the equation of continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1$ it follows that

$$\Pi \frac{\partial v}{\partial y} = \Pi \frac{\partial u}{\partial x} = 1,$$

hence $\Pi v = \delta'$ and consequently

$$\Pi \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} = \delta'$$

and

$$\Pi \frac{\partial v}{\partial t} = \delta'.$$

Let us write down the equations of motion and the corresponding evaluations of the orders of magnitude of their terms.

$$\begin{array}{llll} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), & & & \\ 1 \quad 1\cdot 1 \quad \delta'\cdot\frac{1}{\delta'}, & & 1 & \frac{1}{\delta'^2} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). & & & \\ \delta' \quad 1\cdot\delta' \quad \delta'\cdot 1 & & \delta' & \frac{1}{\delta'} \end{array} \qquad (3)$$

In the first equation $\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2}$, and in the second $\frac{\partial^2 v}{\partial x^2} \ll \frac{\partial^2 v}{\partial y^2}$. The second derivatives with respect to x can therefore be left out, because they are negligible in comparison with the second derivatives with respect to y.

Furthermore, the forces of viscosity and the forces of inertia are magnitudes of the same order within the limits of the boundary layer. But the first of the equations (3) shows that the order of the inertial forces is 1, while the order of the forces of viscosity equals $\frac{\mathrm{Re}}{\delta'^2}$. Consequently, $\Pi \frac{\mathrm{Re}}{\delta'^2} = 1$, whence $\delta' = \Pi \frac{1}{\sqrt{\mathrm{Re}}}$.

Returning to the physical thickness of the boundary layer $\delta = \delta' l$, we obtain:

$$\delta = \Pi \frac{l}{\sqrt{\mathrm{Re}}}. \qquad (4)$$

Thus, if the motion takes place at high Reynolds numbers, a boundary layer appears at the solid surface limiting the fluid, and its thickness is determined by equation (4).

The order of magnitude of all the remaining terms of equation (3) is δ', and therefore $\frac{\partial p}{\partial y}$ is of the same order. Consequently, the pressure inside the boundary layer can be considered to be approximately equal to the external pressure, whence $\frac{\partial p}{\partial y} = 0$ and $p = p(x)$, i.e., within the limits of the boundary layer the pressure varies only in the direction of the flow, and does not vary normally to the plate immersed in the air stream. Returning to the dimensional variables, we obtain the equation of motion for the viscous fluid in the boundary layer in the following form:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2},$$
$$\frac{\partial p}{\partial y}=0;\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{5}$$

These equations were derived by Prandtl. Instead of Prandtl's first equation it is more expedient to use the integral relationship of Kármán, obtained by integrating this equation over the whole thickness δ of the boundary layer between the limits 0 and δ. The upper surface of the layer is, of course, very conditional, but even with this restriction one of the powerful aspects of Kármán's idea concerning the boundary layer can be demonstrated. After integration we obtain:

$$\frac{\partial}{\partial t}\int_0^\delta \rho u\,dy+\frac{\partial}{\partial x}\int_0^\delta \rho\frac{u^2}{2}dy+\int_0^\delta \rho v\frac{\partial u}{\partial y}=$$
$$=-\int_0^\delta \frac{\partial p}{\partial x}dy+\mu\left(\frac{\partial u}{\partial y}\right)_\delta-\mu\left(\frac{\partial u}{\partial y}\right)_0. \tag{6}$$

Since p is independent of y within the limits of the boundary layer, we have

$$-\int_0^\delta \frac{\partial p}{\partial x}dy=-\frac{\partial p}{\partial x}\cdot\delta.$$

The motion is still vortex-free, or potential, at the outer surface of the layer; therefore

$$\mu\left(\frac{\partial u}{\partial y}\right)_\delta=0.$$

At the inner surface the same equation expresses the force of friction, referred to unit area of the plate

$$\mu\left(\frac{\partial u}{\partial y}\right)_\delta=\tau.$$

Integrating by parts, we obtain

$$\int_0^\delta v\frac{\partial u}{\partial y}dy=[v\cdot u]_0^\delta-\int_0^\delta u\cdot\frac{\partial v}{\partial y}\cdot dy.$$

But

$$[v\cdot u]_0^\delta=v_\delta u_\delta=\bar{u}\int_0^\delta \frac{\partial v}{\partial y}\cdot dy;$$

in addition, according to the equation of continuity $-\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}$. Consequently

$$-\int_0^\delta u\frac{\partial v}{\partial y}dy=\int_0^\delta u\frac{\partial u}{\partial x}dy=\frac{\partial}{\partial x}\int_0^\delta \frac{u^2}{2}dy.$$

By substituting all the obtained expressions in equation (6), we obtain the integral relationship, expressing Kármán's theorem:

$$\frac{\partial}{\partial t}\int_0^\delta \rho u\,dy+\frac{\partial}{\partial x}\int_0^\delta \rho u^2\,dy-\bar{u}\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy=-\delta\frac{\partial p}{\partial x}-\tau. \tag{7}$$

Let us explain the physical meaning of each term in equation (7).

We will consider for this purpose a volume element of the fluid, enclosed within the boundary layer (which we assume to be of unit thickness) and bounded by the solid plate, by the exterior surface of the layer, and by the planes orthogonal to the axis OX, whose abscissas respectively equal x and x + 1.

The first term of equation (7) $\frac{\partial}{\partial t}\int_0^{\delta} \rho u\, dy$ represents the variation of momentum in a unit column of the boundary layer. The second term $\frac{\partial}{\partial x}\int_0^{\delta} \rho u^2\, dy$ is the rate of change of the momentum of the fluid enclosed in the volume under consideration, due to its flow through the lateral sides of the volume element.

The third term, $-\bar{u}\frac{\partial}{\partial x}\int_0^{\delta} \rho u\, dy$ gives the rate of change of momentum of the fluid which is due to its flow through the upper boundary of the volume element.

Consequently, the left side of the equation gives the total rate of change of the momentum of the fluid contained in a unit column of the boundary layer .

The term $-\delta \cdot \frac{\partial p}{\partial x}$ gives the sum of the forces of pressure, acting on the lateral sides of the volume element under consideration.

τ is the force of friction, acting on the unit surface area which forms the basis of the unit column of the boundary layer.

Thus Kármán's theorem is nothing else but the law of conservation of momentum, applied to a unit column of the fluid in a boundary layer.

For our case, namely the appearance of a boundary layer near a flat plate, Kármán assumes that $\frac{\partial p}{\partial x} = 0$ (the pressure must be constant, since the unperturbed velocity far from the plate is constant).

Different values for δ and $\int_0^{x} \tau\, dx$ are obtained in a stationary boundary layer for different distributions of the velocity. Some results which can be obtained without difficulty from Kármán's equations are given below:

Velocity distribution law	Thickness of the boundary layer δ	Friction against the plate $\tau\, dx$
$u = \bar{u}\cdot y$ for $0 \leqslant y < \delta$ $u = \bar{u}$ for $y > \delta$ (Figure 78, a)	$\delta = 3.46\sqrt{\frac{\nu x}{\bar{u}}}$	$1.155\sqrt{\rho\mu\bar{u}^3 \cdot x}$
$u = \bar{u}\cdot\frac{(2\delta - y)\cdot y}{\delta^2}$ (Figure 78, b)	$\delta = 5.4\sqrt{\frac{\nu x}{\bar{u}}}$	$0.926\sqrt{\rho\mu\bar{u}^3 \cdot x}$
$u = \bar{u}\cdot\frac{(3\delta^2 - y^2)\cdot y}{2\delta^3}$ (Figure 78, c)	$\delta = 4.64\sqrt{\frac{\nu x}{\bar{u}}}$	$0.65\sqrt{\rho\mu\bar{u}^3 \cdot x}$

It should be pointed out that no sharp external termination of the boundary layer can be traced, because the perturbed velocity in the boudary layer asymptotically approaches the unperturbed velocity.

All the obtained relations are correct, of course, only for a laminar boundary layer. Turbulence is developed in the boundary layer, if $Re > Re_{cr}$.

In Table 25 we give thicknesses of boundary layers for a laminar flow of air over a polished flat surface; this thickness which depends on the covered distance x and on the value of the unperturbed velocity u,

$$\delta = 3.46\sqrt{\frac{\nu x}{u}} \quad \text{where } \nu = 0.133\ \text{cm}^2/\text{sec}.$$

Table 25

Thickness of the boundary layer for air flow past a flat plate

x / u	10 cm	30 cm	50 cm	75 cm	100 cm
1 m/sec	0.40	0.69	0.89	1.09	1.26 cm
2 »	0.28	0.49	0.63	0.77	0.94 cm
5 »	0.18	0.31	0.40	0.49	— cm
10 »	0.13	0.22	—	—	— cm

The unfilled entries in the table correspond to high values of Re and consequently to development of turbulence in the boundary layer.

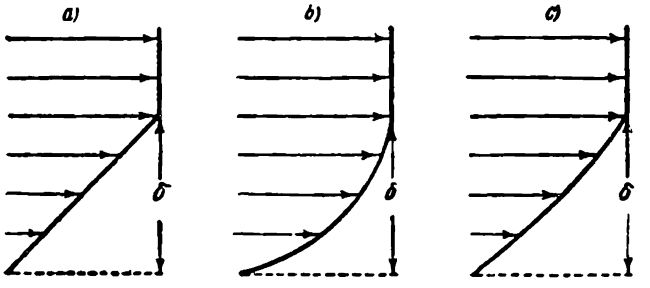

Figure 78. The velocity profile in the boundary layer

The assumptions we made concerning the constancy of pressure within the boundary layer are correct only on the condition that the boundary layer is very thin and the solid surface past which the air flows is flat. Otherwise a pressure difference is developed within the boundary layer; this pressure difference gives rise to backward currents, causing the formation of vortexes, "obstructing" the upperturbed potential flow. We shall only consider the qualitative aspect of this process.

Let us imagine that the unperturbed stream runs against some obstacle (Figure 79) such as a hill. If there were no viscosity (this may manifest itself by lack of adhesion to the boundary surface) the streamlines would have been free of stagnation points (i.e., points of zero velocity); in addition, the lines are concentrated where the velocity is greater, and sparse in the regions of smaller velocities (Figure 79a).

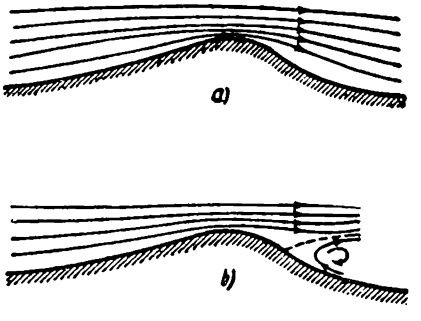

Figure 79. Flow around an obstacle, formed by: a) a perfect fluid; b) a viscous fluid

Since air is viscous (though only slightly), a boundary layer is formed on the surface of a body surrounded by air streams. On account of the divergence of the streamlines over the lee side of the hill the velocities there will be smaller, and the pressure greater (according to Bernoulli's equation). The slow air particles will therefore meet with increasing pressure on their way and will eventually lose all motion.

If we follow the phenomenon which occurs to the right of the stagnation point, where the velocity vanishes (Figure 79, b), we will find that a backward current is developed in the boundary layer(in respect to the fundamental unperturbed stream), because the pressure increases to the right. The geometric locus of the stagnation points gives the surface of separation, where the stream becomes discontinuous. This surface of separation is unstable and generates vortexes. The vortexes engendered on the surface of separation will be borne-off into the region of the unperturbed potential flow, and this region will be gradually filled up with vortexes.

However, taking into consideration the negligible thickness of the boundary layer, and consequently the very small size of the vortexes engendered in it, it will be clear that the role of the boundary layer in the obstruction of the atmosphere by vortexes is not very important, and that the majority of the vortexes congesting the atmosphere are of another type.

§6 The Boundary Layer in the Temperature Field and in the Humidity Field

Let us consider an unperturbed stream of air, having a temperature T, into which is introduced a plate of temperature T_0. The temperature field near the plate will then be disturbed. The layer in which the disturbance of the temperature is observed is called the boundary layer of the temperature field.

The distribution of temperature also affects the motion of gas. This influence will be studied below; here we will restrict ourselves to the examination of the boundary layer in the temperature field, regarding the gas as incompressible.

We will proceed from the analogy between heat transfer and momentum transfer, substituting the enthalpy $\rho c_p T$, for the momentum ρu in equation (5) § 5 of the boundary layer, and the coefficient of thermal conductivity k for the coefficient of viscosity μ; by analogy with Prandtl's equation (5) § 5, the equation of heat conduction for the boundary layer can be written as

$$\rho c_p \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2}. \tag{1}$$

By repeating the reasoning of the previous section, we obtain the following expression for the thickness of the temperature boundary layer δ_T

$$\text{II}\ \delta_T = \sqrt{\frac{kl}{\rho c_p \cdot u}}. \tag{2}$$

The thickness of the temperature boundary layer for dry air is connected with the thickness of the flow field boundary layer by the following relationship:

$$\delta_T = \delta \cdot \sqrt{1.3} = 1.14\delta, \tag{3}$$

i.e., the thickness of the temperature boundary layer is greater by 14 % than that of the flow boundary layer.

Let us assume now that air flows in a laminar manner over a flat water surface.

A boundary layer will develop in the humidity field over this surface. Using the analogy between the transport of humidity and the transfer of momentum, we can write the following equation for the diffusive transport of vapor by substituting the specific humidity s for the momentum density and the coefficient of diffusion D for the coefficient of viscosity μ:

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = D\frac{\partial^2 s}{\partial y^2}. \tag{4}$$

Evaluating the thickness of the boundary layer in the humidity field, we obtain

$$\text{II}\,\delta_s = \text{II}\,\sqrt{\frac{kl}{u}}. \tag{5}$$

For the diffusion of water vapor in air

$$\delta_s = \sqrt{1.6}\cdot\delta = 1.3\,\delta. \tag{6}$$

Thus, the thicknesses of the flow, temperature, and humidity boundary layers satisfy the inequality

$$\delta < \delta_T < \delta_s. \tag{7}$$

This result has a considerable importance in the theory of instruments for the measurement of evaporation.

The obtained results are related to the stationary laminar boundary layer, whose thickness is very small. The thickness of the boundary layer rarely exceeds 1 centimeter even in the case of a plate placed parallel to the stream.

However, the existence of wind gusts rules out the assumption that the air flow is stationary. The boundary layer in air therefore continually collapses and is constantly reformed.

§ 7. The Turbulent Boundary Layer

Experiments in cylindrical tubes have shown that laminar flow becomes turbulent as the number Re increases. It is true that the study of flow patterns in cylindrical tubes has little bearing on meteorological problems. The studies of stream perturbations over plane surfaces and plates are of considerably greater interest. It is natural to expect (and these expectations in fact prove to be correct) that the flow past a certain surface ceases to be laminar and becomes turbulent at definite values of the number Re. Owing to the lack in reliable data concerning the flow over flat surfaces, we are obliged to apply data obtained by experiments with tubes.

We explained in the previous section that a boundary layer with thickness δ is formed on the surface of the horizontal plate. The flow over a plane surface can be approximately regarded as similar to the flow in a tube having a diameter of $2\,\delta$. Notwithstanding, a complete analogy cannot be drawn here, since the flow in a cylindrical tube is either laminar or turbulent for the whole length of the tube. The current flowing around a flat plate can be laminar near its upstream edge, and turbulent behind it. The thickness of the laminar boundary layer changes, as we have seen, proportionally to $x^{\frac{1}{2}}$; the thickness of the turbulent boundary layer changes according to a different law.

Experiments have shown that the turbulent boundary layer and the laminar boundary layer differ above all in the velocity near the wall, which increases much

more slowly in a turbulent than in a laminar layer.

Prandtl adopted the "1/7 power law" for the velocity distribution in a turbulent layer,

$$u = u_\delta \left(\frac{z}{\delta}\right)^{\frac{1}{7}}. \tag{1}$$

This law is semiempirical. It is obtained on the basis of the following assumptions:

1) The velocity distribution near the wall depends not only on the physical properties of the fluid (the viscosity μ and the density ρ), but also on the shear stress τ_0 on the wall.

2) It is independent of the dimensions of the tube, and of the shape of its walls.

3) The resistance for polished tubes (according to experimental data) is proportional to the 7/4 power of the velocity.

Applying the method of dimensions and using assumptions 1) and 2) we find that the required velocity distribution law shall have the form

$$u = B \cdot \rho^l \mu^m \tau^n z^k, \tag{2}$$

where the unknown exponents l, m, n, k, must satisfy the principle of dimensional homogeneity, and B is a dimensionless coefficient. We rewrite the equation (2) using dimensional symbols

$$[LT^{-1}] = [ML^{-3}]^l [ML^{-1}T^{-1}]^m [ML^{-1}T^{-2}]^n L^k. \tag{3}$$

By equating the exponents of every symbol on the left and right-hand sides of the equation, we obtain:

$$\begin{gathered} l + m + n = 0, \\ -3l - m - n + k = 1, \\ m + 2n = 1. \end{gathered}$$

Solving these equations for n we obtain:

$$m = 1 - 2n; \quad l = n - 1; \quad k = 2n - 1. \tag{4}$$

By substituting (4) in (2) we find:

$$u = B \cdot \rho^{n-1} \mu^{1-2n} \cdot \tau^n \cdot z^{2n-1},$$

from which

$$\tau = C \cdot u^{\frac{1}{n}} \cdot \rho^{\frac{1-n}{n}} \cdot \mu^{\frac{2n-1}{n}} z^{\frac{1-2n}{n}},$$

where

$$C = B^{-\frac{1}{n}}.$$

Let us assume, according to the third condition, that $\frac{1}{n} = \frac{7}{4}$, then $n = \frac{4}{7}$ and therefore $k = \frac{1}{7}$.

Thus the velocity distribution law assumes the form

$$u = B\rho^{-3/7}\mu^{-1/7}\tau^{4/7}z^{1/7}$$

or

$$u = B\left(\frac{\tau}{\rho}\right)^{4/7}\left(\frac{z}{\delta}\right)^{4/7}. \tag{5}$$

For τ we obtain the formula

$$\tau = C\rho u^{7/4}z^{-1/4}\nu^{1/4}, \tag{6}$$

and for the thickness of the boundary layer δ, for C = 0.0225, the formula

$$\delta = 0.37 \cdot x^{4/5} \cdot \nu^{1/5}u^{-1/5}. \tag{7}$$

Thus, the thickness of the turbulent boundary layer increases considerably faster than the thickness of the laminar layer. This occurs, because eddies break away from the walls during the turbulent motion and are gradually carried into the unperturbed stream along inclined trajectories (with respect to the surface).

Subsequent measurements in tubes have shown that the 1/7 power law is valid no longer for very high values of Re, and is replaced by a more general 1/n power law, n being equal to 10 for Re = 3,000,000.

When one speaks about a velocity at any point of the turbulent stream, a mean value of the velocity at this point is meant, of course. The actual velocity fluctuates around this mean value. This fluctuation of the velocity decreases near the wall; directly next to the wall the mean velocity is given by: $\tau_0 = \mu\left(\frac{du}{dz}\right)_{z=0}$.

However, if we accept the law of the 1/n power for the turbulent boundary layer near the wall, we obtain an infinite shear stress, because

$$\tau_0 = \mu\left(\frac{du}{dz}\right)_{z=0} = \mu u_\delta \frac{1}{\delta^{1/n}}\left(z^{-\frac{n-1}{n}}\right)_0 = \infty.$$

This conclusion, which is in contradiction with experiment, can be avoided by assuming the 1/n power law to hold for the turbulent stream nearly up to the wall, but to lose its validity at the wall itself. Experimental data corroborate the formation of a thin laminar layer in the immediate proximity of the wall.

Thus, we arrive at the following conclusion on the basis of numerous velocity measurements of turbulent flow in tubes: the flow velocity in a very thin layer lining the wall first increases linearly very rapidly and then varies quite slowly.

This means that a laminar boundary layer always underlies the turbulent boundary layer. Such a separation of the velocity field into two fields (a turbulent field of mean velocities, obeying the 1/7 power law, and a linear laminar velocity field) becomes more acutely apparent the greater is the number Re.

§ 8. Orders of Magnitude of the Meteorological Elements

Let us now consider some meteorological element, say, the pressure. Observations of the pressure distribution in the lower layers of the atmosphere show that the different values of the pressure lies within a very narrow range. Since the pressure changes very little, its order of magnitude can be completely determined by one number – the numerical value of the pressure, rounded to the nearest power of 10.

Let us now consider another meteorological element, the wind velocity, for example. The value of the wind velocity varies within a wider range. An estimate of the order of magnitude of the wind velocity can therefore be obtained only by the limits 1 and 10 of its range of variation. The wind velocity exceeds these limits only in very rare cases.

Fridman and Hesselberg adopted the following fundamental rule, in deriving simplified equations of the atmospheric motions: any term of the order $10^{n-2} - 10^{n-1}$ can be rejected, if it enters an equation containing summands of order $10^{n} - 10^{n+1}$. The error incurred thereby rarely exceeds 2%.

Fridman and Hesselberg have studied in detail the problem of the order of magnitude, not only for the fundalmental meteorological elements u, v, w, $p, \alpha = \frac{1}{\rho}, T$, but also for their derivatives. However, it is necessary to use empirical data for the evaluation of the orders of magnitude of the derivatives of meteorological elements. Finite difference quotients are therefore usually substituted for the derivatives of meteorological elements. Fridman and Hesselberg take 10 km = 10^4 m for the horizontal difference, and 100 m = 10^2 m for the vertical difference. The considerable variance in the choice of the horizontal and vertical differences is explained by the fact that the corresponding vertical variation of the meteorological elements is 100 – 1,000 times greater than the horizontal variation. One hour = 3,600 sec is adopted as the time difference.

In Table 26 we give the data for the orders of magnitude of the meteorological elements u, v, w, p, α, T, in MTS units, as well as the orders of magnitude of the corresponding derivatives.

Table 26

Orders of magnitude of the meteorological elements and their derivatives

	u, v	w	p	α	T
	$10^0 - 10^1$	$10^{-2} - 10^{-1}$*	10^2	10^3	10^2
$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$	$10^{-5} - 10^{-4}$	$10^{-7} - 10^{-6}$*	$10^{-7} - 10^{-6}$	$10^{-5} - 10^{-4}$	$10^{-5} - 10^{-4}$
$\frac{\partial}{\partial z}$	$10^{-3} - 10^{-2}$	$10^{-5} - 10^{-4}$	10^{-2}	10^{-1}	$10^{-3} - 10^{-2}$
$\frac{\partial}{\partial t}$	$10^{-4} - 10^{-3}$	$10^{-6} - 10^{-5}$*	$10^{-5} - 10^{-4}$	$10^{-4} - 10^{-3}$	$10^{-4} - 10^{-3}$
$\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}$	$10^{-10} - 10^{-9}$	$10^{-10} - 10^{-9}$	$10^{-12} - 10^{-11}$	$10^{-10} - 10^{-9}$	$10^{-10} - 10^{-9}$
$\frac{\partial^2}{\partial z^2}$	$10^{-5} - 10^{-4}$	$10^{-8} - 10^{-7}$	10^{-6}	10^{-5}	$10^{-5} - 10^{-4}$
$\frac{\partial^2}{\partial x \partial z}, \frac{\partial^2}{\partial y \partial z}$	$10^{-9} - 10^{-7}$	$10^{-10} - 10^{-9}$	$10^{-10} - 10^{-9}$	$10^{-8} - 10^{-7}$	$10^{-8} - 10^{-7}$
$\frac{\partial^2}{\partial t^2}$	$10^{-8} - 10^{-7}$	$10^{-10} - 10^{-9}$*	$10^{-9} - 10^{-8}$	$10^{-8} - 10^{-7}$	$10^{-8} - 10^{-7}$
$\frac{\partial^2}{\partial x \partial t}, \frac{\partial^2}{\partial y \partial t}$	$10^{-8} - 10^{-7}$	$10^{-11} - 10^{-10}$*	$10^{-10} - 10^{-9}$	$10^{-9} - 10^{-8}$	$10^{-9} - 10^{-8}$
$\frac{\partial^2}{\partial z \partial t}$	$10^{-7} - 10^{-6}$	$10^{-9} - 10^{-8}$	$10^{-9} - 10^{-8}$	$10^{-7} - 10^{-6}$	$10^{-7} - 10^{-6}$

* This value must be increased 10–100 times for a layer lying lower than 500 m.

Pressure is one of the meteorological elements which can be measured with the greatest accuracy. At the earth's surface, pressure can be measured with an accuracy of 0.1 mb, which is approximately 0.0001 of its value. In the layer extending from 0 up to 4,000 m the pressure stays within the limits 110 cb–55cb. The order of magnitude of the pressure in this layer may be taken as 10^2.

An exact determination of the temperature is very difficult, and a high degree of accuracy in the determination of temperature in the above mentioned layer is unattainable. It is nevertheless known that the value of the temperature in this layer remains between the limits 220°–320°K. Thus, the order of magnitude of the absolute temperature is 10^2.

The specific volume α is not measured in meteorology; it is calculated by means of Clapeyron's equation. The order of magnitude of the specific volume is $10^3\ m^3\ ton^{-1}$ for the same layer, extending from 0 up to 4,000 m.

The value of the wind velocity changes within broader limits; it can reach values from 0 (calm) to 50 m sec^{-1}. The wind velocity can exceed 50 m sec^{-1} only in exceptional cases. The order of magnitude of the wind velocity should therefore be put equal to 1–10 m sec^{-1}.

Our knowledge of the value of the vertical velocity in the atmosphere is less reliable. Regular measurements of the vertical velocity have not been carried out yet, owing to the lack of a simple and dependable method of measurement. Nevertheless, it has been confirmed on the basis of sporadic measurements that the order of magnitude of 2 is 10^{-1} – 1 m sec^{-1}. The vertical velocity exceeds 5 m sec^{-1} only in very rare cases.

The orders of magnitude of local time derivatives are evaluated by Fridman from data of observations carried out in the same place at different times. For example:

$$\frac{\partial f}{\partial t} = \frac{f_1 - f_0}{3600}; \quad \frac{\partial^2 f}{\partial t^2} = \frac{f_2 - f_1 - (f_1 - f_0)}{(3600)^2}.$$

The values of horizontal space derivatives are found from terrestrial or high altitude weather charts. For example

$$\frac{\partial f}{\partial x} = \frac{f_1 - f_0}{10^4}; \quad \frac{\partial^2 f}{\partial x^2} = \frac{f_2 - f_1 - (f_1 - f_0)}{10^8} \ldots$$

Mixed derivatives of the type $\frac{\partial^2 f}{\partial x\, \partial t}$ ...are determined by means of terrestrial and high altitude maps.

The vertical derivatives are determined according to the data obtained from vertical sounding of the atmosphere. For example

$$\frac{\partial f}{\partial z} = \frac{f_1 - f_0}{10^2}; \quad \frac{\partial^2 f}{\partial z^2} = \frac{f_2 - f_1 - (f_1 - f_0)}{10^4}.$$

Mixed derivatives of the type $\frac{\partial^2 f}{\partial z\, \partial x}$ ... are determined from high altitute weather charts, plotted for different levels.

Mixed derivatives of the type $\frac{\partial^2 f}{\partial z\, \partial t}$ are determined from high altitude isopleth maps, plotted for different levels.

Individual derivatives of meteorological elements form part of many weather equations. It is necessary to consider products of the type $u\frac{\partial f}{\partial x}$, $v\frac{\partial f}{\partial y}$ when evaluating the magnitudes of the individual derivatives.

For example, the order of magnitude of the product $\Pi\left(u\frac{\partial p}{\partial x}\right) =$ $= (10^{0} - 5.10^{1})\,(10^{-7} - 5.10^{-6})$ and hence has a lower limit 10^{-7}, and an upper limit 10^{-4}. However, a direct estimate of the values $u\frac{\partial p}{\partial x}$, on the basis of an analysis of weather charts, shows that $\Pi\left(u\frac{\partial p}{\partial x}\right) = 10^{-6} - 10^{-5}$. In Table 27 are given the orders of magnitude of the individual derivatives of meteorological elements.

Table 27

Orders of magnitude of individual derivatives of the meteorological elements

	u, v	w	p	α	T
$\frac{\partial}{\partial t}$	$10^{-4} - 10^{-3}$	$10^{-6} - 10^{-5}$ *	$10^{-5} - 10^{-4}$	$10^{-4} - 10^{-3}$	$10^{-4} - 10^{-3}$
$u\frac{\partial}{\partial x}$, $v\frac{\partial}{\partial y}$	$10^{-4} - 10^{-3}$	$10^{-6} - 10^{-5}$ *	$10^{-6} - 10^{-5}$	$10^{-4} - 10^{-3}$	$10^{-4} - 10^{-3}$
$w\frac{\partial}{\partial z}$	$10^{-4} - 10^{-3}$ *	$10^{-6} - 10^{-5}$ *	$10^{-4} - 10^{-3}$ *	$10^{-3} - 10^{-2}$ *	$10^{-4} - 10^{-3}$ *
$\frac{d}{dt}$	$10^{-4} - 10^{-3}$ *	$10^{-8} - 10^{-5}$ *	$10^{-4} - 10^{-3}$ **	$10^{-3} - 10^{-2}$	$10^{-4} - 10^{-3}$ *

* This value must be decreased 10-100 times for altitudes under 500 m.

** This value must be decreased 10 times for altitudes under 500 m.

Logarithmic derivatives of the meteorological elements are also often encountered. We shall therefore also give, in Table 28, the orders of magnitude of the logarithmic derivatives of some of the meteorological elements.

We also refer to Table 29, which gives the orders of magnitude of the divergence of the velocity, and of the components of the vorticity vector.

Table 28

Orders of magnitude of the logarithmic derivatives of meteorological elements

	$\ln p$	$\ln \alpha$	$\ln T$
$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$	$10^{-9} - 10^{-8}$	$10^{-8} - 10^{-7}$	$10^{-8} - 10^{-7}$
$\frac{\partial}{\partial z}$	10^{-4}	$10^{-5} - 10^{-4}$	$10^{-5} - 10^{-4}$
$\frac{\partial}{\partial t}$	$10^{-7} - 10^{-6}$	$10^{-7} - 10^{-6}$	$10^{-7} - 10^{-6}$
$\frac{d}{dt}$	$10^{-6} - 10^{-5}$	$10^{-6} - 10^{-5}$	$10^{-7} - 10^{-6}$
$\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial y^2}$	$10^{-14} - 10^{-13}$	$10^{-13} - 10^{-12}$	$10^{-13} - 10^{-12}$
$\frac{\partial^2}{\partial z^2}$	10^{-8}	$10^{-8} - 10^{-7}$	$10^{-8} - 10^{-7}$
$\frac{\partial^2}{\partial x\,\partial z}$, $\frac{\partial^2}{\partial y\,\partial z}$	$10^{-12} - 10^{-11}$	$10^{-11} - 10^{-10}$	$10^{-11} - 10^{-10}$
$\frac{\partial^2}{\partial t^2}$	$10^{-11} - 10^{-10}$	$10^{-11} - 10^{-10}$	$10^{-11} - 10^{-10}$
$\frac{\partial^2}{\partial x\,\partial t}$, $\frac{\partial^2}{\partial y\,\partial t}$	$10^{-12} - 10^{-11}$	$10^{-12} - 10^{-11}$	$10^{-12} - 10^{-11}$
$\frac{\partial^2}{\partial z\,\partial t}$	$10^{-11} - 10^{-10}$	$10^{-10} - 10^{-9}$	$10^{-10} - 10^{-9}$

Finally, we will also give the orders of magnitude of some physical constants, which enter in the equation of motion (Table 30).

Table 29

Orders of magnitudes of the velocity divergence, and of the components of the vorticity

	$\operatorname{div} v$	$\operatorname{curl}_x v, \operatorname{curl}_y v$	$\operatorname{curl}_z v$
	$10^{-6}-10^{-5}$	$10^{-8}-10^{-2}$	$10^{-5}-10^{-4}$
$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$	$10^{-11}-10^{-10}$	$10^{-8}-10^{-7}$	$10^{-10}-10^{-9}$
$\frac{\partial}{\partial z}$	$10^{-9}-10^{-8}$	$10^{-5}-10^{-4}$	$10^{-8}-10^{-7}$
$\frac{\partial}{\partial t}$	$10^{-10}-10^{-9}$	$10^{-7}-10^{-6}$	$10^{-9}-10^{-8}$
$\frac{d}{dt}$	$10^{-10}-10^{-9}$	$10^{-7}-10^{-6}$	$10^{-9}-10^{-8}$

Table 30

Orders of magnitude of some constants entering in the equations of motion

ω_x, ω_y	ω_z	μ	g	$\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$	$\frac{\partial g}{\partial z}$	R
10^{-4}	10^{-4}	$10^{-3}-10^{-2}$	10^{1}	10^{-8}	10^{-6}	10^{2}

We shall now consider rules for the use of orders of magnitude both of the meteorological elements and of their derivatives, and apply these rules in order to simplify the equations of motion of the atmosphere:

It is sufficient to examine an equation comprising three terms:

$$A+B=C. \tag{1}$$

1. It is obvious that any such equation must contain at least two terms of order of magnitude not lower than that of the remaining term. Otherwise we have a contradiction.

2. The simplification of an equation is impossible, if all its terms are of the same order of magnitude.

3. If an equation contains terms with different orders of magnitude, among which one has an order of magnitude at least 100 times smaller than the order of the highest term, the smaller term can be rejected. For example, if A and B are terms of an order $10^{n}-10^{n+1}$, and C of an order $10^{n-2}-10^{n-1}$, the simplified equation can be written thus:

$$A+B=0.$$

It should be pointed out that the rejected term is not at all equal to zero:

$$C\neq 0;$$

it may be rejected because it is small in comparison with the other terms.

4. If A and B have the same order of magnitude, their sum cannot have a higher order.

5. It is often necessary to evaluate the order of magnitude of a product of two or more terms. In this case it is necessary to apply the following procedure: Let the term of order $10^m - 10^{m+1}$ be multiplied by a term of order $10^n - 10^{n+1}$. It is found by direct multiplication that its order ranges from $10^{m+n} - 10^{m+n+3}$. Attention should be drawn to the fact that 10^{m+n+3}, and not 10^{m+n+2}, is taken as the upper bound of the product, because terms of type $5 \cdot 10^{m+1}$, $5 \cdot 10^{n+1}$ appear in the factors, and obviously their product has an order of 10^{m+n+3}. It is very often possible to narrow the range of the order of the product down to $10^{m+n+1} - 10^{m+n+2}$ For greater accuracy it is recommended, when evaluating the order of magnitude of this term, to determine it by direct empirical data.

6. In addition it should be pointed out that the simplified equations thus obtained can be neither differentiated, nor integrated. Only algebraic operations can be carried out on them. It is always necessary to apply precise equations when differentiating or integrating. Simplifications are only possible after the operations of differentiation or integration have been carried out; otherwise false conclusions may be obtained.

We will determine the order of magnitude of the terms entering in the equations of motion by the preceding tables:

$$\Pi\left(\frac{du}{dt}\right) = \Pi\left(\frac{dv}{dt}\right) = 10^{-4}—10^{-3}; \quad \Pi\left(\frac{dw}{dt}\right) = 10^{-6}—10^{-5};$$

$$\Pi\left(\alpha\frac{\partial p}{\partial x}\right) = \Pi\left(\alpha\frac{\partial p}{\partial y}\right) = 10^{-4}—10^{-3}; \quad \Pi\left(\alpha\frac{\partial p}{\partial z}\right) = 10^{1}; \quad \Pi(g) = 10^{1};$$

$$\Pi(2\omega_z v) = \Pi(2\omega_z u) = 10^{-4}—10^{-3};$$

$$\Pi(2\omega_x w) = \Pi(2\omega_y w) = 10^{-6}—10^{-5};$$

$$\Pi\left(\alpha\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)\right) = \Pi\left(\alpha\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)\right) = 10^{-4}—10^{-3};$$

$$\Pi\left(\alpha\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)\right) = 10^{-7}—10^{-6}.$$

Thus, the equations of motion can be written in the simplified form:

$$\frac{du}{dt} = -\alpha\frac{\partial p}{\partial x} + 2\omega_z v + \alpha\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right),$$

$$\frac{dv}{dt} = -\alpha\frac{\partial p}{\partial y} - 2\omega_z u + \alpha\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right),$$

$$0 = -\alpha\frac{\partial p}{\partial z} + g.$$

The first two equations are worked out correct to one part in a hundred while the third (hydrostatic) equation is correct to one in a thousand.

Let us try to simplify yet further the equation of continuity, writing it in the form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{d\ln\alpha}{dt}.$$

We find from the tables:

$$\Pi\left(\frac{\partial u}{\partial x}\right) = \Pi\left(\frac{\partial v}{\partial y}\right) = 10^{-5}—10^{-4}, \quad \Pi\left(\frac{\partial w}{\partial z}\right) = 10^{-5}—10^{-4},$$

$$\Pi\left(\frac{d\ln\alpha}{dt}\right) = 10^{-6}—10^{-5}.$$

Thus, according to the principles mentioned above, it is impossible to simplify the equation of continuity any further. The equation of continuity can be used in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

only in isolated cases. Let us examine the possibility of simplifying the individual derivatives of meteorological elements. Referring to the data collected in Table 28, it is not difficult to understand that the derivatives $\frac{du}{dt}$, $\frac{dv}{dt}$, $\frac{dw}{dt}$, $\frac{dT}{dt}$, $\frac{da}{dt}$ cannot be simplified for a free atmosphere. Only the derivative $\frac{dp}{dt}$ can be simplified and used in the form

$$\frac{dp}{dt}=\frac{\partial p}{\partial t}+w\frac{\partial p}{\partial z}.$$

The derivatives $\frac{dw}{dt}$, $\frac{da}{dt}$, $\frac{dp}{dt}$ cannot be simplified for conditions prevalent near the earth's surface. Only the following derivatives can be simplified:

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y};$$
$$\frac{dv}{dt}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y};$$
$$\frac{dT}{dt}=\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}.$$

The replacement of an individual by a local derivative should be particularly stressed. Such a substitution is possible in only one case – when the atmosphere is immobile.

§ 9. The Boundary Layer in the Atmosphere

Using considerations of similarity, Helmholtz showed that the viscosity plays a negligible role in large-scale motions. Helmholtz, however, dealt with molecular viscosity, but did not know of eddy viscosity. Nevertheless his considerations remain valid even now, when eddy viscosity is taken into account.

Furthermore, Helmholtz's conclusions are correct only on the conditions that there are no interfaces in the atmosphere.

However, it turns out that the motion within the boundary layer, even of a fluid of low viscosity, differs greatly from the motion of a perfect fluid.

Since the greatest part of the atmosphere can be regarded as a thin envelope 10–20 km thick, covering the huge globe, it is quite clear that the influence of eddy viscosity near the earth's surface cannot be neglected. It is obvious that a boundary layer always exists in the atmosphere, and eddy viscosity plays an essential role within this layer. The thickness of this boundary layer varies both in space and with time. Eddy viscosity plays an insignificant role outside the boundary layer.

Considerable differences of temperature and density may occur in the atmosphere even over short distances, as a result of nonuniform heat flow which, in their turn, can lead to considerable velocity variations, and consequently to an increase of turbulence.

We have mentioned previously the difficulties arising in the derivation of the equations of motion for a turbulent atmosphere. In order to study the turbulent motion of the atmosphere, Kochin used the Navier–Stokes equation of motion (in spherical coordinates) with a constant coefficient of viscosity, regarding it as the coefficient of eddy viscosity on a scale of the general circulation.

The nature of the coefficient of eddy viscosity has been little studied until now. Measurements of the coefficient of viscosity in the free atmosphere

demonstrate its great variability. It depends on the wind velocity, on the lapse rate, on the altitude and on the roughness of the underlying surface. These measurements show that the order of magnitude of the coefficient of eddy viscosity varies from 10^{-3} to 10^{-2} MTS.

The values of μ for atmospheric air were mainly obtained by measurements of the variations in the wind: it is obvious that they correspond to turbulence on a scale which is negligible in comparison with the scale of the general circulation. Horizontal large-scale turbulence, to which cyclones and anticyclones belong, must also be taken into consideration in the study of large-scale motions (of an order of the general circulation of the atmosphere). The coefficient of eddy viscosity for the general circulation of the atmosphere must therefore be several times greater than the vertical coefficient of eddy viscosity. The entire atmosphere can then be regarded as a boundary layer.

Kochin's fundamental premise consists in the application of a method derived from boundary layer theory to the general circulation of the atmosphere. As we have seen, this method is based on the simplification of the fundamental equations of hydrodynamics by rejecting the terms of secondary importance.

However, the system of equations obtained by Kochin differs from the system presented in the usual theory of the boundary layer, where the action of the external forces on the fluid can be disregarded, and the inertial forces are of an order of magnitude comparable with the viscosity forces. The inertial forces encountered in problems of general circulation are small as compared with the Coriolis forces. The viscosity forces must therefore be compared not with the inertial forces, but with the Coriolis forces.

Thus, the initial equations of Kochin's problem are the Navier-Stokes equations in spherical coordinates r, θ, ψ, where r is the distance from the earth's center; θ - the complement of the latitude ($\theta = \frac{\pi}{2} - \varphi$), the southern latitude being negative; ψ - the longitude, which increases from west to east.

In the particular case of a stationary zonal circulation i.e., a motion of the atmosphere which is independent of the time t and the longitude ψ, Kochin assumes that

$$\frac{\partial v_r}{\partial t} = \frac{\partial v_\theta}{\partial t} = \ldots = 0.$$

Such a circulation could be realized if the earth's axis were perpendicular to its orbital plane, the orbit itself were circular, and if the surface of the earth were uniform.

The equation of stationary zonal circulation has the same form as in the general case of a nonstationary and nonzonal circulation, with the sole difference that none of the equations contains the derivatives with respect to ψ and t and the coefficients of the equations are independent of ψ and t . Let us again note that the equation of stationary zonal circulation describes the general circulation of the atmosphere, averaged over the time and over the longitude.

Let us write the equation of the stationary zonal circulation:

$$\rho\left(v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta^2+v_\psi^2}{r}\right)=-g\rho-\frac{\partial p}{\partial r}+2\omega\rho\sin\theta\cdot v_\psi+$$
$$+\mu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{2}{r}\frac{\partial v_r}{\partial r}+\frac{\operatorname{ctg}\theta}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-2\frac{v_r}{r^2}-2\frac{\operatorname{ctg}\theta}{r^2}v_\theta\right);$$
$$\rho\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}-\frac{\operatorname{ctg}\theta v_\psi^2}{r}\right)=2\omega\rho\cos\theta\cdot v_\psi-\frac{1}{r}\frac{\partial p}{\partial\theta}+$$
$$\mu\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{2}{r}\cdot\frac{\partial v_\theta}{\partial r}+\frac{\operatorname{ctg}\theta}{r^2}\cdot\frac{\partial v_\theta}{\partial\theta}-\frac{v_\theta}{r^2\sin^2\theta}\right); \qquad (1)$$
$$\rho\left(v_r\frac{\partial v_\psi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\psi}{\partial\theta}+\frac{v_\psi\cdot v_r}{r}+\frac{\operatorname{ctg}\theta\, v_\theta v_\psi}{r}\right)=-2\omega\rho\cos\theta\cdot v_\theta+$$
$$+\mu\left(\frac{\partial^2 v_\psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\psi}{\partial\theta^2}+\frac{2}{r}\frac{\partial v_\psi}{\partial r}+\frac{\operatorname{ctg}\theta}{r^2}\cdot\frac{\partial v_\psi}{\partial\theta}-\frac{v_\psi}{r^2\sin^2\theta}\right);$$
$$\frac{\partial(\rho v_\theta\cdot r\sin\theta)}{\partial\theta}+\frac{\partial(\rho v_r\cdot r^2\sin\theta)}{\partial r}=0.$$

We will evaluate the order of magnitude of the terms entering in equation (2) with a view to their further simplification.

We adopt the notation $\Pi(v)$ for those quantities having the same order of magnitude as v. We shall denote the horizontal component of the velocity by v_s. Equations (1) contain derivatives with respect to θ and r. We will prove that

$$\Pi\left(\frac{\partial v_s}{\partial\theta}\right)=v_s. \qquad (2)$$

Indeed, the values of v_ψ and v_θ vary rather slowly during large-scale motions. These variations become appreciable only at a great distance, comparable to the earth's radius. A latitude arc $\Delta\theta$ of 1 radian, i.e., equal approximately to 57°, corresponds to such distances, and for this reason equation (2) can be applied.

Let us note, however, that the condition (2) is not fulfilled in the presence of a front.

Since the atmosphere is a viscous fluid, all the three components of the velocity vanish at the surface of the earth.

$$v_r=v_\theta=v_\lambda=0 \quad \text{or} \quad r=a, \qquad (3)$$

where a is the earth's radius.

The equation of continuity, integrated for the boundary condition (3), becomes

$$v_r=-\frac{1}{\rho r^2\sin\theta}\int_a^r\frac{\partial(\rho\, v_\theta r\sin\theta)}{\partial\theta}\,dr. \qquad (4)$$

Taking equation (2) into consideration, we arrive at the following fundamental relation:

$$\Pi(v_r)=v_s\cdot\frac{\delta}{a}, \qquad (5)$$

where δ is the altitude of the tropopause.

It follows from equation (5) that in the general circulation of the atmosphere the ratio of the vertical component of the wind velocity to its horizontal component equals the ratio of the thickness of the troposphere to the globe radius.

Evaluating successively the orders of magnitude of all the terms which enter

in the Navier-Stokes equations and rejecting all the small terms containing the factors δ/a and δ^2/a^2, we arrive at the following equations

$$-\rho\frac{(v_\theta^2+v_\psi^2)}{r^2}=-g\rho-\frac{\partial p}{\partial r}+2\omega\rho\sin\theta v_\psi+\mu\frac{\partial^2 v_r}{\partial r^2}; \tag{6}$$

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}-\frac{v_\psi^2\operatorname{ctg}\theta}{r}\right)=2\omega\rho\cos\theta\cdot v_\psi-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\frac{\partial^2 v_\theta}{\partial r^2}; \tag{7}$$

$$\rho\left(v_r\frac{\partial v_\psi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\psi}{\partial\theta}+\frac{v_\psi v_\theta\operatorname{ctg}\theta}{r}\right)=-2\omega\rho\cos\theta v_\theta+\mu\frac{\partial^2 v_\psi}{\partial r^2}. \tag{8}$$

The equations of motion (6) - (8) contain five sets of terms, corresponding to the components of different forces:

1) The components of inertial forces

$$\left(\rho\frac{v_\theta^2}{r},\ \frac{\rho v_\psi^2}{r},\ \rho v_r\frac{\partial v_\theta}{\partial r},\ \frac{\rho v_\theta}{r}\frac{\partial v_\theta}{\partial\theta},\ \frac{\rho v_\psi^2\operatorname{ctg}\theta}{r},\ \frac{\rho v_\theta}{r}\frac{\partial v_\psi}{\partial\theta},\ \frac{\rho v_\psi v_\theta\cdot\operatorname{ctg}\theta}{r}\right);$$

these are all of the same order $\frac{\rho v_s^2}{a}$;

2) the components of the pressure gradient $\frac{\partial p}{\partial r}$, $\frac{1}{r}\frac{\partial p}{\partial\theta}$;

3) the force of gravity $g\rho$;

4) the Coriolis forces $(2\omega\rho\sin\theta\cdot v_\psi,\ 2\omega\rho\cos\theta\cdot v_\psi,\ -2\omega\rho\cos\theta\cdot v_\theta)$;

5) the forces of eddy viscosity $\left(\mu\frac{\partial^2 v_r}{\partial r^2},\ \mu\frac{\partial^2 v_\theta}{\partial r^2},\ \mu\frac{\partial^2 v_\psi}{\partial r^2}\right)$.

The action of the external forces is omitted in the general theory of the boundary layer. In Kochin's problem it is necessary to compare four sets of terms: the force of gravity, the Coriolis forces, the inertial forces and the forces of viscosity. The expression which determines the thickness of the boundary layer therefore differs essentially, in the case of the general circulation, from the corresponding expression for flow past a plate.

Indeed, let us compare the inertial forces with the Coriolis forces:

$$\Pi\left(\rho\frac{v_s^2}{a}\right):\Pi(\omega\rho v_s)=\frac{v_s}{a\omega}.$$

But at $v_s \approx 10$ m/sec, $\frac{v_s}{a\omega}=0.03$.

Consequently, the inertial terms in the equations of motion can be rejected. The equations of motion (6), (7) and (8) become:

$$\mu\frac{\partial^2 v_r}{\partial r^2}+2\omega\rho\sin\theta\cdot v_\psi=\frac{\partial p}{\partial r}+g\rho, \tag{9}$$

$$\mu\frac{\partial^2 v_\theta}{\partial r^2}+2\omega\rho\cos\theta\cdot v_\psi=\frac{1}{r}\frac{\partial p}{\partial\theta}, \tag{10}$$

$$\mu\frac{\partial^2 v_\psi}{\partial r^2}-2\omega\rho\cos\theta\cdot v_\theta=0. \tag{11}$$

The last equation proves that the viscosity forces in a zonal stationary circulation have the same order of magnitude as the Coriolis forces.

The order of magnitude of the viscosity forces is equal to that of the expression $\mu\frac{v_s}{\delta^2}$ and the order of magnitude of the Coriolis forces--to the order of the expression

$\omega\rho v_s$. By equating these expressions we obtain an equation, which determines δ_1 —the thickness of the earth's surface layer:

$$\delta_1 = \sqrt{\frac{\mu}{\omega\rho}}. \tag{12}$$

Putting $\rho = 1.3 \times 10^{-3}$ g/cm^3, $\omega = 7.3 \times 10^{-5}$ sec^{-1} and giving to μ the value of the coefficient of molecular viscosity 1.7×10^{-5} g/cm • sec, we obtain: $\delta = 42$ cm.

Thus, the thickness of the laminar boundary layer would be of the order of 1 meter, if there were no turbulence in the atmosphere.

For the actual atmosphere however, it is necessary to use the coefficient of turbulence, which is about 5×10^5 times greater, instead of the coefficient of molecular viscosity. The thickness of the turbulent boundary layer δ will then increase approximately 700 times, and we will obtain $\delta \approx 300$ m.

By evaluating the order of magnitude of the terms of equation (9), Kochin has shown that this equation can be approximated by the hydrostatic equation:

$$\frac{\partial p}{\partial r} + g\rho = 0. \tag{13}$$

Chapter X

SIMPLE ATMOSPHERIC MOTIONS WITHOUT FRICTION

§ 1. Zonal Flow

The mathematical difficulties arising in the attempt to solve even the commonest problem of dynamic meteorology are still insurmountable, and so far no problem of practical importance in meteorology has been fully solved. Therefore, we have to limit ourselves to the discussion of isolated, most elementary problems, sometimes simplifying them in quite a crude fashion.

Ignoring the forces of friction we assume for the present that a moving particle of air does not change its state, i. e., there are no changes in pressure, density, temperature and humidity resulting from its motion. Thereby we limit ourselves to the discussion of such motions which do not involve mutual penetration of masses into each other and in which no vacuum is created, so that the equation of continuity is fulfilled.

We shall first consider a horizontal motion of the atmosphere, all the air particles moving without acceleration in straight or circular trajectories.

In cases of such elementary motions there is no need to integrate the differential equations of motion and it suffices to study them in their differential form, since the trajectory of the particles is given. All our attention should be directed to the study of the dynamic conditions of motion, which can be reduced to the determination of the conditions of equilibrium of forces actually affecting the air particle and inertial forces.

Though this may seem a somewhat oversimplified statement of the problem, the results achieved in solving it—the gradient wind and the slope of the frontal surfaces—are of tremendous practical importance and constitute almost the only quantitative results in atmospheric dynamics operationally used by the weather services.

In addition, our daily experience indicates that air motions on a large scale are almost always horizontal. Formation of areas of vertical motions is accompanied as a rule by the formation of clouds and occurrence of precipitation, or by dispersion of clouds. These areas should therefore be especially studied. However, in the present chapter we will limit ourselves to the study of the horizontal motions.

A simple example of horizontal motion in the atmosphere is the zonal flow, which we shall discuss first.

A zonal flow is that flow in which each air parcel moves regularly on a circle of latitude with an absolute angular velocity ω_a. This angular velocity may be different for different air parcels, but should remain constant for each of them. If the absolute angular velocity ω_a of the parcel differs from the angular velocity of rotation of the earth ω, i. e., if $\omega_a \neq \omega$, then the parcel is in relative motion, moving on a circle of latitude from west to east when $\omega_a > \omega$, and from east to west when $\omega_a < \omega$

(since ω_a and ω are parallel or antiparallel to the earth's axis, depending on the direction of the moving parcel, they can be regarded not as vectors but as scalar algebraic magnitudes).

It is convenient to study the zonal flow using absolute coordinates, since the trajectory of absolute motion is known and the equations of absolute motion have a simpler form than those of relative motion.

In zonal flow the absolute total acceleration of a particle equals the absolute centripetal acceleration:

$$j_a = -\omega_a^2 \cdot R, \tag{1}$$

therefore the centrifugal force will be given by the formula:

$$J_a = \omega_a^2 \cdot R. \tag{2}$$

The equation of the absolute zonal motion is:

$$b + g_a + J_a = 0, \tag{3}$$

where b is the pressure gradient per unit mass and g_a is the earth's gravitational acceleration.

If $\omega_a = \omega$ for all air parcels, $J_a = J_e$, i. e., the atmosphere rotates as an absolutely rigid body and is in equilibrium. Accordingly, the isobaric surfaces should coincide with the geopotential surfaces (cf. § 9 Chapter V).

The centrifugal and gravitational forces act in the meridian plane. Therefore the first vector of equation (3), the pressure gradient b, is also a meridional vector i. e., the isobaric surfaces should be perpendicular to any meridian surface; consequently, they are surfaces of revolution about the earth's axis. The total pressure field can be fully determined by the meridional isobars, i. e., by the intersection lines of the isobaric surfaces and the meridian plane.

The gravitational equipotential surfaces are concentric spheres.

If $\omega_a \neq 0$ the meridional isobars converge together at the poles.

Since ω_a enters quadratically in the equation for the centrifugal force (2), the isobaric surfaces will be the same for any sense of rotation, at a given angular velocity. Dynamically similar, the two opposite senses of rotation of an air parcel do, however, differ kinematically.

If the sense in which the air parcel rotates coincides with the sense of rotation of the earth, ω_a is positive, and when the two rotate in opposite senses ω_a is negative. We will consider the motion of an air parcel in the northern hemisphere. Anyhow, results obtained for stationary currents in the northern hemisphere can easily be extended to the southern hemisphere.

At different ω_a the isobaric surfaces are differently oriented in relation to the level surfaces. Because of the symmetry of the force field in relation to the equator and to the axis of the earth, it is sufficient to examine one quadrant instead of the whole circle.

The zonal motion of the atmosphere is observed from the earth as a relative zonal motion with angular velocity α, where

$$\omega_a = \omega + \alpha \tag{4}$$

If $\alpha > 0$ the zonal wind blows from the west: when $\alpha = 0$ the atmosphere is relatively at rest; if $\alpha < 0$ the zonal wind blows from the east.

If the sense of the relative rotation of the air parcel coincides with the sense of rotation of the earth, the motion of the parcel is cyclonic; in the contrary case the motion is anticyclonic. The terms cyclonic and anticyclonic will be generalized later. In Figure 80 are shown various positions of the isobaric surfaces in relation to the level surfaces for cyclonic and anticyclonic motions. In Table 31 are given explanations of Figure 80.

Figure 80 and Table 31 show, that zonal currents possess two centers of rotation—one at each pole. For a cyclonic current (case a) the center of rotation at any level is a barometric depression. For anticyclonic currents, for which $\alpha - 2\omega$, there is a pressure elevation at the center of rotation, for any level. However, if for the anticyclonic current $\alpha < -2\omega$, depression develops at the center of rotation.

Zonal anticyclonic currents of the e, f, and g types never develop in the atmosphere (although dynamically they are allowed), since none of the existing atmospheric conditions could cause its absolute rotation in a sense opposite to that of the earth.

Analysis of the zonal motion enables us to explain in a very simple manner the physical meaning of the Coriolis acceleration.

The absolute and the relative accelerations of an air parcel in zonal motion are respectively determined by the formulas

$$j_a = -\omega_a^2 R, \tag{5}$$

$$j = -\alpha^2 R. \tag{6}$$

Let us resolve the absolute acceleration

$$j_a = -(\alpha + \omega)^2 R = -\alpha^2 R - 2\alpha\omega R - \omega^2 R$$

and separate the terms giving the transport acceleration and the relative acceleration

$$j_a = j + j_e - 2\omega \cdot \alpha R. \tag{7}$$

The third member of the right part of equation (7) gives the Coriolis acceleration. Since the absolute, the transport, and the relative accelerations are directed along the radius of the latitudinal circle, towards its center, the Coriolis acceleration is likewise directed along this radius; it is directed towards the center of the latitudinal circle when the zonal motion is cyclonic ($\alpha > 0$) and away from the center in case of an anticyclonic motion ($\alpha > 0$). Accordingly, the Coriolis force in a cyclonic zonal motion is directed centrifugally along the radius of the latitudinal circle and in case of an anticyclonic motion the direction is centripetal, along the same radius.

Table 31

The zonal motion and the pressure field

Fig. 80	Relative angular velocity	Absolute angular velocity	Nature of motion	Dynamic condition	Nature of the pressure field
a	$\alpha > 0$	$\omega_a > \omega$	Cyclonic zonal westerly	$I_a > I_e$ $\lvert b\rvert < \lvert g\rvert$	The isobars are flattened at the poles more than the level surfaces. At any level on the equator there is a belt of high pressure, and at the poles a region of low pressure.
b	$\alpha = 0$	$\omega_a = \omega$	The atmosphere is relatively quiescent	$I_a = I_e$ $\lvert b\rvert = \lvert g\rvert$	The isobars are everywhere parallel to the level surfaces of gravitation.
c	$-\omega < \alpha < 0$	$\omega_a < \omega$	Anticyclonic zonal westerly	$I_a < I_e$ $\lvert b\rvert > \lvert g\rvert$	The isobars are flattened at the poles less than the level surfaces. At any level on the equator there is a belt of low pressure, and at the poles there is a cap of high pressure.
d	$\alpha = -\omega$	$\omega_a = 0$	The same	$I_a = 0$ $b + g_a = 0$	Isobars coincide with the spherical level surfaces of gravitation. At any level of gravitation on the equator there is a belt of low pressure and at the pole a cap of high pressure.
e	$-\omega > \alpha > -2\alpha$	$0 > \omega_a > -\omega$	The same	$I_a < I_e$ $\lvert b\rvert > \lvert g\rvert$	The isobars are flattened at the poles less than the level surfaces of gravitation. At any level on the equator there is a belt of low pressure, and at the pole a cap of high pressure.
f	$\alpha = -2\omega$	$\omega_a = -\omega$	"	$I_a = I_e$ $\lvert b\rvert = \lvert g\rvert$	The isobars are parallel to the level surfaces of gravitation.
g	$-2\omega > \alpha$	$\omega_a < -\omega$	"	$I_a > I_e$ $\lvert b\rvert < \lvert g\rvert$	The isobars are flattened at the poles more than the level surfaces. At any level on the equator there is a belt of high pressure, and at the poles a region of low pressure.

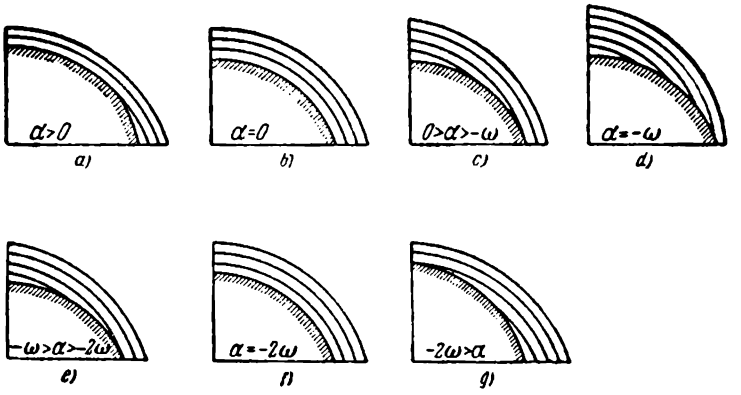

Figure 80. Relative positions of isobaric and equipotential surfaces in zonal motion

§ 2. The Equation of Motion of the Atmosphere, Regarded as a Perfect Fluid, in Intrinsic and in Standard Coordinates

For the study of horizontal motions of the atmosphere one can use either the standard (Figure 81, a) or the intrinsic coordinates (Figure 81, b). The intrinsic coordinates s, n, z, are determined by the directions of the vectors $\boldsymbol{\tau}$, $\boldsymbol{n}$, $\boldsymbol{k}$, forming a right-handed system, where the vector $\boldsymbol{\tau}$ is tangent to the trajectory, the vector $\boldsymbol{k}$ points to the zenith, and the vector $\boldsymbol{n}$ is normal to the trajectory and will be observed from the zenith to the left of the direction of motion.

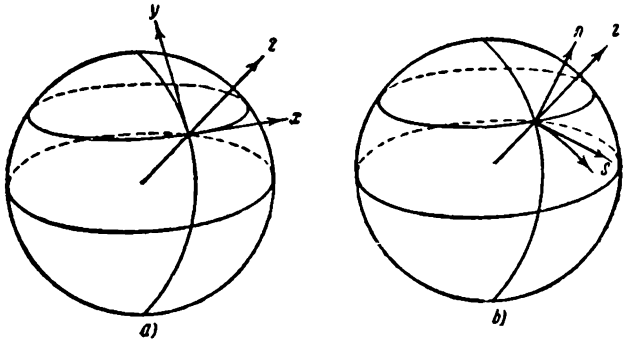

Figure 81. a) Standard, and b) intrinsic systems of coordinates

If the motion is directed eastwards, the intrinsic frame of coordinates s, n, z coincides with the standard frame x, y, z, the x-axis being directed to the east, the y-axis to the north, and the z-axis towards the zenith.

Let us turn now to the equation of motion of the atmosphere as a perfect fluid

$$\frac{dv}{dt} = -\frac{1}{\rho}\operatorname{grad} p - \operatorname{grad}\Phi - 2[\omega, v]. \qquad (1)$$

The acceleration of the particle in arbitrary horizontal motion will obviously be expressed in intrinsic coordinates in the following way

$$\frac{dv}{dt} = \frac{dv}{dt}\tau + \frac{v^2}{r}n,$$

where r is the radius of curvature of the particle trajectory. In standard coordinates this acceleration will be expressed by:

$$\frac{dv}{dt} = \frac{dv_x}{dt} i + \frac{dv_y}{dt} j.$$

The pressure gradient can be expressed in intrinsic coordinates in this form:

$$\operatorname{grad} p = \frac{\partial p}{\partial s} \cdot \tau + \frac{\partial p}{\partial n} n + \frac{\partial p}{\partial z} k,$$

and in standard coordinates:

$$\operatorname{grad} p = \frac{\partial p}{\partial x} i + \frac{\partial p}{\partial y} j + \frac{\partial p}{\partial z} k.$$

The force of gravity obviously does not have a horizontal component. Therefore its only nonvanishing component will be the vertical one

$$g_z = -g.$$

Before we develop the expression for the Coriolis force in intrinsic and in standard coordinates let us restate the definition of the angular velocity of the earth's rotation.

The angular velocity of the rotation of the earth ω is a vector directed along the axis of rotation from south to north. This is a meridional vector, hence its latitudinal component is equal to zero:

$$\omega_x = 0. \tag{2}$$

The other standard components ω_y, ω_z, lying in the plane of the meridian can be easily expressed through the latitude φ. The component ω_y, pointing to the north and lying in the horizontal plane, will be:

$$\omega_y = \omega \cos \varphi, \tag{3}$$

and the vertical component will be:

$$\omega_z = \omega \sin \varphi. \tag{4}$$

The vertical component of the angular velocity of the earth is positive in the northern hemisphere and negative in the southern hemisphere. Therefore, using a negative sign for southern latitudes we can apply equation (4) for both hemispheres.

Now we can develop the vector product in standard coordinates, writing it first in the form of a determinant

$$a = -2[\omega, v] = -2 \begin{vmatrix} i, & j, & k \\ 0, & \omega_y, & \omega_z \\ v_x, & v_y, & 0 \end{vmatrix} \tag{5}$$

or

$$a = 2\omega_z \cdot v_y \cdot i - 2\omega_z \cdot v_y \cdot j + 2\omega_y v_x \cdot k. \tag{6}$$

Thus, in standard coordinates the components of the Coriolis force for horizontal motion will be given by:

$$a_x = 2\omega_z \cdot v_y; \quad a_y = -2\omega_z \cdot v_x; \quad a_z = 2\omega_y \cdot v_x \tag{7}$$

or

$$a_x = 2\omega \sin\varphi \cdot v_y; \quad a_y = -2\omega \sin\varphi \cdot v_x; \quad a_z = 2\omega \cos\varphi \cdot v_x. \tag{8}$$

In intrinsic coordinates the Coriolis force will be given by:

$$a = -2[\omega, v] = -2\begin{vmatrix} \tau, & n, & k \\ \omega_s, & \omega_n, & \omega_z \\ v, & 0, & 0 \end{vmatrix} = -2\omega_z \cdot v \cdot n + 2\omega_n \cdot v \cdot k, \tag{9}$$

hence

$$a_s = 0; \quad a_n = -2\omega_z \cdot v; \quad a_z = 2\omega_n \cdot v. \tag{10}$$

We can see that the Coriolis force does not have a component in the direction of the motion. It merely deflects the particle in the direction of the normal to its trajectory. Therefore, the Coriolis force is also called the deflecting force of terrestrial rotation. The only horizontal component of the Coriolis force is

$$a_n = -2\omega \sin\varphi \cdot v. \tag{11}$$

As can be seen from (11) the Coriolis force deflects the particle to the right of its direction of motion in the northern hemisphere and to its left in the southern hemisphere. At the equator the horizontal component of the Coriolis force vanishes.

Thus, the equations of horizontal motion has the following form in the standard coordinate system:

$$\begin{aligned} \frac{dv_x}{dt} &= -\alpha \frac{\partial p}{\partial x} + 2\omega \sin\varphi \cdot v_y, \\ \frac{dv_y}{dt} &= -\alpha \frac{\partial p}{\partial y} - 2\omega \sin\varphi \cdot v_x, \\ 0 &= -\alpha \frac{\partial p}{\partial z} - g + 2\omega \cos\varphi \cdot v_x. \end{aligned} \tag{12}*$$

These equations are often used for the study of atmospheric dynamics, when it is preferable to have a coordinate frame that is independent of the motion.

However, of wider use are the equations of horizontal motion in intrinsic coordinates:

$$j_s = \frac{dv}{dt} = -\alpha \frac{\partial p}{\partial s}, \tag{13}$$

$$j_n = \frac{v^2}{r} = -\alpha \frac{\partial p}{\partial n} - 2\omega \sin\varphi \cdot v. \tag{14}$$

$$0 = -\alpha \frac{\partial p}{\partial z} + 2\omega_n v - g. \tag{15}$$

The equations of motion (13) and (14) are respectively called the tangential and the normal equations.

To an earthbound observer, measuring the vertical force by means of a spring balance, gravity will appear as the sum of the force of gravitation and the centrifugal force due to terrestrial rotation.

However, if the observer moves horizontally together with the air parcel (on a free aerostat, for instance), the vertical force he measures with a spring balance will be the sum of the force of gravity and the vertical component of the Coriolis force which is exerted on the air parcel, due to its motion relative to

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* [In these and the following equations α stands for the specific volume. Translator's note].

the earth. Nevertheless the observer, moving together with the air parcel, will not be able to distinguish by measurement one component of the total force from the other. The spring balance will merely register the resultant of these component forces, which we perceive as the force of gravity. Let us call the sum of the force of gravity and the vertical component of th. Coriolis force the "virtual" force of gravity g', so that

$$g' = g - 2\omega \cos\varphi\, v_x. \tag{16}$$

As can be seen from (16) the virtual force of gravity as measured by a spring balance moving together with the air parcel, is not equivalent to the force of gravity acting on the same mass when at rest. It depends upon the parcel's speed and its direction and upon the latitude as well.

The vertical component of the Coriolis force is directed upwards if $v_x > 0$, i. e., if the parcel moves to the east, and downwards, if the particle moves to the west. Thus, as a result of the existence of the Coriolis force an air parcel in a westerly is effectively lighter than in an easterly current.

Finally, at a given horizontal velocity the effect of the vertical component of the Coriolis force depends upon the latitude φ, reaching a maximum on the equator and vanishing at the pole.

Such an irregular behavior of the virtual force of gravity might have considerably complicated the study of horizontal currents. However, even at the immense velocity of 100 m/sec, which is but very rarely encountered in the atmosphere, the Coriolis acceleration* at the equator is 0.0146 m/sec^2 while the average acceleration of gravity amounts to 9.81 m/sec^2. Consequently, even in such extreme cases, the correction we would have to introduce in the value of the gravity force, would not attain even 0.2% of the latter's average value. Usually this correction does not amount to even a hundredth part (see Appendix 19).

The equation of motion (15) may be written using the virtual gravity force, in the form:

$$-\alpha \frac{\partial p}{\partial z} - g' = 0, \tag{17}$$

since the vertical component of the pressure gradient is balanced by the virtual force of gravity. This is the so-called generalized equation of atmospheric statics. From equation (17) a generalized barometric formula can be derived. However the error introduced by substituting the ordinary gravity for the virtual gravity is much smaller than the errors unavoidable in atmospheric sounding. Therefore, for horizontal currents the basic equation of atmospheric statics given above remains highly accurate. Consequently, the barometric formula is valid not only for a quiescent atmosphere, but practically for any horizontal current whatsoever.

In the tangential equation, the tangential acceleration is equated with the tangential component of the pressure gradient per unit mass. As can be seen from this equation, the velocity of the air parcel changes only when it crosses the horizontal isobars in the course of its motion; the parcel is accelerated in the direction of low pressure and decelerated in the direction of high pressure.

A horizontal flow along the isobars, which is therefore perpendicular to the pressure gradient, is called gradient flow.

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* [Hereand elsewhere the Russian text has "force" instead of "acceleration". Translator's note].

Gradient flow obtains at points where $\frac{dv}{dt} = 0$, i.e., where the velocity is either constant or extremal (maximal or minimal).

When gradient flow is generally prevalent, the velocity vector is everywhere tangent to the isobars. However, in an ordinary flow pattern, gradient flow exists only at certain isolated points.

To illustrate the instantaneous velocity field, streamlines are drawn. A streamline is a curve to which the velocity is tangent at all of its points. With time the streamlines may change, and if they do not the flow is called steady.

One should not confuse the trajectories with the streamlines. A trajectory is a curve gradually traced by an individual particle. The velocity of the same particle is tangent to the trajectory at any point in different instants of time. A streamline is a curve which can be described by simultaneously considering all points and joining some of them so that the velocity is tangent to the streamline.

Since the trajectory and the streamline at a given point have a common tangent, they are osculating curves. In steady flow the trajectories and the streamlines coincide.

In the atmosphere, the pattern of the streamlines changes, so that the trajectories and the streamlines have different forms. However, the horizontal flows in a free atmosphere change but little with time. If the air motion is accelerated, the flow deviates slightly from the isobars and crosses them in the direction of low pressure. If, on the other hand, the motion is decelerated, the air flows across the isobars in the direction of high pressure. Usually, however, in a free atmosphere the streamlines are close to the isobars. This circumstance is of immense practical importance, allowing to draw an accurate enough picture of the currents in a free atmosphere by the pressure field alone. Thus, even without wind sounding we can get a sufficiently clear idea of the nature of the flow in a free atmosphere from the results of temperature plotted on a chart.

In the normal equation (14) the normal component of the acceleration is equated to the sum of the normal component of the pressure gradient per unit mass and the horizontal component of the Coriolis acceleration.

Thus, the horizontal velocity of flow depends on the latitude of the locality φ, the radius of curvature of the trajectory r and the normal component of the pressure gradient.

By convention we take northern latitude as positive and southern latitude as negative. We shall denote the two latitudes by φ and $-\varphi$ respectively.

We shall also denote by r and $-r$ the radii of curvature at the corresponding latitudes and by the expressions $-\frac{\partial p}{\partial n}$ and $+\frac{\partial p}{\partial n}$ the corresponding normal components of the pressure gradient. This definition of correspondence is obviously equivalent to the substitution of $-n$ in the southern hemisphere for n in the northern hemisphere. This means that a direction to the "left" of the flow in the northern hemisphere is equivalent to a "right" direction in the southern hemisphere.

The physical meaning of this change is clear. The horizontal component of the Coriolis force acts to the right of the current in the northern hemisphere and to the left of it in the southern hemisphere; the normal component of the pressure gradient force and the horizontal centripetal acceleration of the corresponding currents in both hemispheres are similarly oriented in relation to the deflecting force of the earth's rotation. Therefore, equation (14) is equally valid for the northern and southern hemispheres.

Using equation (14) we can classify the uniform horizontal flows in the atmosphere as follows:

1) Geostrophic flow: $j_n = \frac{v^2}{r} = 0.$

2) Inertial flow: $b_n = -\alpha \frac{\partial p}{\partial n} = 0.$

3) Cyclostrophic flow: $a_n = -2\omega \sin \varphi \cdot v = 0.$

In a corresponding analysis of the direction of the normal component of the pressure gradient and of the centripetal acceleration in relation to the Coriolis acceleration, we will single out the following four types of flow:

1) Baric flow: $b_n \downarrow \uparrow a_n$. The normal component of the pressure gradient force is opposite to the horizontal Coriolis force.

2) Antibaric flow: $b_n \uparrow \uparrow a_n$. The normal component of the pressure gradient force has the same direction as the horizontal Coriolis force.

Thus, in the northern hemisphere the flow will be baric if a pressure trough is located to the left of the velocity vector and antibaric, if the pressure trough is to the right. In the southern hemisphere the flow is baric if the pressure trough is to the right of the velocity vector and antibaric, if the pressure trough is to its left.

3) Cyclonic flow: $I \uparrow\uparrow a_n$. The centrifugal force has the same direction as the horizontal Coriolis force.

4) Anticyclonic flow: $I \uparrow\downarrow a_n$. The centrifugal force is directed opposite to the Coriolis force.

In the northern hemisphere a flow deflected to the left is cyclonic, and one deflected to the right is anticyclonic. In the southern hemisphere the cyclonic flow is deflected to the right and the anticyclonic flow to the left.

It is not difficult to see that the definition of the cyclonic and anticyclonic flows given above corresponds to the definition given in § 1.

We will now discuss in greater detail the various types of atmospheric flows.

§ 3. The Geostrophic Wind

A geostrophic wind is a flow of air along the circumference of a great circle. However, the projection of this circumference onto a horizontal plane is a straight line. Therefore, the horizontal curvature of the trajectory and the horizontal centripetal acceleration of the air parcel are equal to zero in case of a geostrophic wind. The normal component of the pressure gradient will then be balanced by the horizontal component of the Coriolis force

$$b_n + a_n = 0. \tag{1}$$

This is the dynamic condition of the geostrophic wind. Therefore, the geostrophic wind is a baric flow. This means that looking in the direction of the geostrophic wind, we shall observe a region of low pressure to the left of the trajectory in the northern hemisphere, and to the right, in the southern hemisphere.

This rule is known as the baric law of wind*.

Since the normal component of the pressure gradient is directed opposite to the horizontal component of the Coriolis force, and the latter is orthogonal to velocity, the pressure gradient is likewise orthogonal to the velocity. Hence, the geostrophic wind is a gradient flow - a gradient wind. It follows that $\frac{\partial p}{\partial s} = 0$; consequently, the tangential acceleration vanishes too ($j_s = 0$).

Thus, by a geostrophic wind we mean a horizontal uniform and rectilinear air current, developing under the action of forces of pressure without the participation of friction forces. The equations of the geostrophic wind in intrinsic coordinates have the following form (Figure 82):

$$\left.\begin{aligned} \frac{\partial p}{\partial s} &= 0; \\ \alpha \frac{\partial p}{\partial n} + 2\omega \sin\varphi \cdot v &= 0; \\ \alpha \frac{\partial p}{\partial z} + g &= 0, \end{aligned}\right\} \tag{2}$$

and in standard coordinates:

$$\left.\begin{aligned} -\alpha \frac{\partial p}{\partial x} + 2\omega \sin\varphi \cdot v_y &= 0, \\ -\alpha \frac{\partial p}{\partial y} - 2\omega \sin\varphi \cdot v_x &= 0, \\ -\alpha \frac{\partial p}{\partial z} - g &= 0. \end{aligned}\right\} \tag{3}$$

The third equation of the geostrophic wind, identical with the equation of atmospheric statics, though approximate, is confirmed in practice to a high degree of accuracy. Therefore, the barometric formulas are fully applicable to horizontal uniform and rectilinear atmospheric flows.

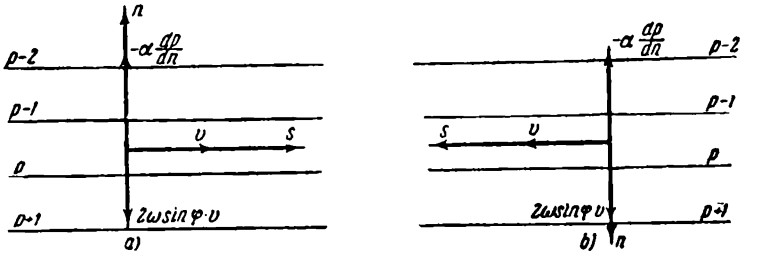

Figure 82. Geostrophic wind: a) northern hemisphere; b) southern hemisphere

We have settled the question of the direction of the geostrophic wind. Let us now find the absolute value of its velocity, designating it by v_g. From (2) we have

$$v_g = \frac{\alpha}{2\omega \sin\varphi} \cdot \frac{\partial p}{\partial n}. \tag{4}$$

The specific volume is not measured in meteorology, being eliminated by the use of the equation of state

$$\alpha = \frac{RT}{p},$$

hence

$$v_g = \frac{RT}{2p \cdot \omega \sin\varphi} \cdot \frac{\partial p}{\partial n}. \tag{5}$$

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* [Buys–Ballot's law. – Translator's note].

Thus the velocity of the geostrophic wind is

1) directly proportional to the pressure gradient $\frac{\partial p}{\partial n}$,

2) directly proportional to the air temperature T,

3) inversely proportional to air pressure p,

4) inversely proportional to the sine of the latitude φ.

Equation (5) is widely used by the weather services. In case of rectilinear isobars it enables to calculate the wind velocity at a given altitude and latitude from a given pressure gradient, providing the effects of friction and acceleration can be neglected.

In synoptic practice it is not customary to express the pressure gradient in the absolute unit (dyne/cm^3), since this is too large, but in other practical units such as millibars per 1° latitude (i. e., mb/111 km), or in millibars per 100 km.

Designating the pressure variation by $\frac{\Delta p}{\Delta \varphi}$ as measured in millibars per 1° latitude (corresponding to 111 km), measuring v_g in m/sec and assuming that $\omega = 7.29 \cdot 10^{-5}$ sec^{-1}, R = 287 m^2 sec^{-2} degree^{-1}, T = 273°, p = 1,000 mb, we obtain the following working formula usually mentioned in the literature, for the calculation of the velocity of the geostrophic wind:

$$v_g = \frac{4.84}{\sin \varphi} \cdot \frac{\Delta p}{\Delta \varphi}, \tag{6}$$

However, the distance between the isobars is more conveniently measured not in degrees of latitude, but in hundreds of kilometers, and thus formula (5) for the determination of the velocity of the geostrophic wind assumes the form.

$$v_g = \frac{K}{\sin \varphi} \cdot \frac{\Delta p}{\Delta n}, \tag{7}$$

where

$$K = \frac{R}{2\omega} \cdot \frac{T}{p} \cdot \frac{1}{10^5} = \frac{287}{2 \cdot 7.29} \cdot \frac{T}{p} \cong 20 \frac{T}{p}. \tag{8}$$

In Table 32 are given the values of K, calculated by formula (8).

It can be seen from this table, within what range the velocity of the geostrophic wind may vary at the given latitude and pressure gradient for different values of T and p. Thus, for example, other conditions being equal, the velocity of the geostrophic wind in summer exceeds that in winter by 20-30%. The velocity of the geostrophic wind increases especially at a reduced pressure. Thus, for example, at an altitude of 5 km, where the pressure is about 500 mb, the velocity of the geostrophic wind, under otherwise identical conditions, is twice the corresponding velocity at the earth's surface.

It can be seen from formula (7), that the velocity of the geostrophic wind diminishes with increasing latitude from infinity at the equator (which means that no geostrophic wind can exist on the equator) to a minimum at the pole, where $v_g = K \frac{\Delta p}{\Delta n}$. In Table 33 are given values of the geostrophic wind velocity for different latitudes and pressure gradients at T = 270°, p = 1,000 mb; this table was calculated by the equation

$$v_g = \frac{5.40}{\sin \varphi} \cdot \frac{\Delta p}{\Delta n}.$$

We see that at the same pressure gradient, the velocity of the geostrophic wind in tropical latitudes is 5 - 6 times greater than at the pole. Conversely, at the same geostrophic wind velocity the pressure gradient is several times lower in the low than in the high latitudes. So, for example, at a geostrophic wind velocity of 15 m/sec, the distance between isobars drawn at 5 mb intervals at a latitude of 20° is 500 km,and at a latitude of 60° it is only 200 km.

Table 32

Values of K = K(T, p)

p \ T	220	240	260	270	280	300	320° K
1000	4.40	4.80	5.20	5.40	5.60	6.00	6.40
900	4.88	5.53	5.77	5.99	6.22	6.66	
700	6.29	6.89	7.43	7.71	8.00	8.89	
500	8.80	9.60	10.40	10.80	11.20		
300	14.64	16.59	17.31	17.97			
200	22.00	24.00	26.00				
100	44.00	48.00					

Table 33

Geostrophic wind velocity at different latitudes and different pressure gradients

Δp mb/100 km	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	6.0
Δn km/5mb	1000	500	333	250	200	167	143	125	111	100	85
$\varphi = 10$	15.5	31.0	46.5	62.0	77.5	93.0	108.5	124.0	139.5	155.0	186.0
20	7.9	15.8	23.7	31.6	39.5	47.4	55.3	63.2	71.1	79.0	94.8
30	5.4	10.8	16.2	21.6	27.0	32.4	37.8	43.2	48.0	54.0	64.8
40	4.2	8.4	12.6	16.8	21.0	25.2	29.4	33.6	37.8	42.0	50.4
50	3.6	7.1	10.7	14.2	17.8	21.3	24.9	28.4	32.0	35.5	42.6
60	3.1	6.2	9.3	12.4	15.5	18.6	21.7	24.8	27.9	31.0	37.2
70	2.9	5.7	8.6	11.4	14.3	17.1	20.0	22.8	25.7	28.5	34.2
80	2.8	5.5	8.3	11.0	13.8	16.5	19.3	22.0	24.8	27.5	33.0
90	2.7	5.4	8.1	10.8	13.5	16.2	18.9	21.6	24.3	27.0	32.4

For the quick determination of the geostrophic wind velocity the so-called gradient rulers are employed. A gradient ruler is a nomogram of velocities of the geostrophic wind, drawn on transparent paper, tracing paper,cellophane or celluloid,on the same scale as the chart with which it is intended to be used. For rough determinations of the geostrophic wind velocity certain average values of the latitude φ and the coefficient K are given. However, it is better to use a precise ruler (Figure 83), which enables the calculation of the geostrophic wind velocity variation with latitude and temperature; the pressure can be taken to have the constant value 1,000 mb, since these rulers are used in practice together with synoptic charts, drawn for sea level. As for altitude charts, the Soviet weather services use pressure contour maps by which the geostrophic wind velocity is being determined, as we shall show below, differently and in a somewhat simpler way. By the way, the ruler just described can be used with pressure charts as well.

Let us assume that $R = 287\ m^2/sec^2$ degree, $\omega = 7.29 \times 10^{-5}\ sec^{-1}$, p = 1,000 mb.

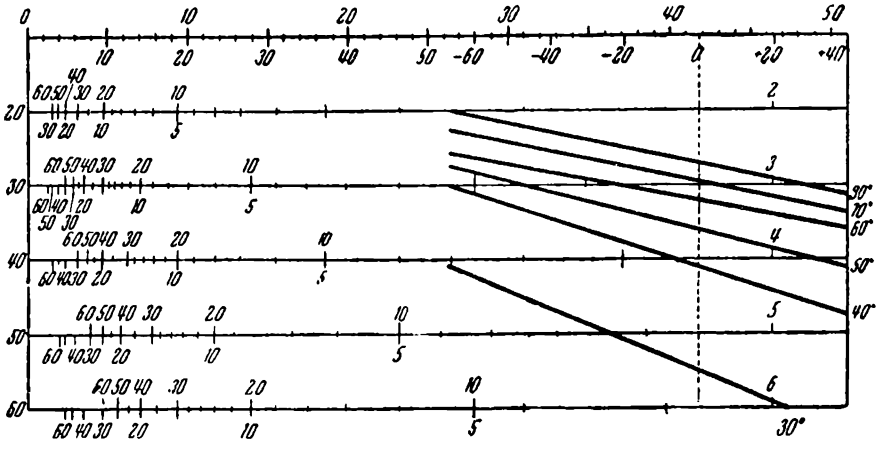

Figure 83. Universal gradient ruler for the determination of the geostrophic wind velocity

On synoptic charts the isobars are usually plotted in 5 mb intervals. Therefore, $\Delta p = 5$ mb. In that case, for the distance between the isobars Δn and the geostrophic wind velocity v_g the following relationship holds:

$$v_g \cdot \Delta n = a,$$

where

$$a = \frac{R}{2\omega} \cdot \frac{\Delta p}{p} \cdot \frac{T}{\sin \varphi} = \frac{287}{2 \cdot 7.29 \cdot 10^{-5}} \cdot \frac{5}{1000} \cdot \frac{T}{\sin \varphi}.$$

If v_g is measured in m/sec and Δn in hundreds of kilometers, we have

$$a \cong 0.1 \frac{T}{\sin \varphi}.$$

The temperatures occurring in the Soviet Union and in countries of the Eurasian continent, which the synoptician encounters in his practice, may change within quite a range—from 200° to 320° K, and the latitude—from 30° to 80°. Therefore a assumes widely different values, from a = 20 (T = 200; $\varphi = 90^\circ$) to a = 60 (T = 300°; $\varphi = 30^\circ$) and even more. Thus at a constant density of isobars the geostrophic wind velocity may change (depending on the value of a) by 200 - 300%, from which follows that one may not ignore the effect of T and φ on the velocity of the geostrophic wind.

The gradient ruler proposed by us consists of six different strips having two different scales each. The upper scale is calibrated to fit charts drawn on a scale of 1:10, 000, 000, and the lower one to charts on a scale of 1:20, 000, 000. Strips 2, 3, 4, 5, 6 correspond to the a values of 20, 30, 40, 50, 60. They are all constructed in exactly the same manner, namely, for each integral value of the velocity the corresponding distance between the isobars is determined and marked out on the scale, the graduations bearing the corresponding figures of the velocity v_g. The velocity values 5, 10, 20, 30, 40, 50, 60 are graduated by longer strokes and bear the explicitly printed corresponding figures. The other velocity values are marked by short strokes. The first strip is graduated in divisions which are equal to the path, relative to the chart scale, traveled in 6 hours by a particle moving with the given velocity, considered not to vary in magnitude and direction during this period. The marks and figures on this scale show the magnitude of the given geostrophic wind velocity in meters per second.

The preliminary determination of the number of the strip which should be used for work on the synoptic chart at given values of air temperature and latitude, is facilitated by a group of straight lines, corresponding to certain latitudes (30, 40, 50, 60, 70, 80, 90) and by an auxiliary temperature scale plotted on the right side of the gradient ruler. The technique of working with the ruler on synoptic charts is the following:

1) The number of the working strip is determined from the average values of T and φ in the considered region.

2) The left edge of the ruler is laid on the isobar, and thus the corresponding working strip is directed along the pressure gradient (or ascendent). The intersection of the working strip with nearest isobar, drawn at an interval of 5 mb, gives the geostrophic wind velocity, which is read off the upper scale if the synoptic chart scale is 1:10, 000, 000, and off the lower scale, if the scale of the chart is 1:20, 000, 000.

Examples:

In order to determine on synoptic charts the geostrophic wind velocity in the vicinity of Spitzbergen ($\varphi \approx 80^{\circ}$) at an average temperature $t = -50^{\circ}$ strip No 2 is used; for the Rostov region ($\varphi \approx 45^{\circ}$) at $t \approx 0^{\circ}C$, strip No 4 is used; for the central parts of Persia ($\varphi \approx 30^{\circ}$) at $t \approx 30^{\circ}C$ one works with strip No 6.

The graphic determination of the geostrophic wind velocity is more exact, the closer are the isobars to parallel straight lines. If the curvature of the isobars is appreciable or in case they sharply diverge, the result may be unsatisfactory, which is understandable, since in these cases the wind is not geostrophic.

Since in the case of a geostrophic wind the pressure gradient is orthogonal to the velocity, the work done by the pressure forces is zero, as is also the work of the deflective force of the earth's rotation.

We will also note, that at the equator the deflective force of the earth's rotation also vanishes; therefore at the equator the flow always proceeds along the pressure gradient, i.e., a geostrophic wind is impossible there. The pressure forces cause a mass transport in the direction of the gradient and before long the pressure is equalized.

We shall see later that in the lower atmospheric layers outside equatorial latitudes, the air flow is also not directed along the isobars, but under the effect of friction forces it is deflected away from the isobar towards the pressure low. Therefore in the lower layers the pressure tends to equalize. In this case work is done by the pressure forces, while in case of a geostrophic wind their work is zero. Flows very similar to the geostrophic can be found at extratropical latitudes, usually at altitudes exceeding 500 m, when there is no storm or other atmospheric disturbance. This should be understood in the sense, that at any altitude above 500 m the velocity of a more or less regular wind is close to the velocity of a geostrophic wind, as calculated for the given latitude air density and pressure gradient, and is directed along the isobar.

§ 4. The Slope of the Isobaric Surfaces. Geostrophic Wind on Pressure Charts

The velocity of a geostrophic wind is closely connected with the slope of the isobaric surface, the equation of which may be written in the following form

$$p(s, n, z) = \text{const.} \tag{1}$$

or in differential form

$$dp = \frac{\partial p}{\partial s} \cdot ds + \frac{\partial p}{\partial n} dn + \frac{\partial p}{\partial z} dz = 0. \qquad (2)$$

Designating the angle between the isobaric and the horizontal surfaces by θ_p we obtain from the equation of the geostrophic wind

$$\operatorname{tg} \theta_p = + \frac{dz}{dn} = - \frac{\frac{\partial p}{\partial n}}{\frac{\partial p}{\partial z}} = - \frac{2\omega \sin \varphi}{g} v_g.$$

The minus sign in (3) shows, that in the northern hemisphere ($\varphi > 0$) at $v > 0$, the isobaric surface forms an obtuse angle with the positive direction of the normal n, i.e., the isobaric surfaces rise to the right of the direction of flow (Figure 84, a); in the southern hemisphere ($\varphi < 0$) these surfaces rise to the left (Figure 84, b); this conclusion is in perfect agreement with the baric law of wind*.

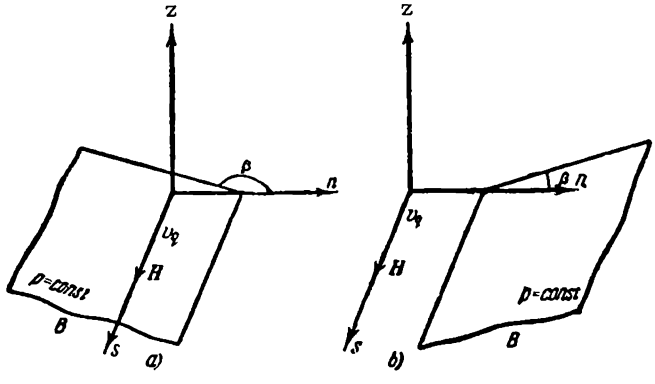

Figure 84. The slope of the isobaric surface: a) northern hemisphere, b) southern hemisphere

We should notice that the slope of the isobaric surface is entirely independent of the pressure p, the temperature T, or the density ρ, and is determined only by the ratio of the pressure derivatives with respect to n and to z, or by the latitude φ and the geostrophic wind velocity v_g.

In Appendix 21 are given the values of the slope of the isobaric surface for different latitudes and velocities of the geostrophic wind, calculated according to equation (3).

This table shows, that the slope of the isobaric surfaces is extremely small. Even at high values of the geostrophic wind velocity ($\approx$ 50 m/sec) at high latitudes the slope angle of the isobaric surfaces does not exceed 2' - 3'. Under ordinary conditions this angle rarely exceeds 0.5', becoming zero at the equator.

At the latitude of Moscow ($\varphi \approx 56^\circ$), and at a wind velocity of 10 m/sec the isobaric surface rises 12 m above ground level over a distance of 100 km from the isobar.

The formula determining the angle the isobaric surface makes with the horizon can be used (after a certain transformation) for the determination of the geostrophic wind velocity on contour maps of the isobaric surfaces . A contour

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* [Buys-Ballot's law. - Tr. note].

map of the isobaric surface is obtained by projecting the intersection lines of the given isobaric surface with chosen standard isopotential surfaces on a horizontal plane (Figure 85). These lines are called the isohypses of the dynamic altitudes of the isobaric surface, or simply isohypses. Replacing dz in equation (3) by $\frac{d\Phi}{g}$, we obtain

$$\frac{d\Phi}{dn} = -2\omega \sin\varphi \cdot v_g.$$

Passing to finite differences we obtain the following formula for the determination of the geostrophic wind velocity on pressure contour maps

$$v_g = \frac{1}{2\omega \sin\varphi} \cdot \frac{\Delta\Phi}{\Delta n}. \qquad (4)$$

The direction of the geostrophic wind will obviously be determined in the following manner: if one were to stand looking in the direction of the geostrophic wind, the geopotential elevation would be to his right in the northern hemisphere (Figure 86) and to his left—in the southern.

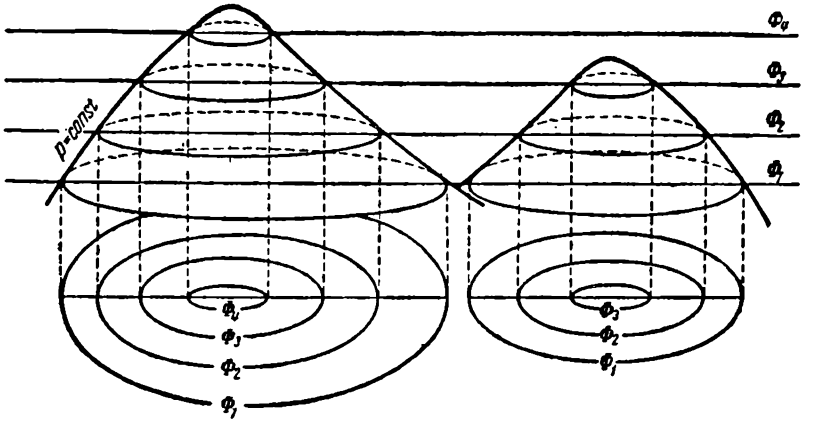

Figure 85. Spatial configuration of an isobaric surface, illustrated by a contour map

Formula (4) shows that the velocity of the geostrophic wind on contour maps does not depend at all on the air density, but is determined solely by the closeness of isohypses and by the latitude of the locality. This is one of the main advantages of pressure contour maps as compared with altitude isobar maps. As already shown above, the geostrophic wind velocity depends not only on the closeness of the isobars and on the altitude, but to a considerable degree also on the air density.

On contour maps the isohypses of the main isobaric surfaces (900, 700, 500, 300 etc) are usually drawn at a distance of 40 berk from each other, thus, in the formula (4) $\Delta\Phi$ = 40 berk. Measuring v_g in meters per second and Δn in hundreds of kilometers, we obtain the following working formula for the determination of the geostrophic wind velocity on pressure contour maps:

$$v_g = \frac{274}{\sin\varphi} \cdot \frac{1}{\Delta n}. \qquad (5)$$

In Table 34 are given values of geostrophic wind velocity on pressure contour maps, computed by formula (5) for various latitudes and various densities of the isohypses.

Formula (5) shows, that the determination of geostrophic wind velocity on pressure contour maps from the distance between isohypses is equivalent to its determination by the distance between isobars on the synoptic chart at a temperature of 274°K, or approximately 0°C.

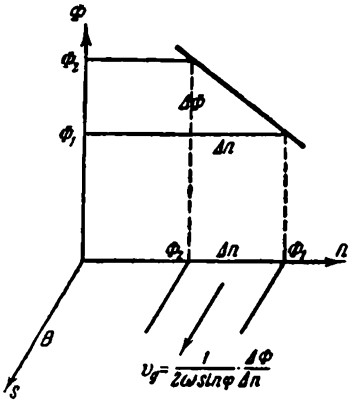

Figure 86. The geostrophic wind velocity is determined on pressure contour maps by the position of the isohypses

Since a dotted line is marked on the gradient ruler, corresponding to $0^{\circ}C$, the geostrophic wind is determined on pressure contour maps by the use of the gradient ruler in exactly the same manner as on synoptic charts drawn for sea level, with the only difference, that in the determination of the number of the working strip to be used, we look for the intersection point of the fixed dotted line with the line corresponding to the average value of φ.

Table 34

The velocity of geostrophic wind on pressure contour maps

φ \ Δn	100	300	500	700	1000 km
10°	150.0	52.7	31.6	22.6	15.8 m/sec
20°	80.5	26.6	16.1	11.5	8.1 »
30°	55.0	18.3	11.0	7.9	5.5 »
40°	42.8	14.3	8.6	6.1	4.3 »
50°	35.9	12.0	7.2	5.1	3.6 »
60°	31.8	10.6	6.4	4.5	3.2 »
70°	29.3	9.8	5.9	4.2	2.9 »
80°	27.8	9.2	5.6	3.9	2.7 »
90°	27.4	9.1	5.5	3.9	2.7 »

§ 5. Inertial Air Flow

If the pressure gradient is zero, the horizontal flow of air is inertial and the centripetal acceleration equals the horizontal component of the Coriolis acceleration (Figure 87); hence the inertial flow equation assumes the form

$$j_n = a_n \tag{1}$$

or

$$\frac{v^2}{r} = -2\omega \sin\varphi\, v. \tag{2}$$

Equation (2) has two roots:

$$1)\ v = 0 \text{ (quiescence)} \tag{3}$$

and

$$2)\ v_i = 2r\omega \sin \varphi, \tag{4}$$

where v_i is the velocity of the inertial motion.

Since the centripetal acceleration has the same direction as the Coriolis force, inertial flow is anticyclonic (Figure 87). The radius of curvature of the trajectory is determined by the formula

$$r = \frac{v_i}{2\omega \sin \varphi}. \tag{5}$$

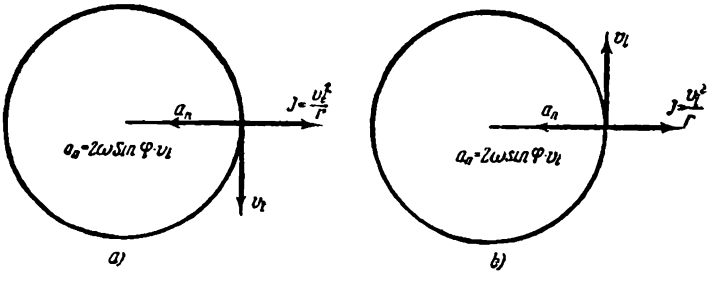

Figure 87. Inertial motion of a point: a) northern hemisphere, b) southern hemisphere

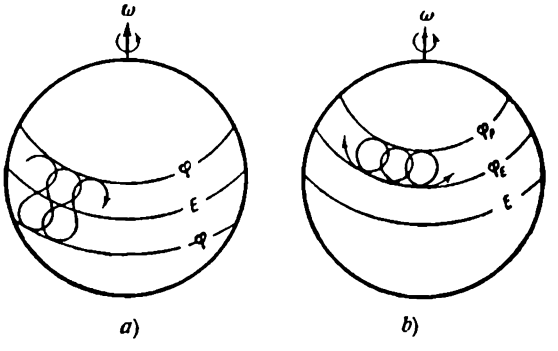

Figure 88. The orbit of a particle in inertial motion, relative to the rotating earth: a) the orbit crosses the equator; b) the orbit does not cross the equator

If we disregard variations in the latitude φ along the orbit and recall the constancy of the velocity of inertial motion, the radius of curvature r is seen to be constant, and the orbit is a circle, known in meteorology as the inertial circle. It can be seen from (5), that the inertial circle is larger, the higher the velocity and the smaller the latitude.

In Appendix 20 are given values of the radius of the inertial circle, or, to be exact, the values of the radius of curvature of the orbit described by an air particle in inertial flow.

If changes of the latitude φ, are taken into account, the orbit will differ from a circle, though only very slightly. An exception is the equatorial zone, since on crossing the equator the sign of the curvature of the inertial trajectory changes. The inertial orbit which intersects the equator is enclosed between the circles of

latitude φ and $-\varphi$, , as shown in Figure 88, a. These latitudes are determined by the velocity of the inertial flow v_i. The inertial trajectory which does not intersect the equator is enclosed between two latitudes φ_p and φ_g (Figure 88, b), which are also determined by the inertial flow velocity v_i. In this case the orbit forms loops, since at high latitudes its curvature is greater than at low ones.

§ 6. Cyclostrophic Wind

In the discussion of currents in the northern or in the southern hemispheres the direction of the earth's rotation and the ensuing Coriolis force should be taken into account. On the equator, on the other hand, the angular velocity of rotation around the local zenith and the horizontal component of the Coriolis force are equal to zero. Therefore, the normal equation of motion will assume the form

$$j_n = b_n. \tag{1}$$

Equation (1) shows, that if the normal component of the pressure gradient is other than zero, there exists a centripetal acceleration. Such flow is called cyclostrophic (Figure 89).

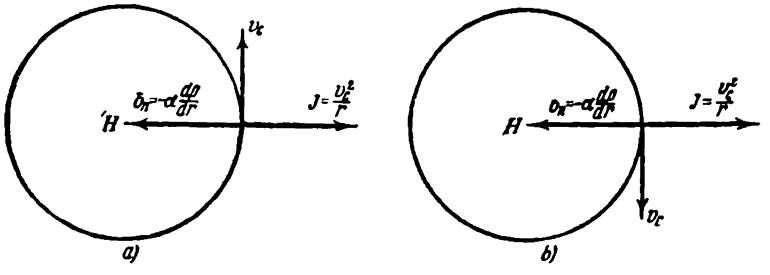

Figure 89. Cyclostrophic flow: a) cyclonic; b) anticyclonic

Introducing the velocity of the cyclostrophic wind v_c, equation (1) will be transformed into

$$\frac{v_c^2}{r} = a\frac{\partial p}{\partial r}, \tag{2}$$

hence

$$v_c = \pm\sqrt{ra\frac{\partial p}{\partial r}}. \tag{3}$$

Formula (3) determines the cyclostrophic wind velocity, necessary for counterbalancing the centrifugal force and the pressure gradient of the steady circular flow.

Table 35

Velocity of the cyclostrophic wind

$\frac{\partial p}{\partial r}$ mb/100 km \ r	10	20	30	50	100	200 km
1	2.5	3.5	4.3	17.5	24.9	35.0
2	3.5	4.9	6.1	24.5	35.0	49.0
3	4.3	6.1	7.5	30.5	43.0	61.0
4	5.6	7.8	9.6	39.0	56.0	78.0
5	7.8	11.0	13.5	55.0	78.5	110.0

As can be seen from (3), a cyclostrophic wind is possible only when $\frac{\partial p}{\partial r} > 0$, i.e., it can occur only in a region of low pressure, although the direction of flow may be either clockwise or counterclockwise.

In Table 35 are given values of cyclostrophic wind velocities at $\rho = 1.276 \times 10^{-3}$ for different r and $\frac{\partial p}{\partial r}$.

§ 7. Arbitrary Horizontal Flow

In the foregoing sections we have discussed simple cases of horizontal air flow. Let us now turn to the discussion of the general case of an arbitrary horizontal current.

The equation of horizontal air flow in the absence of friction forces has the form

$$j_s = \frac{dv}{dt} = b_s \tag{1}$$

$$j_n = b_n + a_n. \tag{2}$$

Let the current be inclined to the isobar at an angle β (Figure 90). This angle is equal to the angle formed between the horizontal pressure gradient and the normal to the particle trajectory. We will consider the sense of β positive, if looking from the local zenith it is read counterclockwise from the pressure gradient b to the unit normal n.

Then

$$b_n = b \cdot \cos\beta;$$
$$b_s = b \sin\beta.$$

Since, according to (1) $b_s = j_s = \frac{dv}{dt} = \dot{v}$, the normal component of the pressure gradient can be expressed by the horizontal pressure gradient and the rate of change of the velocity

$$b_n^2 = b^2 - b_s^2 = b^2 - \dot{v}^2, \tag{3}$$

At any point where gradient flow obtains, β $b_s = \dot{v} = 0$. At these points the wind is directed along an isobar with a constant or extremal velocity. In general, however, the current deviates from the isobar.

If the flow is accelerated, then $\frac{dv}{dt} > 0$ and

$$b_s = b \sin\beta > 0,$$

that is, the flow deviates from the isobar towards the pressure low (Figure 90, a). If the flow is decelerated, $\frac{dv}{dt} > 0$ and

$$b_s = b \sin\beta < 0$$

i.e., the flow deviates from the isobar in the direction of the pressure high (Figure 90, b). At any instant of time and for any flow the forces [per unit mass] a_n, b_n and c_n are in equilibrium. Types of flow possible in the northern hemisphere and which are determined by different ratios between these forces, are shown in Figure 91.

Let us regard the wind velocity and the latitude as given. Then the Coriolis force will be simply determined. Depending on the direction and magnitude of the normal component of the pressure gradient, the atmospheric flow may assume the following forms:

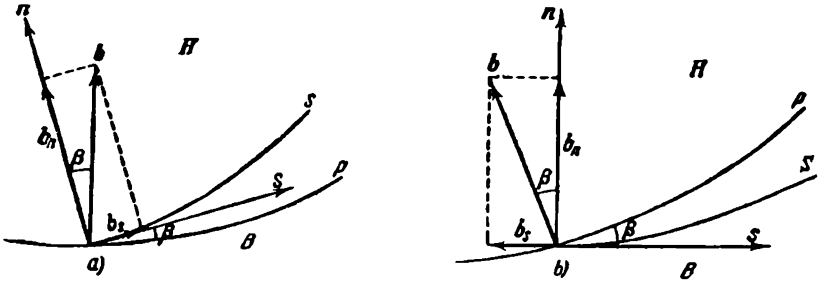

Figure 90. An arbitrary horizontal current deviates from the isobar: a) accelerated current is deflected towards the pressure low; b) decelerated current is deflected towards the pressure high

If the normal component of the pressure gradient is antiparallel to the Coriolis force, three cases are distinguished:

1) Cyclonic flow, $I_n \uparrow\uparrow a_n$, in which the absolute value of the normal component of the pressure gradient is greater than the Coriolis force ($|b_n| > |a_n|$). In this case the flow is cyclonic and a pressure trough exists (Figure 91, a).

2) Anticyclonic flow $I_n \uparrow\downarrow a_n$, where the absolute value of the normal component of the pressure gradient is smaller than the Coriolis force ($|b_n| < |a_n|$). This case corresponds to an anticyclone, or a pressure high (Figure 91, b).

3) A rectilinear current $I_n = 0$, in which the pressure gradient balances the Coriolis force (Figure 91, c).

If the flow is antibaric, i.e., if the normal component of the pressure gradient parallels the Coriolis force, then theoretically only an anticyclonic flow is possible. The baric law of wind* is clearly not applicable to the antibaric flow, since it is only an empirical rule and not a strict law (Figure 91, e).

Antibaric flow is possible only when the centrifugal force is strong enough to counterbalance the resultant of the normal component of the pressure gradient and the horizontal component of the Coriolis force:

$$\frac{v^2}{r} = b_n + 2\omega \sin \varphi \cdot v.$$

Conditions favorable for the appearance of antibaric flow are high wind velocities, small radii of curvature of the trajectory and low latitudes. Let us point out, that antibaric flow has hitherto not been encountered in the pressure patterns traced on synoptic charts. It is possible that such flows may take place in tornadoes. Sandstorms with antibaric flow have been observed.

Finally, when the normal component of the pressure gradient vanishes the following are possible:

1) inertial flow with an anticyclonic trajectory curvature, when $|a_n| = |I_n|$ (Figure 91, h);

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* [Buys-Ballot's law. - Translator's note].

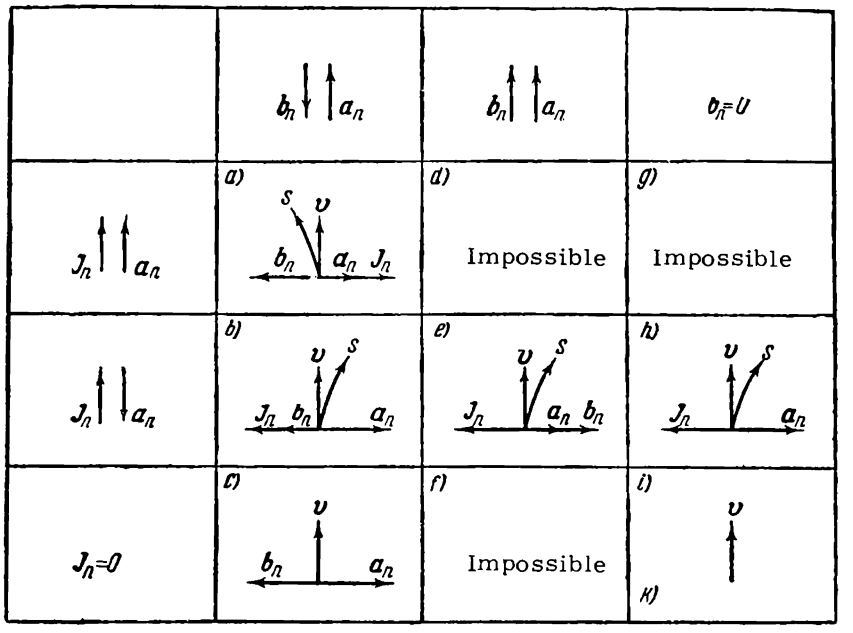

Figure 91. Different types of horizontal flow in the northern hemisphere: a) cyclonic baric flow (cyclone, or pressure trough); b) anticyclonic baric flow (anticyclone, or a pressure elevation); c) geostrophic wind; d) anticyclonic antibaric flow (anomalous cyclone); e) inertial flow; f) relative quiescence; g) rectilinear flow on the equator

2) relative quiescence, when $a_n = I_n = v = 0$ and $\varphi \neq 0$;

3) rectilinear flow on the equator, when (Figure 91, i)

$$a_n = I_n = \varphi = 0 \text{ and } v \neq 0.$$

We will now examine in greater detail the problem of the magnitude of the horizontal current velocity. We start as before, from the normal equation of flow (2), which we solve for b_n:

$$b_n = 2\omega v \sin\varphi \pm \frac{v^2}{r}. \tag{4}$$

The first term on the right side is often called the geostrophic component of the pressure gradient, and the second term the cyclostrophic component. It follows from (4), that at the same wind velocity the pressure gradient should be greater in case of a cyclonic trajectory curvature than at an anticyclonic curvature, since in the former case the cyclostrophic component is added to the geostrophic one and in the latter case it is deducted from it. In case of a geostrophic wind the pressure gradient has a certain intermediate value. Thus, in order to obtain the same wind velocity the isobars should be most dense at a cyclonic curvature and least dense at an anticyclonic one. The same results are obtained from the analysis of the diagrams in Figure 91.

Regarding b_n as a function of the velocity v, the latitude φ, and the radius of curvature r, it can be shown that b_n, generally speaking, possesses a maximum.

Since there exists a direct relationship between b_n and v, which is determined by (4), the maximal wind velocity corresponds to the maximal value of the pressure gradient. Differentiating b_n with respect to v and equating the result to zero we get the following equation for the determination of v max.:

$$\frac{\partial b_n}{\partial v} = 2\omega \sin\varphi \pm 2\frac{v}{r} = 0,$$

hence

$$v_{max} = r\omega \sin\varphi = \frac{v_i}{2}. \qquad (5)$$

In Table 36 are given maximal values of the gradient wind velocity for different values of φ and different trajectory radii r.

It follows from Table 36, that the gradient wind in anticyclones cannot reach high values. These velocities are particularly low at low latitudes. Formula (5) shows that anticyclonic flow on the equator is impossible. Only on the periphery of anticyclones are considerable wind velocities realizable, but only at high latitudes. The obtained results agree well with the widely known fact of small wind velocities in anticyclones, especially near the center, where practically dead calm prevails.

Table 36

Maximum gradient wind velocity in an anticyclone

r \ φ	10	20	30	40	50	60	70	80	90
100	1.3	2.5	3.6	4.7	5.6	6.3	6.8	7.2	7.3
200	2.5	5.0	7.3	9.3	11.1	12.6	13.7	14.3	14.6
300	3.8	7.5	10.9	14.0	16.7	18.9	20.5	21.5	21.9
500	6.3	12.4	18.2	23.3	27.8	31.5	34.1	35.8	36.4
1000	12.7	24.9	36.4	46.7	55.7	63.0	68.3	71.6	72.9

Substituting the maximum velocity (5) in formula (4) which determines the magnitude of the pressure gradient, we get a formula for the maximal value of the normal component of this gradient

$$b_{n\,max} = r\omega^2 \sin^2\varphi; \qquad (6)$$

it is evident that the pressure gradient is bounded only in case of an anticyclonic curvature of the isobars.

Table 37, giving the maximal value of the pressure gradient in anticyclones, was calculated for $\rho = 1.276 \times 10^{-3}$ by the formula $\left|\frac{\partial p}{\partial r}\right|_{max} = 6.75 \times 10^{-3} r \sin^2\varphi$, where r is measured in kilometers and $\left|\frac{\partial p}{\partial r}\right|$ in millibars per 100 km.

Formula (6) and Table 37 show, that relatively high pressure gradients may occur only in the periphery of the anticyclone. At a distance of 300 km from the center, even at the pole, the pressure gradient cannot exceed 2 mb/100 km. The magnitude of the maximal pressure gradient decreases very rapidly with the latitude. Thus, for example, at a latitude of 10°, even on the periphery of the anticyclone, isobars drawn at 1 mb intervals are separated from each other not less than 500 km. We should also note, that the maximal value of the pressure gradient also depends on the air density ρ. Therefore, the isobars in winter are generally denser than in summer and on pressure charts they are more sparse over rising than over low terrain.

Table 37

Maximum pressure gradients in anticyclones

r \ φ	10	20	30	40	50	60	70	80	90
100	0.02	0.07	0.17	0.28	0.39	0.51	0.59	0.65	0.68 mb/100 m
200	0.04	0.16	0.34	0.56	0.79	1.01	1.19	1.31	1.35 mb/100 m
300	0.06	0.24	0.51	0.84	1.18	1.52	1.78	1.96	2.03 mb/100 m
500	0.10	0.40	0.85	1.40	1.97	2.52	2.97	3.27	3.38 mb/100 m
1000	0.20	0.79	1.69	2.80	3.94	5.05	5.94	6.54	6.75 mb/100 m

In cyclones neither the magnitude of the pressure gradient nor the gradient wind velocity are bounded. Thus, there is no lower limit to the pressure in cyclones, while the intensity of anticyclones is limited.

For practical calculations of wind velocity in the absence of friction forces it is convenient to use the velocities introduced above: the geostrophic velocity v_g, the inertial velocity v_i and the cyclostrophic velocity v_c; regarding these velocities as independent parameters:

$$v_g = \frac{b_n}{2\omega \sin \varphi}; \qquad v_i = -2r\omega \sin \varphi; \qquad v_c^2 = rb_n. \tag{7}$$

The parameters v_g, v_i and v_c need not conform with the special conditions fulfilled by geostrophic, inertial and cyclostrophic currents. Thus, for example, v_i which cannot be negative, being an inertial velocity, may nevertheless assume negative values, describing an arbitrary cyclonic flow as a parameter.

The three parameters v_g, v_i and v_c are not really independent, but connected by the relationship

$$-v_g v_i = v_c^2. \tag{8}$$

Using two of the three parameters, one can put the equation of gradient wind in one of the following forms. Multiplying (4) by r and using (7), we obtain

$$v^2 - v_i v - v_c^2 = 0. \tag{9}$$

Eliminating v_c^2 by the use of (8), we get:

$$v^2 - v_i (v - v_g) = 0. \tag{10}$$

Dividing (9) by v_c^2 we obtain :

$$\frac{v^2}{v_c^2} + \frac{v}{v_g} = 1. \tag{11}$$

From equations (9), (10) and (11) we have the essential conditions for the existence of the following winds:

1) for the inertial wind: $v = v_i$; $v_c^2 = 0$, i.e., $b_n = 0$;

2) for the geostrophic wind: $v = v_g$; $v_c^2 = \infty$, i.e., $r = \infty$ *;

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* [The Russian text has $R = \infty$, an obvious misprint. - Translator's note].

3) for the cyclostrophic wind: $v = v_c$; $v_i = 0$, i.e., $\omega \sin \varphi = 0$.

In general, however, v_g, v_i, and v_c differ from zero and one can express by these magnitudes the velocity of an arbitrary horizontal current. Each of the equations (9), (10) and (11) includes an unknown function v and known parameters. For the calculation of v it is most convenient to use specially constructed nomograms.

Formula (10), which can be written in the form

$$v_g = v - \frac{v^2}{v_i}, \tag{12}$$

shows, that at the same isobar density and at otherwise identical conditions the velocity of an arbitrary horizontal current at an anticyclonic trajectory curvature ($v_i > 0$) is larger and at a cyclonic trajectory curvature ($v_i < 0$) is smaller than the corresponding geostrophic wind velocity.

This assertion apparently contradicts the fact that comparatively weak winds prevail in the anticyclone. The reason for this, as was previously indicated, is the absence of high pressure gradients in anticyclonic regions, which precludes the possibility of strong winds except on the periphery of the anticyclone.

Table 38 gives the magnitude of the geostrophic wind velocity v_g which corresponds to the wind velocity in case of a cyclonic trajectory curvature, at different velocities of the inertial flow. This table was calculated by formula (12).

Table 38

Geostrophic wind velocities, corresponding to the wind velocity for cyclonic trajectory curvature

v_i \ v	1	2	3	5	10	20	30	50 m/sec
2	1.5	4.0	7.5	17.5	60.0	220.0	—	—
10	1.1	2.4	3.9	7.5	20.0	60.0	120.0	
20	1.1	2.2	3.5	6.3	15.0	40.0	75.0	—
40	1.0	2.1	3.2	5.6	12.5	30.0	52.5	—
100	1.0	2.0	3.1	5.3	11.0	24.0	39.0	75.0m/sec

As can be seen from Table 38, the geostrophic velocity is very close to the actual wind in case of a cyclonic trajectory curvature, but under conditions corresponding to higher values of $v_i = 2r\omega \sin \varphi$, i.e., at high latitudes and at large radii of curvature r. In this case the geostrophic velocity may be taken as an approximation to the cyclonic wind velocity; only at very high velocities should we take into account the trajectory curvature. However, under conditions corresponding to low values of $v_i = 2r\omega \sin \varphi$, i.e., for appreciable curvatures (small r) and at low latitudes the geostrophic wind differs greatly from the cyclonic, and therefore cannot serve even as a rough first approximation in the estimation of the latter.

Table 39, calculated by formula (12), gives the magnitude of the geostrophic velocity v_g, corresponding to the wind velocity v at an anticyclonic curvature and for different values of the inertial flow velocity v_i.

It is evident from Table 39, that if the actual wind velocity v at an anticyclonic trajectory curvature is far from the limiting value $v_{max} = v_i/2$, i.e.,

Table 39

Geostrophic velocities corresponding to wind velocities for anticyclonic trajectory curvature

v_i \ v	1	2	3	5	10	20	30	50 m/sec
4	0.7	1.0	—	—	—	—	—	—
10	0.9	1.6	2.1	2.5	—	—	—	—
20	0.95	1.8	2.5	3.7	5.0	—	—	—
40	0.97	1.90	2.8	4.4	7.5	10.0	—	—
100	0.99	1.96	2.91	4.75	9.00	16.00	21.00	25.00
200	0.995	1.98	2.95	4.87	9.50	18.00	25.50	37.50

if $v \ll v_i/2$, it can be replaced, in a first approximation, by the geostrophic wind velocity, the substitution being more appropriate, the smaller v is. However, if v is close to $v_{max} = v_i/2$, the geostrophic wind velocity is almost half the wind velocity at an anticyclonic trajectory curvature, and in the extreme case $v = v_{max}$, we get:

$$v_g = \frac{1}{2} v_{max} = \frac{v_i}{4}. \qquad (13)$$

Thus, at high wind velocities (close to the maximum velocity) for anticyclonic trajectory curvature it is necessary to take the curvature into account. However, the wind velocities actually observed in the anticyclone fall considerably short of the maximum. Hence, at an anticyclonic curvature one can practically always use the geostrophic wind. In cyclones, on the other hand, the actual wind velocities observed are close to and sometimes even higher than the maximal anticyclonic velocities; therefore, one ought to apply the geostrophic relationships to cyclones with great caution. Geostrophic relationships are obviously entirely unapplicable to tropical cyclones, characterized by their small dimensions and encountered at low latitudes, for which $v \gg \frac{1}{2} v_i$, and they are likewise unapplicable to extratropical cyclones of small dimensions.

§ 8. Trajectories, Streamlines and Isobars

In dealing with an arbitrary horizontal air flow, friction forces being absent, one should distinguish between the following three families of curves: 1) trajectories of air parcels, 2) streamlines, and 3) isobars. In an arbitrary horizontal flow these curves generally differ, though in case of a gradient wind they do coincide. We will use the term gradient wind for a horizontal, uniform and steady air flow. This flow may proceed either in rectilinear trajectories (then the wind is geostrophic), or in circular ones. Since for a gradient wind $v = \text{const.}$ and $\frac{dv}{dt} = 0$, the tangential component of the pressure gradient also vanishes ($b_s = 0$). Thus the gradient wind blows along the isobar, which coincides with the parcel trajectory, and since the gradient wind is a steady flow, the streamline also coincides with these two curves.

Thus, in case of a gradient wind, the trajectories, streamlines and isobars coincide. We can therefore use all the formulas of the preceding section, derived from the normal equation of flow, substituting the isobar curvature for the trajectory curvature.

Strictly speaking, however, a gradient wind never exists in the atmosphere. This we are taught by our daily synoptical practice, showing that the pressure field incessantly changes: with it change also the streamlines and trajectories.

Therefore practical application of the said formulas to an arbitrary current requires that the trajectory curvature be known.

But obviously one cannot directly observe the trajectory of an air parcel. Admittedly there are in existence methods for approximate determination of these trajectories, which ultimately reduce to the observation of the free horizontal flight of an aerostat, or the extrapolation of the subsequent trajectory from one synoptic chart to another; however, regular prolonged horizontal flights of free aerostats are practically unfeasible and plotting a trajectory on contiguous synoptic charts often leads to unsatisfactory results due to the incompleteness of data.

The trajectory curvature can be obtained more readily and simply from the curvature of the streamlines. The horizontal curvature of the streamlines can be determined from altitude charts. The trajectory and streamline are osculating curves, differing however in their curvature.

The relationship between the trajectory curvature K and the streamline curvature K_L at an arbitrary point M can be obtained in the following way:

Let MS be the trajectory of an air particle located initially at the point M (Figure 92). The wind velocity vector at the initial moment is tangent to the curve MS at the point M. Let us further assume, that ML is a streamline passing through the point M. The wind velocity is also tangent to the streamline at the point M.

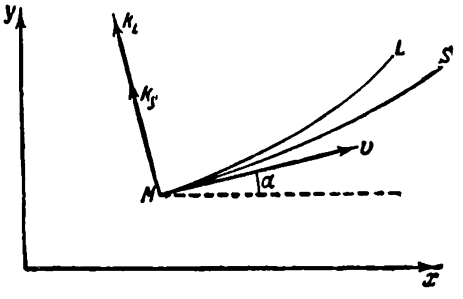

Figure 92. Comparison of the trajectory curvature with the streamline curvature

Let us designate by α the angle of the wind's direction. We shall read it counterclockwise from the x-axis of the standard coordinate system. Obviously, it is a function of the arc length L of the streamline: $\alpha=\alpha(L)$. If the flow is steady, α is independent of the time, since the streamline in this case does not change in time, but if the flow is unsteady the wind direction is a function of the time t:

$$\alpha=\alpha(L,\ t).$$

As the particle moves in its trajectory the direction of the wind changes by $\frac{d\alpha}{dt}$ per unit time. However $\frac{d\alpha}{dt}$ is the velocity of rotation of the vector of the wind velocity, and equals $\frac{v}{r}$, where r is the trajectory radius of curvature and v is the wind velocity. Therefore,

$$\frac{d\alpha}{dt}=\frac{v}{r}=K\cdot v. \tag{1}$$

On the other hand, $\frac{d\alpha}{dt}$ is the individual derivative of the angle; hence

$$\frac{d\alpha}{dt}=\frac{\partial\alpha}{\partial t}+v\cdot\frac{\partial\alpha}{\partial L}. \tag{2}$$

But $\frac{\partial\alpha}{\partial L}$ is the streamline curvature $K_L = 1/r_L$; therefore

$$\frac{d\alpha}{dt}=\frac{\partial\alpha}{\partial t}+\frac{v}{R_L}=\frac{\partial\alpha}{\partial t}+K_L\cdot v, \tag{3}$$

where r_L is the radius of curvature of the streamline.

Eliminating $\frac{d\alpha}{dt}$ between (1) and (3) we obtain the following relationship between the trajectory curvature K and the streamline curvature K_L:

$$K=K_L+\frac{1}{v}\frac{\partial\alpha}{\partial t}. \tag{4}$$

The derivative $\frac{\partial\alpha}{\partial t}$ represents a local change in the wind direction, observed from a fixed point. The trajectory curvature coincides with the streamline curvature only upon the condition that the wind direction at the given point does not change in time, i.e., when $\frac{\partial\alpha}{\partial t} = 0$. In this case $K = K_L$ and $r = r_L$. In stationary flow, particularly, $\frac{\partial\alpha}{\partial t} = 0$ at any time instant.

However, the direction of wind at any given point usually does change. In that case the streamline curvature and the trajectory curvature are different. If the wind rotates to the left $\left(\frac{\partial\alpha}{\partial t}>0\right)$ the trajectory curvature is greater than the streamline curvature $(K>K_L)$ and if it rotates to the right $\left(\frac{\partial\alpha}{\partial t}<0\right)$ the trajectory curvature is smaller than the streamline curvature $(K<K_L)$, where the curvature is taken with its proper algebraic sign. The two curvatures may even be of a different sign.

Therefore, one should use with caution the terms "cyclonic", "geostrophic" and "anticyclonic", keeping in mind that they refer, strictly speaking, to trajectories and not to streamlines or isobars.

Table 40

Values of $\Delta K=\frac{1}{v}\frac{\partial\alpha}{\partial t}$.

$\frac{\partial\alpha}{\partial t}$ \ v	10	20	30	50	100 km/h
1	0.0017	0.0008	0.0006	0.0003	0.0002 km^{-1}
2	0.0034	0.0017	0.0011	0.0007	0.0003 km^{-1}
3	0.0051	0.0026	0.0017	0.0010	0.0005 km^{-1}
5	0.0085	0.0043	0.0028	0.0017	0.0009 km^{-1}
10	0.0170	0.0085	0.0057	0.0034	0.0017 km^{-1}
20	0.0341	0.0170	0.0114	0.0068	0.0034 km^{-1}
30	0.0511	0.0256	0.0170	0.0102	0.0051 km^{-1}
50	0.0852	0.0426	0.0284	0.0170	0.0035 km^{-1}

In Table 40 are given the values of $\Delta K=\frac{1}{v}\cdot\frac{\partial\alpha}{\partial t}$ for various current velocities v and different rates of change of the wind direction $\frac{\partial\alpha}{\partial t}$ °/h. Using

this table and formula (4), we will illustrate by an example how real the difference is between the trajectory curvature and streamline curvature.

Let us assume that the streamlines are straight, and the wind velocity is 30 km/h ($\approx$ 9m/sec). In the absence of wind rotation the flow becomes geostrophic in due course and the trajectories coincide with the streamlines. In case of a right-handed wind rotation $\left(\frac{\partial \alpha}{\partial t} < 0\right)$ at the same straight streamlines the trajectories bend anticyclonically and if the wind rate of rotation equals 5°/h, then $\Delta K = 0.0028$ and the trajectory radius of curvature is 357 km. Under the same conditions, but with a cyclonic streamline curvature of 357 km radius, the trajectory is a straight line.

Since in synoptic practice one deals with an instantaneous streamline, it is useful to introduce the streamline curvature into the normal equation of flow. Substituting (4) [of this section] in equation (4) § 6 we obtain :

$$K_L v^2 + \left(2\omega \sin\varphi + \frac{\partial \alpha}{\partial t}\right) \cdot v - b_n = 0, \qquad (5)$$

where b_n remains the same as in equation (4) § 6, i. e., the component of the pressure gradient normal to the trajectory, which at the given point coincides with the normal to the streamline.

Let us now discuss the relationships between the isobar curvature and the trajectory curvature in several cases.

1. Cyclonic center. Let us consider the flow of air around a cyclonic center, advancing with velocity c. Let us direct the x–axis parallel to the velocity of the center. Let us designate by $\frac{\delta \alpha}{\delta t}$ the temporal variation in the wind direction in a coordinate system moving together with the cyclonic center. Then we have

$$\frac{\delta \alpha}{\delta t} = \frac{\partial \alpha}{\partial t} + c\frac{\partial \alpha}{\partial x}. \qquad (6)$$

Here $\frac{\delta \alpha}{\delta t}$ stands for the changes in wind direction, caused by inner processes occurring in the moving cyclone: its abatement or intensification, and c $\frac{\partial \alpha}{\partial x}$ for that change in the wind direction which is caused by the shift of the cyclone. If we assume the inner changes in the cyclone to be small and ignore the term $\frac{\delta \alpha}{\delta t}$, we obtain:

$$\frac{\partial \alpha}{\partial t} = -c\frac{\partial \alpha}{\partial x}. \qquad (7)$$

However $\frac{\partial \alpha}{\partial x} = \frac{\partial \alpha}{\partial c} \cdot \frac{\partial c}{\partial x} = K_L \cdot \cos\psi$; consequently

$$\frac{\partial \alpha}{\partial t} = -c \cdot K_L \cdot \cos\psi. \qquad (8)$$

Substituting (8) in (4) we finally get

$$K = K_L\left(1 - \frac{c}{v} \cdot \cos\psi\right), \qquad (9)$$

where ψ is the angle between the wind and the direction of the x–axis.

Above the friction level, i. e., above the terrestrial boundary layer, the streamline curvature may be replaced by the isobar curvature, considering the flow to be uniform but not steady. Then

$$K = K_l\left(1 - \frac{c}{v} \cdot \cos\psi\right). \qquad (10)$$

It follows from (10) that the trajectory curvature may be equal to the isobar curvature in a nonstationary flow only under the condition, that $\cos\psi = 0$, i.e., when the isobars are orthogonal to the trajectory of the cyclone center (Figure 93). To the left of the path of the cyclone center, where $\cos\psi < 0$, the trajectory curvature will be greater than the isobar curvature, and the curvature of the particle trajectory attains its maximum where the isobars are antiparallel to the cyclone center trajectory ($\psi = 180°$). To the right of this trajectory the curvature of the particle trajectory is smaller than the isobar curvature, and the minimum of curvature is reached at the point where the isobars are parallel to the trajectory of the center of the cyclone ($\psi = 0$).

We should note that the difference between the trajectory curvature and the isobar curvature depends upon the ratio c v. Usually the velocity of a moving cyclone is less than the particle velocity, i.e., $c < v$. Then the trajectory curvature has the same sign as the isobar curvature. However in certain cases it may happen that $c > v$. This may occur at points not too far from the cyclone center. At such

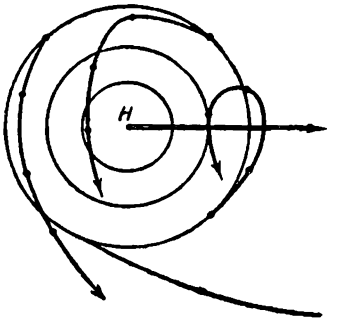

Figure 93. Trajectory curvature of air particles in a moving cyclone

points the trajectory and isobar curvatures may have opposite signs, i.e., at these points the trajectory may possess an anticyclonic curvature.

2. Anticyclonic center. Proceeding analogously to the case of the cyclonic center, we reach the following results (Figure 94). In the anterior and posterior parts of the anticyclone, where the isobars are orthogonal to the trajectory of the anticyclone center, the trajectory curvature coincides with the isobar curvature. To the left of the anticyclone center trajectory the absolute value of the particle trajectory curvature is less than the isobar curvature and reaches a minimum at points where the particle trajectory is parallel to the isobar. To the right of the anticyclone center trajectory the absolute value of the particle trajectory curvature is greater than the isobar curvature. It reaches a maximum at points where the trajectory is antiparallel to the isobars.

3. Pressure high. Let us assume, that a pressure trough moves uniformly in a straight line in a direction perpendicular to the trough axis. Repeating the above reasoning, we find that at points located on the trough axis the trajectory curvature reaches a minimum, and remains always smaller than the isobar curvature at these points.

4. Pressure high. With the same assumptions, the trajectory curvature at any point of a pressure high is greater in absolute value than the isobar curvature at these points and reaches a maximum on the high axis.

5. Warm sector of a cyclone. Careful analysis of synoptic maps reveals the fact, that in the warm sector of a cyclone the isobars are straight lines running parallel to the cyclone center trajectory. The isobars are also rectilinear behind the cold and before the warm fronts. Therefore, independent of the ratio

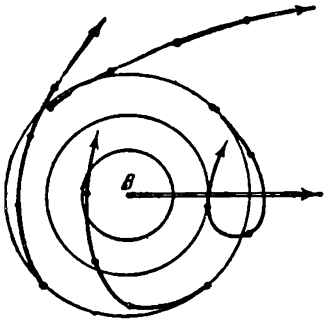

Figure 94. Curvature of trajectories of air particles in a moving anticyclone

c/v, in the warm sector of the cyclone and also behind the cold and before the warm fronts, air particle trajectory curvatures coincide with isobar curvatures i.e., the trajectories of the particles are straight lines bending at the warm and cold fronts.

§9. Ageostrophic flows

Geostrophic and gradient winds are only first approximations to a real wind. Real atmospheric flows are certainly not uniform, since the applied forces and the inertial forces are not in equilibrium. When one force prevails over the others, considerable accelerations may occur, which increase with the departure of the wind from a geostrophic flow. Flows not governed by the geostrophic equations will be called ageostrophic flows.

The equations of horizontal flow are

$$\left.\begin{aligned}\frac{du}{dt}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+lv,\\ \frac{dv}{dt}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}-lu.\end{aligned}\right\}, \tag{1}$$

where $l = 2\omega \sin\varphi$

Let us replace the pressure gradient components by the velocity components of the geostrophic wind u_g and v_g

$$lv_g=\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad lu_g=-\frac{1}{\rho}\frac{\partial p}{\partial y}.$$

We then obtain:

$$\left.\begin{aligned}\frac{du}{dt}&=l(v-v_g)=lv',\\ \frac{dv}{dt}&=-l(u-u_g)=-lu',\end{aligned}\right\} \tag{2}$$

where u' and v' are components of the ageostrophic deviation [geostrophic departure]

It follows from (2), that

$$\sqrt{\left(\frac{du}{dt}\right)^2+\left(\frac{dv}{dt}\right)^2}=l\sqrt{u'^2+v'^2}, \tag{3}$$

$$\left|\frac{dc}{dt}\right|=l\,|c'|.$$

Thus the ageostrophic deviation is directly proportional to the acceleration. Let us designate by α the angle formed by the vector v' and the x-axis and by β, the angle between the x-axis and the acceleration (Figure 95). Then

$$\operatorname{tg}\beta = \frac{\frac{dv}{dt}}{\frac{du}{dt}} = -\frac{u'}{v'} = -\frac{1}{\operatorname{tg}\alpha}. \tag{4}$$

Hence, the acceleration is orthogonal to the ageostrophic deviation and it is directed to its right, which is obvious from the signs of equations (2).

Let us also determine the relationship between the acceleration and the direction of the ageostrophic wind. Multiplying equations (2) by u and v respectively and adding the results we obtain

$$\frac{d}{dt}\left(\frac{u^2+v^2}{2}\right) = l(vu_g - uv_g).$$

Designating by θ_g and θ the angles formed with the x-axis by the geostrophic and ageostrophic wind velocities respectively, we obtain (Figure 96).

$$\frac{d}{dt}\left(\frac{c^2}{2}\right) = lcc_g \sin(\theta - \theta_g). \tag{4'}$$

Hence,

$$\frac{dc}{dt} = lc_g \sin(\theta - \theta_g). \tag{5}$$

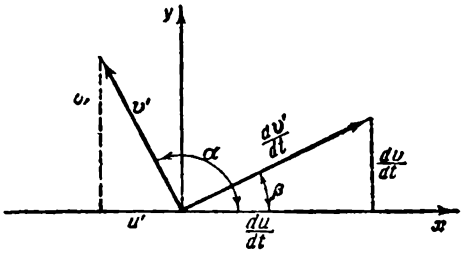

Figure 95. Ageostrophic deviation* directed orthogonally to the acceleration

Formula (4') shows, that in ageostrophic flow the rate of change of the kinetic energy of the individual particle is directly proportional to the area of the parallelogram formed by the vectors of the geostrophic and ageostrophic wind velocities. The proportionality coefficient depends on the latitude of the locality, namely, the higher the latitude the faster the kinetic energy of the particle changes. It is further evident from this formula, that the flow will be accelerated if the ageostrophic wind velocity deviates to the left of the geostrophic one ($\theta - \theta_g < 0$), and decelerated in the opposite case ($\theta - \theta_g > 0$).

Finally, formula (5) shows that the acceleration of the ageostrophic wind is always smaller than the Coriolis acceleration, its magnitude being independent of the velocity of the ageostrophic wind and determined solely by its direction (i.e., by the angle $\theta - \theta_g$), and also by the velocity of the geostrophic wind and by the latitude of the locality.

Let us now consider local changes of wind velocity

$$\frac{\partial u}{\partial t} = \frac{\partial u'}{\partial t}; \quad \frac{\partial v}{\partial t} = \frac{\partial v'}{\partial t}.$$

*[Translator's note: Geostrophic departure is the usual term]

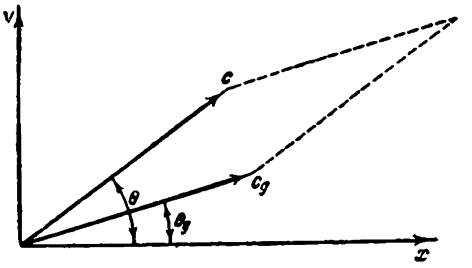

Figure 96. In ageostrophic flow the rate of change of kinetic energy of an individual particle is directly proportional to the area of the parallelogram formed by the vectors c and c_g

It follows from (1)

$$\left.\begin{aligned} u &= -\frac{1}{l\rho}\frac{\partial p}{\partial y} - \frac{1}{l}\frac{dv}{dt}, \\ v &= \frac{1}{l\rho}\frac{\partial p}{\partial x} + \frac{1}{l}\frac{du}{dt}. \end{aligned}\right\} \tag{6}$$

Let us differentiate both equations (6), regarding l and ρ as constants

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{1}{l\rho}\frac{\partial}{\partial t}\frac{\partial p}{\partial y} - \frac{1}{l}\frac{\partial}{\partial t}\frac{dv}{dt}, \\ \frac{\partial v}{\partial t} &= \frac{1}{l\rho}\frac{\partial}{\partial t}\frac{\partial p}{\partial x} + \frac{1}{l}\frac{\partial}{\partial t}\frac{du}{dt}. \end{aligned}\right\} \tag{7}$$

We assume that the last terms in equations (7) are small in comparison with the first ones. Ignoring these terms and changing the order of differentiation, we get:

$$\frac{\partial u}{\partial t} = -\frac{1}{l\rho}\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial y}\right), \qquad \frac{\partial v}{\partial t} = \frac{1}{l\rho}\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial x}\right). \tag{8}$$

In formulas (8) appears $\frac{\partial p}{\partial t}$ the local rate of change of the pressure. To a certain, not very high, degree of approximation the field of $\frac{\partial p}{\partial t}$ may be regarded as represented by the isallobaric field, obtained by recording on a chart the isobaric tendencies, which give the pressure change at three hour intervals. Equations (8) show, that the isallobaric gradient causes a local acceleration, directed along the isallobar in such a manner, that the region of low pressure remains to the left of the isallobar.

It is not difficult to see, that the magnitude of the local acceleration can be easily obtained by the use of the previously described universal gradient ruler.

Substituting (8) in (2) and ignoring the convective terms, we obtain:

$$u' = -\frac{1}{l^2\rho}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial t}\right); \qquad v' = -\frac{1}{l^2\rho}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial t}\right). \tag{9}$$

Consequently, the ageostrophic deviation is proportional to the isallobaric gradient. Therefore, the ageostrophic deviation is sometimes called isallobaric wind. The isallobaric wind blows orthogonally to the isallobars towards the low tendencies. While the divergence of the geostrophic wind is always zero, the isallobaric wind may have a nonvanishing divergence, due to the spatial variation of the isallobaric gradient. As we shall see later, the regions of rise and fall of pressure correspond with the regions of convergence and divergence of the wind. In particular, precipitation may take place at an isallobaric convergence. The velocity of the isallobaric wind may reach 5 m sec^{-1}.

In the derivation of formulas (9) we have ignored the convective terms in (2). Strictly speaking, there is no justification for this procedure, since the convective terms are of the same order as the isallobaric term.

Statistical studies on the deviations of pilot balloons from the isobars have shown, that the isallobaric wind is not directed along the isallobaric gradient, but parallels the isallobars in such a way, that the depression remains to the right, if observed in the direction of the isallobaric wind. Incidentally, we should adopt a cautious attitude towards such a method of checking the theoretical considerations, developed above, since pilot balloon soundings are possible only in clear weather, and hence only at an anticyclonic curvature of the isobars. However, when the isobars are curved the expressions (9) are no more applicable to the isallobaric wind, because of the disregarded terms, which depend on the spatial change of wind velocity and pressure.

Chapter XI

THE GEOSTROPHIC WIND VARIATION WITH ALTITUDE

§ 1. Basic Equations

The geostrophic wind equations show that as a result of vertical variations in the density and the pressure gradient, the velocity and direction of the geostrophic wind are dependent on the altitude. This conclusion remains true also in the absence of horizontal variations of the wind.

In the following sections we shall study the dependence of the geostrophic wind on the altitude.

Assuming that $l = 2\omega \sin \varphi$, we write the geostrophic wind equations in the following form:

$$u = -\frac{1}{l\rho}\frac{\partial p}{\partial y}; \quad v = \frac{1}{l\rho}\frac{\partial p}{\partial x}; \quad -\frac{1}{\rho}\frac{\partial p}{\partial z} - g = 0. \tag{1}$$

Eliminating the density by the equation of state, we obtain:

$$\frac{u}{T} = -\frac{R}{l}\frac{\partial \ln p}{\partial y}; \quad \frac{v}{T} = \frac{R}{l}\frac{\partial \ln p}{\partial x}; \quad \frac{g}{T} = -R\frac{\partial \ln p}{\partial z}. \tag{2}$$

Differentiating the first two equations (2) with respect to z, and the third with respect to y and also separately with respect to x, and eliminating mixed derivatives, we find:

$$\frac{\partial}{\partial z}\left(\frac{u}{T}\right) = \frac{1}{l}\frac{\partial}{\partial y}\left(\frac{g}{T}\right); \quad \frac{\partial}{\partial z}\left(\frac{v}{T}\right) = -\frac{1}{l}\frac{\partial}{\partial x}\left(\frac{g}{T}\right). \tag{3}$$

Performing the differentiation and collecting terms, we obtain the following precise formulas, determining the altitude variation of the geostrophic wind components:

$$\begin{aligned} \frac{\partial u}{\partial z} &= \frac{u}{T}\frac{\partial T}{\partial z} - \frac{g}{lT}\frac{\partial T}{\partial y}, \\ \frac{\partial v}{\partial z} &= \frac{v}{T}\frac{\partial T}{\partial z} + \frac{g}{lT}\frac{\partial T}{\partial x}. \end{aligned} \tag{4}$$

We will now clarify under what conditions the geostrophic wind does not vary with altitude, i. e., under what condition $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$. Eliminating u, v, g, between (4) and (1) we get

$$\begin{aligned} \frac{\partial u}{\partial z} &= -\frac{R}{lp}\left(\frac{\partial p}{\partial y}\cdot\frac{\partial T}{\partial z} - \frac{\partial p}{\partial z}\cdot\frac{\partial T}{\partial y}\right), \\ \frac{\partial v}{\partial z} &= \frac{R}{lp}\left(\frac{\partial p}{\partial x}\cdot\frac{\partial T}{\partial z} - \frac{\partial p}{\partial z}\cdot\frac{\partial T}{\partial x}\right). \end{aligned} \tag{5}$$

It follows from (5), that for the geostrophic wind to remain vertically constant the following condition should be fulfilled:

$$\frac{\partial p}{\partial x} : \frac{\partial p}{\partial y} : \frac{\partial p}{\partial z} = \frac{\partial T}{\partial x} : \frac{\partial T}{\partial y} : \frac{\partial T}{\partial z}. \tag{6}$$

That means that the isobaric surfaces must also be isothermic, in other words, the atmosphere should be barotropic. If this is not the case the atmosphere is baroclinic and the geostrophic wind velocity changes with altitude.

We now integrate (3) between the limits z_0 and z

$$\left.\begin{aligned} u &= \frac{T}{T_0} u_0 - \frac{gT}{l} \int_{z_0}^{z} \frac{1}{T^2} \frac{\partial T}{\partial y} dz, \\ v &= \frac{T}{T_0} v_0 + \frac{gT}{l} \int_{z_0}^{z} \frac{1}{T^2} \frac{\partial T}{\partial x} dz. \end{aligned}\right\} \tag{7}$$

Equations (7) show that the geostrophic wind at a given level can be regarded as the resultant of two components: 1) the wind caused by the presence of a pressure gradient at the lower level and being equal to the geostrophic wind at the same level, multiplied by the factor $\frac{T}{T_0}$ and 2) the wind depending on the horizontal temperature gradient in the given layer. The latter is usually called "thermal" wind. We shall designate it by Δc_T (Δu_T, Δv_T). We shall call the first component "baric" by analogy and designate it by c_B (u_B, v_B).

It is obvious that

$$c = c_B + \Delta c_T; \quad u = u_B + \Delta u_T; \quad v = v_B + \Delta v_T. \tag{8}$$

Since we consider the wind to be geostrophic, its variation should correspond to the vertical variation of the pressure gradient. Substituting u and v in (7) according to (1), we get the following relationships, determining the vertical variations of the components of the pressure gradient:

$$\left.\begin{aligned} -\frac{\partial p}{\partial x} &= -\frac{p}{p_0}\left(\frac{\partial p}{\partial x}\right)_0 - g\rho T \int_{z_0}^{z} \frac{1}{T^2} \frac{\partial T}{\partial x} dz, \\ -\frac{\partial p}{\partial y} &= -\frac{p}{p_0}\left(\frac{\partial p}{\partial y}\right)_0 - g\rho T \int_{z_0}^{z} \frac{1}{T^2} \frac{\partial T}{\partial y} dz. \end{aligned}\right\} \tag{9}$$

Equations (9) show that the pressure gradient at the given upper level may also be regarded as the resultant of two components: 1) the pressure gradient due to the pressure distribution at the lower level, and 2) the pressure gradient, due to the temperature distribution in the given layer.

By analogy with the "thermal" and "baric" winds, we call the second component the "thermal" component of the pressure gradient and designate it by $\Delta\mathfrak{G}_T$ ($\Delta\mathfrak{G}_{Tx}$, $\Delta\mathfrak{G}_{Ty}$), and the first one the "baric" component, denoted by $\mathfrak{G}_B$ ($\mathfrak{G}_{Bx}$, $\mathfrak{G}_{By}$). Evidently,

$$\mathfrak{G} = \mathfrak{G}_B + \Delta\mathfrak{G}_T; \quad \mathfrak{G}_x = \mathfrak{G}_{Bx} + \Delta\mathfrak{G}_{Tx}; \quad \mathfrak{G}_y = \mathfrak{G}_{By} + \Delta\mathfrak{G}_{Ty}, \tag{10}$$

where $\mathfrak{G}$ ($\mathfrak{G}_x$, $\mathfrak{G}_y$) is the pressure gradient at the upper level.

We will assume for the time being that in the layer in which the altitude variation of the geostrophic wind is being considered there is no horizontal temperature gradient

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = 0,$$

i.e., we first consider the altitude variation of the "baric" wind.

According to (7) and (8) the baric wind velocity is determined by the formulas:

$$u_B = \frac{T}{T_0} u_0; \quad v_B = \frac{T}{T_0} v_0, \tag{11}$$

Hence

$$c_B = \frac{T}{T_0} c_0, \quad \operatorname{tg} \theta_B = \operatorname{tg} \theta, \tag{12}$$

i.e.,

$$c_B = \frac{T}{T_0} c. \tag{13}$$

That means, that the direction of the baric wind (determined by the angle θ_B between the wind and the x-axis) does not change with altitude, coinciding throughout with the direction of the geostrophic wind at the lower level; only the absolute value of the velocity changes.

Writing $\Delta c_B = c_B - c_0$, $\Delta T = T - T_0 = -\gamma \Delta z$, where $\gamma = -\frac{\partial T}{\partial z}$ is the lapse rate, we obtain:

$$\frac{\Delta c_B}{\Delta z} = -\gamma \cdot \frac{c_0}{T_0}. \tag{14}$$

Thus, at a positive lapse rate ($\gamma > 0$) the baric wind velocity diminishes ($\Delta c_B < 0$); for a vertically isothermic distribution ($\gamma = 0$) it remains constant ($\Delta c_B = 0$) and it increases in the inversion layer ($\gamma < 0$). However, the variation in the absolute value of the baric wind is very small. For instance, in a layer 1 km thick it does not exceed 1–2 m/sec even at immense velocities of 50 m/sec, a variation well within the limits of error of the existing methods of measurement of wind velocities in the free atmosphere. Therefore, we may completely ignore the velocity variations of the baric wind, regarding it at any altitude as equal in magnitude and direction to the geostrophic wind at the lower level $c_B = c_0$, consequently

$$c = c_0 + \Delta c_T. \tag{15}$$

We shall now consider the altitude variations of the "baric"component of the pressure gradient. According to (9) and (10) the baric component is determined by the formulas

$$\mathfrak{G}_{Bx} = \frac{p}{p_0} \mathfrak{G}_{0x}, \quad \mathfrak{G}_{By} = \frac{p}{p_0} \mathfrak{G}_{0y}, \tag{16}$$

Hence

$$\mathfrak{G}_B = \frac{p}{p_0} \mathfrak{G}_0, \quad \operatorname{tg} \psi_B = \operatorname{tg} \psi_0, \tag{17}$$

where ψ_B and ψ_0 are respectively the angles between the vectors $\mathfrak{G}_B$ and $\mathfrak{G}_0$ and the x-axis. Consequently,

$$\mathfrak{G}_B = \frac{p}{p_0} \mathfrak{G}_0. \tag{18}$$

That means that with altitude the baric component of the gradient does not change it direction but only its absolute value.

Assuming that $\Delta \mathfrak{G}_B = \mathfrak{G}_B - \mathfrak{G}_0$ and $\Delta p = p_0$, we obtain from (17)

$$\frac{\Delta \mathfrak{G}_B}{\mathfrak{G}_B} = \frac{\Delta p}{p} \tag{19}$$

or using the hydrostatic equation $\Delta p = -g \rho_m \Delta z$ and the equation of state $p = \rho R T$ and adopting the approximation $\frac{\rho_m}{\rho_0} \cong 1$, we find

$$\frac{\Delta \mathfrak{G}_B}{\Delta z} = -\frac{g}{RT} \mathfrak{G}_0. \tag{20}$$

Thus, the value of the baric component of the gradient always diminishes with altitude.

The variation of the baric component of the pressure gradient may attain an appreciable magnitude and therefore cannot be ignored. We may further notice that in the absence of a horizontal temperature gradient, the isobars will gradually diverge with increasing altitude, although the slope of the isobaric surface will change only slightly; this follows from the relation

$$\operatorname{tg}\beta = \frac{l}{g}c = \frac{l}{g}\frac{T}{T_0}c = \frac{T}{T_0}\operatorname{tg}\beta_0. \tag{21}$$

§ 2. The Thermal Wind and the Thermal Gradient

In the foregoing section we have shown that in the absence of a horizontal temperature gradient, the magnitude of the geostrophic wind changes only very slightly with altitude and its direction does not change at all.

We now turn to a more detailed examination of the effect of a horizontal temperature gradient on the altitude variation of the geostrophic wind. We shall assume, that there is no pressure gradient at the lower level, i.e., that

$$\left(\frac{\partial p}{\partial x}\right)_0 = \left(\frac{\partial p}{\partial y}\right)_0 = 0$$

and therefore

$$u_0 = v_0 = 0.$$

The change with altitude of the geostrophic wind and of the pressure gradient will then be determined only by the horizontal temperature gradient. In analogy with the "thermal" wind we shall call the horizontal temperature gradient, the "thermal" gradient. Designating the thermal wind vector by Δc_T, and its components by Δu_T, Δv_T, we obtain

$$\Delta u_T = -\frac{gT}{l}\int_{z_0}^{z}\frac{1}{T^2}\frac{\partial T}{\partial y}\,dz,$$

$$\Delta v_T = \frac{gT}{l}\int_{z_0}^{z}\frac{1}{T^2}\frac{\partial T}{\partial x}\,dz. \tag{1}$$

Owing to the irregularity of the lapse rate, we cannot carry out the integration on the right side of equations (1) in the general case; therefore we restrict ourselves to their average values

$$\int_{z_0}^{z}\frac{1}{T^2}\frac{\partial T}{\partial y}\,dz = \frac{1}{T_m^2}\left(\frac{\partial T}{\partial y}\right)_m \Delta z, \qquad \int_{z_0}^{z}\frac{1}{T^2}\frac{\partial T}{\partial x}\,dz = \frac{1}{T_m^2}\left(\frac{\partial T}{\partial x}\right)_m \Delta z, \tag{2}$$

where, to a first approximation, T_m may be regarded as an arithmetic mean temperature of the layer; $-\left(\frac{\partial T}{\partial x}\right)_m$, and $-\left(\frac{\partial T}{\partial y}\right)_m$ as arithmetic means of the components of the thermal gradient; Δz is the thickness of the layer. We denote by Γ the vector of the mean thermal gradient in the layer, and its components by

$$\Gamma_x = -\left(\frac{\partial T}{\partial x}\right)_m \text{ and } \Gamma_y = -\left(\frac{\partial T}{\partial y}\right)_m.$$

Substituting these values in (1) and cancelling the approximately equal T_m and T, we obtain (omitting the index m):

$$\Delta u_T = \frac{g\Delta z}{lT}\cdot\Gamma_y,$$

$$\Delta v_T = -\frac{g\Delta z}{lT}\cdot\Gamma_x. \tag{3}$$

The absolute value of the velocity of the thermal wind will be

$$\Delta c_T = \frac{g \Delta z}{lf} \Gamma, \qquad (4)$$

and its direction will be determined in the following manner. We direct the x-axis along the isotherm and the y-axis in the direction of decreasing temperature (Figure 97).

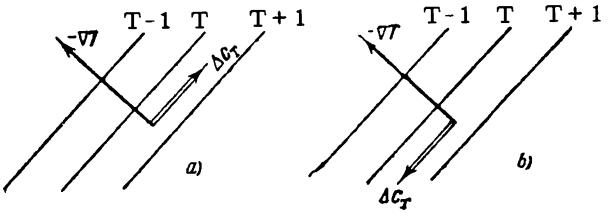

Figure 97. The thermal wind is directed along the isotherm, whereby the cold region remains to the left in the northern hemisphere (a) and to the right in the southern hemisphere (b)

In this case, $\Gamma_x = 0$ and therefore $\Delta v_T = 0$. In the northern hemisphere at $\varphi > 0$ and $\Gamma > 0$, we have $\Delta u > 0$. Thus we have the following rule for the determination of the thermal wind direction.

The thermal wind blows along the isotherm, so that the region of low temperatures remains to the left in the northern hemisphere and to the right in the southern hemisphere. A complete analogy obviously exists between this rule, determining the direction of the thermal wind and the baric law of wind*, determining the direction of the geostrophic wind.

In Table 41 we give the build-up of the thermal wind with altitude in a 1 km layer, calculated on the assumption that T = 273°, by the formula

$$\frac{\Delta c}{\Delta z} = 2.5 \frac{\Gamma}{\sin \varphi}, \qquad (5)$$

where Γ is measured in degrees per 100 km.

Table 41

Growth of thermal wind with altitude

Γ \ φ	10°	20°	30°	40°	50°	60°	70°	80°	90°
1°/100 km	14.4	7.3	5.0	3.9	3.3	2.9	2.7	2.6	2.5
2°/100 »	28.8	14.6	10.0	7.8	6.6	5.8	5.4	5.2	5.10
3°/100 »	43.2	21.9	15.0	11.7	9.9	8.7	8.1	7.8	7.5
5°/100 »	72.2	36.7	25.0	19.5	16.3	14.5	13.3	12.7	12.5

Formula (4) and Table 41 show that the thermal wind sharply decreases with increasing latitude from infinity at the equator (which shows that there can be no thermal wind there) to a minimum at the pole, where formula (5) assumes the form

$$\frac{\Delta c}{\Delta z} = 2.5 \Gamma.$$

---

* [Buys-Ballot's law. - Tr. note].

Only at very small horizontal temperature variation ($\Gamma < 1^\circ/100$ km) may one disregard the thermal wind and even this only at high latitudes. However, at low latitudes (and with more considerable thermal gradients also at high latitudes), the thermal wind cannot be ignored since in this case its value is of the same order as the geostrophic wind velocity.

According to (9) of § 1 and (2) the component of the pressure gradient in the direction of the thermal wind is determined by the formulas

$$\Delta G_{Tx} = \frac{g\Delta z}{RT_m} \cdot \frac{p}{T_m} \Gamma_x,$$
$$\Delta G_{Ty} = \frac{g\Delta z}{RT_m} \cdot \frac{p}{T_m} \Gamma_y. \qquad (6)$$

Formulas (6) show that the absolute value of the component of the pressure gradient ΔG_T in the direction of the thermal gradient Γ, is directly proportional to the absolute value of the thermal gradient

$$\Delta G_T = \frac{g\Delta z}{RT_m} \cdot \frac{p}{T_m} \Gamma. \qquad (7)$$

Measuring the magnitude of ΔG_T in millibars per 100 km, Γ in degrees per 100 km and Δz in kilometers and putting $g = 10$ m/sec, $T = 273^\circ$, $R = 287\ m^2 sec^{-2} deg^{-1}$, we obtain the following values for the variation of the component of the pressure gradient in the direction of the thermal gradient in a layer 1 km thick, given in Table 42 and calculated by the formula

$$\frac{\Delta G_T}{\Delta z} = 0.00046 \cdot p \cdot \Gamma.$$

Table 42

Altitude variation of the component of the pressure gradient in the direction of the thermal gradient

Γ \ p mb	1000	900	700	500	300	200	100 mb
1°/100 km	0.46	0.41	0.32	0.23	0.14	0.09	0.05
2°/100 »	0.9	0.8	0.6	0.5	0.3	0.2	0.1
3°/100 »	1.4	1.2	0.9	0.7	0.4	0.3	0.1
5°/100 »	2.3	2.0	1.6	1.1	0.7	0.5	0.2

With increasing altitude the thermal component of the pressure gradient increases more and more, which corresponds to the steeper slope of the isobaric surface, determined by the formula

$$\operatorname{tg} \beta = \frac{\Gamma}{T} \Delta z. \qquad (8)$$

It follows from (8), that the isobaric surfaces will always be rather less steep than the corresponding isothermal surfaces. These angles fulfill the equality

$$\frac{\operatorname{tg} \beta}{\operatorname{tg} \beta_T} = \frac{\Delta T}{T}, \qquad (9)$$

where β_T is the slope angle of the isothermal surface and ΔT is the temperature difference between the boundaries of the layer in question.

§ 3. Altitude Variations of the Geostrophic Wind due to the Relative Position of the Pressure and Thermal Gradients

The relative position of the pressure and thermal gradients or, differently put, the relative position of the isobars and isotherms, influences the altitude dependence of the geostrophic wind.

In the foregoing sections we have shown that at high altitudes the geostrophic wind velocity vector may be regarded as a resultant of two vectors: c_0, i. e., the geostrophic wind below and the vector Δc_T, i. e., the thermal wind caused by the mean thermal gradient in the layer:

$$c = c_0 + \Delta c_T. \tag{1}$$

The vector c_0 is determined by the pressure gradient $\mathfrak{G}_0$, both in magnitude and direction

$$c_0 = \frac{1}{l\rho}\mathfrak{G}_0, \tag{2}$$

and the vector Δc_T is determined both in magnitude and direction by the thermal gradient Γ, whereby

$$\Delta c_T = \frac{g\lambda z}{lT}\cdot\Gamma. \tag{3}$$

Finally, the vector c is determined by the pressure gradient $\mathfrak{G}$

$$\mathfrak{G} = \frac{p}{p_0}\mathfrak{G}_0 + \Delta\mathfrak{G}_T. \tag{4}$$

Thus, the pressure gradient $\mathfrak{G}$ is the vectorial sum of the pressure gradient $\mathfrak{G}_0$, multiplied by $\frac{p}{p_0}$, and the thermal component of the pressure gradient $\Delta\mathfrak{G}_T$ which according to (7) § 2 equals

$$\Delta\mathfrak{G}_T = \frac{g\lambda z}{RT_m}\cdot\frac{p}{T_m}\Gamma. \tag{5}$$

With increasing thickness of the layer the effect of the first components in (1) and (4) decreases and the second components predominate. Accordingly, the direction of the vector c approaches more and more that of Δc, with increasing altitude and the direction of the vector $\mathfrak{G}$ approaches that of the vector Γ. Thus, with increasing altitude, the direction of the geostrophic wind approaches the direction of the isotherm and the pressure gradient approaches the thermal gradient.

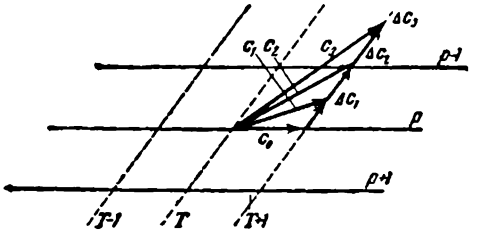

Figure 98. With increasing altitude the geostrophic wind gradually approaches the isotherm

Figures 98 and 99 show schematically the altitude variation of the geostrophic wind

velocity and the pressure gradient. It is clear from these figures that the altitude variation of the wind velocity depends on the relative position of the pressure and thermal gradients.

We have dealt above with the altitude variation of the thermal wind vector Δc_T, which in turn determines the altitude variation of the geostrophic wind. In meteorological practice the wind is often regarded not as a vector, but as a pair of scalars, namely, the magnitude of the air velocity and the angle determining its direction. We shall derive the corresponding formulas, giving the altitude variation of the wind speed and direction as functions of the angle α between the pressure and the thermal gradients.

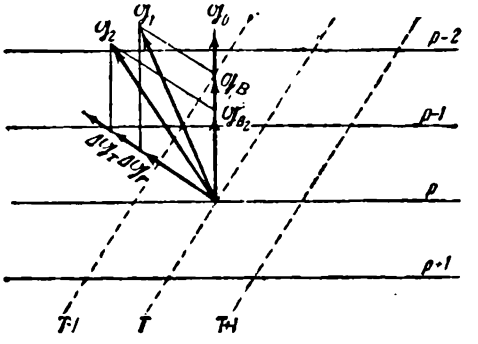

Figure 99. Variation of the pressure gradient with altitude

We direct the x-axis along the isobar and the y-axis along the pressure gradient at the lower level (Figure 100). The thermal wind Δc_T will be directed along the isotherm, forming an angle α with the isobar. The wind velocity c at the upper level equals the vectorial sum $c_0 + \Delta c_T$. Let us denote by θ the angle formed between the velocity c and the x-axis (the isobar of the lower level).

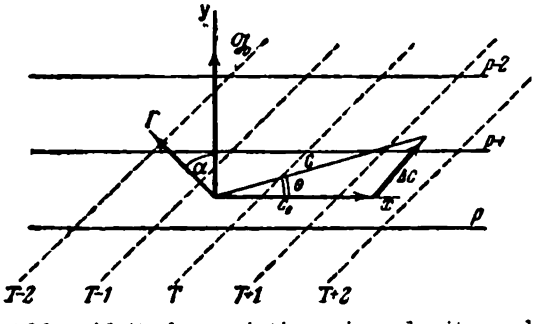

Figure 100. Altitude variations in velocity and direction of the geostrophic wind, depending on the angle between the pressure and the thermal gradients

It is evident, that

$$u = c\cos\theta,$$
$$v = c\sin\theta \tag{6}$$

and

$$\Delta u_T = \Delta c_T \cos\alpha,$$
$$\Delta v_T = \Delta c_T \sin\alpha. \tag{7}$$

Differentiating (6) we find

$$\Delta u = \Delta c \cos\theta - c \sin\theta \Delta\theta,$$
$$\Delta v = \Delta c \sin\theta + c \cos\theta \Delta\theta. \quad (8)$$

Having ignored the wind variation due to the baric component of the gradient, we find

$$\Delta u = \Delta u_T,$$
$$\Delta v = \Delta v_T.$$

Consequently,

$$\Delta c \cos\theta - c \sin\theta \Delta\theta = \Delta c_T \cos\alpha,$$
$$\Delta c \sin\theta + c \cos\theta \Delta\theta = \Delta c_T \sin\alpha, \quad (9)$$

whence, solving (9) with respect to Δc and $\Delta\theta$, we get

$$\Delta c = \Delta c_T \cdot \cos(\alpha - \theta),$$
$$c\Delta\theta = \Delta c_T \cdot \sin(\alpha - \theta) \quad (10)$$

or, considering (3)

$$\frac{\Delta c}{\Delta z} = \frac{g}{lT} \cdot \Gamma \cos(\alpha - \theta),$$
$$\frac{\Delta\theta}{\Delta z} = \frac{g}{lT} \frac{1}{c} \Gamma \sin(\alpha - \theta). \quad (11)$$

Formulas (11) show that the altitude variation of the geostrophic wind essentially depends on the angle $(\alpha - \theta)$, formed between the isotherms and the isobars. If $(\alpha - \theta)$ i.e., if the isobars and isotherms coincide, $\frac{\Delta\theta}{\Delta z} = 0$ and there is no wind rotation with altitude. $\frac{\Delta\theta}{\Delta z} = 0$ also when $(\alpha - \theta) = 180^\circ$. If, however, $(\alpha - \theta) = \pm 90^\circ \frac{\Delta c}{\Delta z} = 0$, i.e., only the wind direction changes but not its speed, whereby at $(\alpha - \theta) > 0$ the wind rotates to the left and at $(\alpha - \theta) < 0$ it rotates to the right. In general, the smaller the angle $(\alpha - \theta)$, the faster does the wind speed vary and the slower the change of its direction. Let us note that if the layer is not very thick we can substitute $(\alpha - \theta)$ for the angle between the thermal gradient and the pressure gradient at the lower level. In case of a considerable thickness of the layer we ought to accept for the angle $(\alpha - \theta)$ the angle formed between the pressure gradient and the thermal gradient at the middle of the layer. Similarly, for c in that case we must take its value at the middle of the layer.

Table 43 gives the altitude variation of the geostrophic wind velocity for various values of $(\alpha - \theta)$ and Γ; is is calculated for T = 273°, φ = 60°, Δz = 1 km,

Table 43

Variation of the geostrophic wind velocity

Γ \ α−θ	0	10	20	30	40	50	60	70	80	90
1°/100 km	2.9	2.9	2.7	2.5	2.2	1.9	1.4	1.0	0.5	0.0
2°/100 »	5.8	5.7	5.4	5.0	4.4	3.7	2.9	2.0	1.0	0.0
3°/100 »	8.7	8.6	8.1	7.5	6.6	5.6	4.3	3.0	1.5	0.0
5°/100 »	14.5	14.3	13.6	12.6	11.1	9.3	7.2	5.0	2.5	0.0

according to the formula

$$\frac{\Delta c}{\Delta z} = 2.9 \cdot \Gamma \cos(\alpha - \theta).$$

In Table 44 is given the variation of wind direction per 1 km altitude for

different values of $(\alpha - \theta)$ and c. It was calculated for $\varphi = 60^{\circ}$, T = 273, $\Gamma = 5^{\circ}/100$ km, by the formula

$$\frac{\Delta\theta}{\Delta z} = 83 \cdot \Gamma \frac{\sin(\alpha - \theta)}{c}.$$

Table 44

Variations in the geostrophic wind direction

Γ \ α−θ	0	10	20	30	40	50	60	70	80	90
1 m/sec	0	14	29	41	59	63	72	78	82	83
3 »	0	5	10	14	18	21	24	26	27	28
5 »	0	3	6	8	11	13	14	15	16	17
10 »	0	1	3	4	5	6	7	8	8	8
20 »	0	1	1	2	3	3	4	4	4	4
30 »	0	0	1	1	2	2	2	3	3	3

Table 44 shows that at high wind velocities (i.e., at high pressure gradients) the wind direction varies little with altitude. On the other hand, the direction of a weak wind varies rapidly with the altitude. Thus, for example, in a layer 1 km thick, a wind of velocity 20 m/sec will rotate only by 4°, while wind of velocity of 3 m/sec will rotate under the same conditions by 28°. A slow wind becomes soon aligned with the isotherms as the altitude increases, regardless of its direction below. At high velocities this variation is slower. In a uniform air mass at high altitudes the air still tends to flow in the direction of the isotherms, the velocity increment being considerable and in this case coinciding in direction with the isotherm. Since a real wind in the higher layers of the atmosphere often does not depart very much from a geostrophic wind, the prevalence of a general westerly current in the troposphere of the middle latitudes may be explained by the decrease of temperature from the equator towards the pole throughout the troposphere. In the stratosphere the westerly component of the wind diminishes with altitude together with the rotation of the thermal gradient.

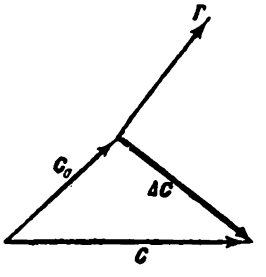

Figure 101. Determination of the horizontal temperature gradient by the altitude variation of the geostrophic wind

The above considerations also allow the inverse problem to be solved: to determine the horizontal temperature distribution by altitude variations of the wind speed and direction.

Let us assume, for instance, that from pilot balloon observations a wind velocity is deduced at the lower level c_0 (Figure 101) and at the upper level c; determining vectorially $\Delta c = c - c_0$, we obtain all that is necessary for the determination of the thermal gradient Γ in this layer. Indeed, the thermal gradient Γ will be orthogonal to Δc and directed to its left (in the northern hemisphere) and its magnitude may be determined by the formula

$$\Gamma = \frac{lT}{g} \cdot \frac{\Delta c}{\Delta z}. \qquad (12)$$

These considerations can be used for the extrapolation of synoptic charts. By means of an operating network of pilot balloon stations, one can get a good enough picture of the distribution of pressure and temperature beyond the borders of the synoptic chart.

However, applying these conclusions in practice one should remember that all was said in reference to the geostrophic wind, and therefore is meaningful only for homogeneous layers, lying above the terrestrial turbulent layer, i.e., for altitudes exceeding 500-1,000 m.

§ 4. Changes of Temperature and Pressure due to Altitude Variation of the Geostrophic Wind Direction

The variation of air temperature at a given point is

$$\frac{\partial T}{\partial t} = \frac{dT}{dt} - u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} - w\frac{\partial T}{\partial z}. \tag{1}$$

Let us assume that the air masses move with constant temperature, i.e., $\frac{dT}{dt} = 0$. Then, since the geostrophic wind is horizontal

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} = c\Gamma\cos(c, \Gamma) = -c\Gamma\sin(\alpha - \theta).$$

Substituting finite differences, we obtain:

$$\frac{\Delta T}{\Delta t} = -c\Gamma\sin(\alpha - \theta), \tag{2}$$

where $\frac{\Delta T}{\Delta t}$ is the temperature variation per unit time and $(\alpha - \theta)$ is the angle between the thermal and pressure gradients (or between the isobar and the isotherm), read from the pressure to thermal gradient.

It is evident from (2) that at a fixed point the temperature variation per unit time due to an influx of air of different temperature is directly proportional to the thermal gradient and wind velocity and also to the sine of the angle between the wind direction and the isotherm. The wind direction is determined by the relative position of the gradients: if the thermal gradient is to the left of the pressure gradient $\left(\frac{\Delta T}{\Delta t} < 0\right)$, the air is cooled $(\alpha - \theta > 0)$, and if the thermal gradient lies to the right of the pressure gradient ($(\alpha - \theta < 0)$) it gets warmer $\left(\frac{\Delta T}{\Delta t} > 0\right)$.

Combining formulas (2) and (11) § 3 enables us to connect the advective variation of the temperature with the altitude variation of the direction of the geostrophic wind. In fact, eliminating $\sin(\alpha - \theta)$, we obtain:

$$\frac{\Delta T}{\Delta t} = -\frac{l}{g}Tc^2\frac{\Delta\theta}{\Delta z}. \tag{3}$$

Formula (3) shows that in the northern hemisphere a backing of the wind with altitude $\left(\frac{\Delta\theta}{\Delta z} > 0\right)$ causes a temperature decrease with time $\left(\frac{\Delta T}{\Delta t} < 0\right)$ and on the other hand, at a veering of the wind $\left(\frac{\Delta\theta}{\Delta z} < 0\right)$ the temperature rises $\left(\frac{\Delta T}{\Delta t} > 0\right)$. The rate of temperature variation increases with the wind velocity and decreases with increasing altitude variation in the wind direction.

It is obvious from formula (3) that even if the wind rotation is slight, but its

velocity high, considerable temperature changes may take place. For low wind velocities however, even at a large rotation, the temperature changes very little by advection. An influx of air differing in temperature, and hence in density, is usually accompanied by a change of air pressure, whereby heating is usually accompanied by a decrease in pressure, and cooling – by an increase. Therefore the wind rotation with altitude can be related to the temporal pressure variations.

If we regard the pressure p_h at a certain altitude h as constant, then from the barometric formula

$$p_h = p_0 \cdot e^{-\frac{gh}{RT_h}}, \tag{4}$$

which applies in the case of geostrophic wind, we obtain by differentiation with respect to t

$$\frac{\Delta p_0}{\Delta t} = -\frac{gh}{RT} \cdot \frac{p_0}{T} \cdot \frac{\Delta T}{\Delta t} \tag{5}$$

or, substituting for $\frac{\Delta T}{\Delta t}$ the expression (3)

$$\frac{\Delta p_0}{\Delta t} = \frac{l}{RT} \cdot p_0 c^2 h \cdot \frac{\Delta \theta}{\Delta z}. \tag{6}$$

Formula (6) shows that when the wind veers with altitude $\left(\frac{\Delta \theta}{\Delta z} < 0\right)$ the pressure falls, owing to advection of air masses of different temperature $\left(\frac{\Delta p_0}{\Delta t} < 0\right)$ and when the wind backs $\left(\frac{\Delta \theta_0}{\Delta z} > 0\right)$ it rises $\left(\frac{\Delta p_0}{\Delta t} > 0\right)$. The second power of the velocity in (6) is a reminder for the immense effect of wind velocities on pressure changes.

§ 5. Altitude Variations of the Wind in Occluded Cyclones and Anticyclones

The results obtained in the foregoing sections may also be applied as a first approximation in the case of curvilinear isobars, if their curvature is sufficiently small.

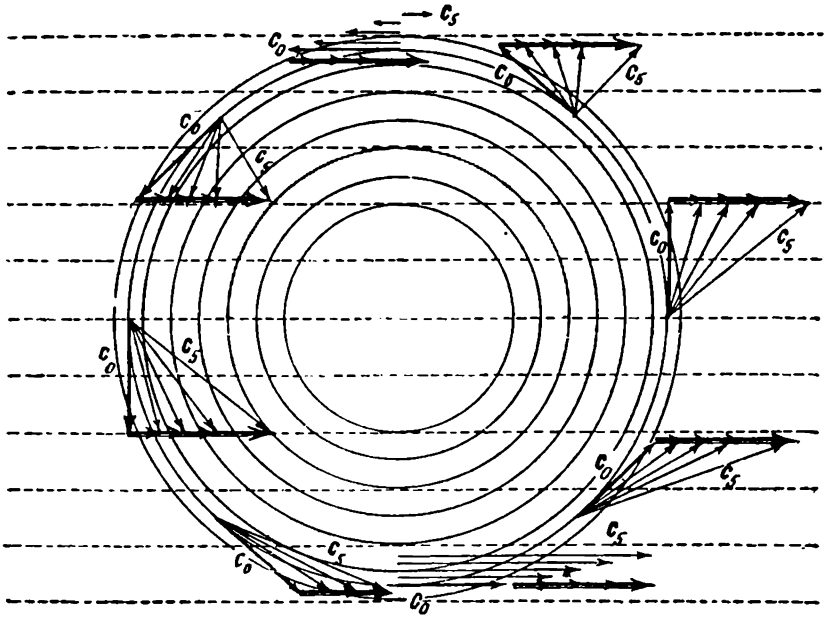

Figure 102. Altitude variation of geostrophic wind in different parts of the cyclone periphery

Let us imagine an occluded symmetrical cyclone with circular isobars (Figure 102). We make the simplifying assumption that the temperature gradient is everywhere constant and points from south to north, and therefore the isotherms pass latitudinally from west to east. The wind direction at the lower level with respect to the cardinal points will differ in the different parts of the cyclone; the direction of the thermal wind, however, will be everywhere the same (from west to east); its magnitude will also be spatially constant. In this case the nature of the altitude variation of the wind depends on the position within the cyclone, as is immediately apparent from the figure.

Temporal variations of temperature and pressure also depend on the position.

In Figure 102 is given a schematic outline of the geostrophic wind altitude variation for the different positions on the periphery of an occluded cyclone, where the isobar curvature is so small, that its radius reaches up to 500 km. The wind velocity at the lower level is taken as 6 m/sec, which corresponds to a pressure gradient of 1 mb/100 km. The thermal gradient is 1°/100 km.

The thermal wind developing in a 1 km layer as a result of this gradient is taken as 3 m/sec at any altitude.

At a point to the south of the cyclone center the wind velocity increases, together with the pressure gradient, i.e., the isobars become denser with increasing altitude. There are no temperature or pressure changes.

At a point lying to the north of the cyclone center the wind velocity decreases with altitude; at a certain altitude it reaches zero and then increases again in the opposite direction. The pressure gradient decreases with altitude, reaches zero and then increases in the opposite direction. There are no temperature or pressure changes.

In the eastern semicircle of the periphery the wind veers with increasing altitude, whereby on the northern sector of this arc the rotation is particularly marked. In connection with this the temperature rises in the eastern half of the periphery and the pressure falls.

In the western half, on the contrary, the wind backs with increasing altitude, this rotation being again more marked in the northern sector. The temperature thereby drops and the pressure rises. Thus the pressure minimum gradually shifts from west to east in the direction of pressure decrease. The pressure minimum progresses along the isotherm in such a manner that the low pressure region remains to the left. In this very direction the air moves at high altitudes, forming a current which according to S. I. Troitskii is termed the leading stream .

Figures 103 describe a pressure field at altitudes of a) 1 km, b) 2 km and c) 5 km, obtained by the superposition of a rectilinear pressure field, due to the thermal gradient, on a circular pressure field of the lower level. The distances between the circular isobars are 110, 130 and 260 km respectively. The distances between the rectilinear isobars are 220, 120 and 70 km.

With altitude the pressure pattern gradually changes; the isobars straighten and approach the isotherms. The depression shifts with altitude towards the region of low temperatures, so that the cyclone axis is not vertical and inclines towards the cold region, forming a very small angle with the horizon.

The change in the isobaric configuration will be more appreciable, the greater the thermal and the smaller the pressure gradients at the lower level.

The anticyclonic picture constitutes, as it were, a mirror image of what we have just seen in the cyclone. Namely, at the same positions of the isobars and isotherms, at a point lying to the south of the anticyclone center, the wind velocity decreases with altitude, vanishes at a certain altitude and then increases again, reversing its direction. The pressure gradient decreases with altitude, reaches zero and then increases again in the opposite direction. There are no temperature or pressure changes.

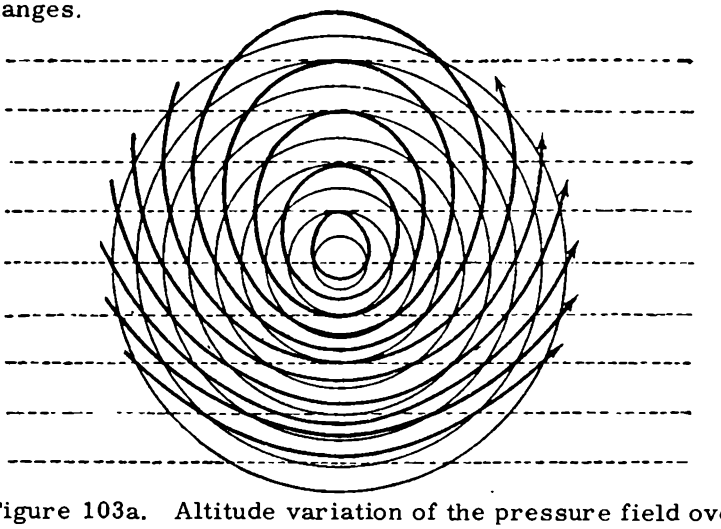

Figure 103a. Altitude variation of the pressure field over a cyclone

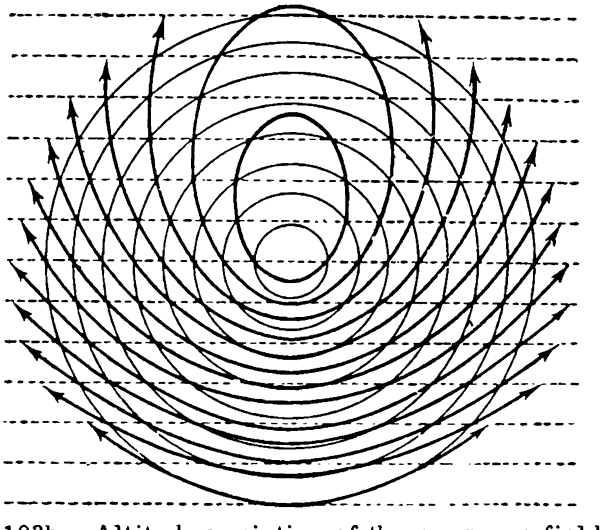

Figure 103b. Altitude variation of the pressure field over a cyclone

At a point northern to the anticyclone center the wind velocity and the pressure gradient increase with altitude, preserving their direction, so that the isobars become denser with altitude. The temperature and the pressure do not change.

In the eastern semicircle of the anticyclone periphery the wind backs with increasing altitude, accompanied by cooling and pressure increase. In the western half, on the other hand, the wind veers, the air warms and pressure drops. Thus, the anticyclone moves along the isotherm from west to east too, leaving the low temperature region to the left. The anticyclone axis is also inclined to the horizon at a very small angle, already in the high temperature region.

The results obtained are basically valid for an arbitrary orientation of the thermal gradient with respect to the cardinal points. In that case it is sufficient to refer to the posterior and anterior parts of the cyclone or anticyclone, instead of the eastern and western halves.

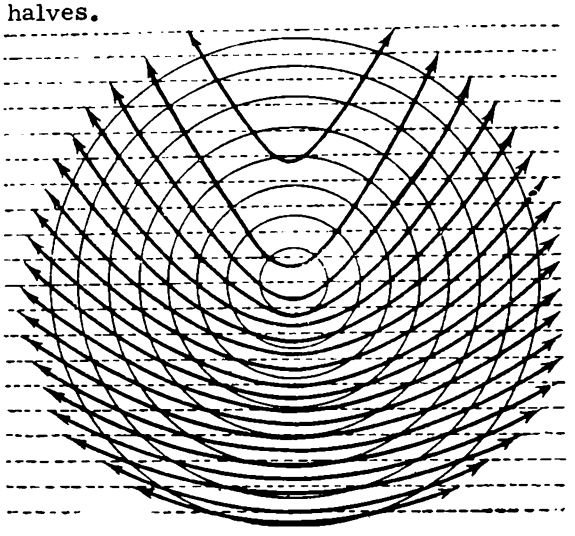

Figure 103c. Altitude variation of the pressure field over a cyclone

Thus, the anterior part of a cyclone and the posterior part of an anticyclone are characterized by a backing of the wind and by heating, whereas the posterior part of the anticyclone and the anterior of the cyclone show veering wind and cooling. Since at high altitudes the wind direction approaches the isotherms, a general leading stream is established at a great altitude above moving cyclones and anticyclones, in the direction of which shifts the entire pressure system in question. The above results remain valid also for nonfrontal parts of nonoccluded cyclones. Only in the frontal region might the pattern basically differ from the one described here, due to the departure of the pressure gradient from the temperature gradient.

Finally let us consider the altitude variation of the geostrophic wind in stationary pressure systems. A pressure system is stationary if the pressure does not change and the system itself does not move. The condition of stationarity expres - sed by the equation

$$\frac{\Delta p_0}{\Delta t} = \frac{l}{RT} p_0 c^2 \cdot h \frac{\Delta \theta}{\Delta z} = 0, \tag{1}$$

whence, according to (11) § 3,

$$\frac{\Delta \theta}{\Delta z} = \frac{g}{lT} \cdot \frac{1}{c} \cdot \Gamma \sin(\alpha - \theta) = 0, \tag{2}$$

which is possible in two cases:

1) when $\Gamma = 0$ (3)

2) when $\sin(\alpha - \theta) = 0$ (4)

i.e., when the pressure and the thermal gradients are parallel.

Thus, the stationary state of a pressure system, characterized by the constancy of the wind direction, is realized in three cases:

1) when the temperatures have equalized horizontally, as happens in a mature occluded cyclone;

2) when the depression coincides with the cold region and the pressure high coincides with the warm region, as in the so-called high cyclones and anticyclones, where wind velocity increases with altitude without changing its direction;

3) when the depression coincides with the warm region and the pressure high with the cold region, in which case the wind weakens with altitude and after reaching zero reverses its direction.

The axes of stationary cyclones and anticyclones are almost vertical.

It may happen that the stationary conditions (3) and (4) are fulfilled only for a certain part of the pressure system while the rest of it is not stationary.

All this applies only to pressure and temperature variations due to advection of air of a different temperature. Other processes may considerably distort the picture given above, especially in case the heat advection is not vigorous enough. This explains the low coefficients of correlation between the theoretically deduced temperature variations and those actually observed.

CHAPTER XII

THE MECHANISM OF PRESSURE VARIATION

§1. The Connection Between Temperature and Pressure Variations in an Air Column

One of the most important porblems of modern dynamic and synoptic meteorology is the clarification of the causes of local pressure changes.

This problem is directly involved in the problem of the origin and development of cyclones and anticyclones, which occupies a central position in modern meteorology.

So far no satisfactory theory of the origin and development of pressure centers has been generally accepted. The major, but by far not the only obstacle faced are the yet undiscovered causes of the accumulation of huge quantities of air in anticyclones and their expulsion from cyclones.

Our understanding of the physical causes of local pressure variations near the ground has progressed but little since the early first investigations.

Some meteorologists remain who have not yet completely abandoned the notion that cold and warm air affect differently the pressure near the ground, although it is quite obvious that temperature changes may affect only the height of the air column, but not its mass.

The entire situation is dominated by the fact that pressure changes at the earth's surface are connected with the spatial distribution of pressure, temperature and wind velocity.

The most rational way to solve this problem is to integrate the equations of hydrodynamics together with the equation of energy balance. This method was used by I. A. Kibel' (Kibel's results will be discussed in detail in Chapter XX).

In the present chapter we shall give in greatest possible detail the results obtained on the basis of the so called "tendency equation", which enables to approach the problem of local pressure variations. Before we get down to the derivation and analysis of the tendency equation we will discuss certain points connected with incorrect use of the barometric formula for the explanation of the causes of pressure variations.

In the present section we shall discuss the relationships between the temperature variations in an air column and pressure variations at its bottom on the basis of the barometric formula.

In Chapter V we have shown that in the calm atmosphere the pressure is measured by the weight of a unit air column, extending from the observation point to the upper level of the atmosphere. Therefore, the pressure at a certain level changes when the air mass above this level changes. This fact should guide us in the analysis of certain formulas, which are often misinterpreted.

Let us imagine the air layer in question divided into n layers of equal thickness h. Denoting the pressures at the boundaries of these layers by $p_0, p_1, \ldots, p_n$, and the mean temperatures of the layers by $T_1, T_2, \ldots, T_n$ and applying the barometric formula. we obtain

$$\ln p_0 = \ln p_1 + \frac{gh}{RT_1}; \quad \ln p_1 = \ln p_2 + \frac{gh}{RT_2}; \ldots; \ln p_{n-1} = \ln p_n + \frac{gh}{RT_n}.$$

Adding these equations term by term and cancelling the identical terms, we find

$$\ln p_0 = \ln p_n + \frac{gh}{R} \sum_{i=1}^{n} \frac{1}{T_i}. \qquad (1)$$

By differentiation we get

$$\frac{\partial p_0}{\partial t} = \frac{p_0}{p_n} \frac{\partial p_n}{\partial t} - \frac{gh}{R} \cdot p_0 \sum_{i=1}^{n} \frac{1}{T_i^2} \frac{\partial T_i}{\partial t}. \qquad (2)$$

Local time derivatives, that can be interpreted as temperature and pressure changes over 12 and 24 hours, can be determined by aerological sounding.

We now turn to the examination of formula (2), keeping in mind that it holds throughout the whole layer in question only in the case of static equilibrium.
Let us first assume, that the temperature of the intermediary layers does not vary. Then

$$\frac{\partial p_0}{\partial t} = \frac{p_0}{p_n} \cdot \frac{\partial p_n}{\partial t}. \qquad (3)$$

Thus, the pressure variation at ground level is $\frac{p_0}{p_n}$ times greater than that in the upper layer. This was taken to imply that pressure variations at ground level are determined by the changes occuring in the upper layers. This conviction became even more firmly rooted when aeorological flights have shown that diurnal pressure variation at high altitudes are of the same order as those at ground level.

However equation (3) cannot mean that a pressure variation of $\frac{\partial p_n}{\partial t}$ aloft, where the pressure is p_n, brings about a pressure variation of $\frac{p_0}{p_n} \cdot \frac{\partial p_n}{\partial t}$ below. This is incorrect, because a pressure change of δp_n only shows that above the level where the pressure rises the mass of the air column of unit cross section has increased by $\frac{\delta p_n}{g}$; but if there is no air advection in the lower layers, the total air mass increases by $\frac{\delta p_n}{g}$ and not by $\frac{p_0}{p} \cdot \frac{\delta p_n}{g}$.

This question is fundamental enough to merit an explanation from still another viewpoint. Let us assume that the pressure at the earth's surface varies by δp_0, taking the variation as positive,to be definite. This pressure rise means that the mass of the air column is increased by $\frac{\delta p_0}{g}$. Let us imagine, that the air column is surrounded on all its sides by absolutely impermeable walls up to the height where the pressure is p_n (Figure 104). Then the mass increment of $\frac{\delta p_0}{g}$ in the air

column is due only to advection at and above the level H, so that the air column below this level is compressed and a part of the air originally located above the H level will descend to it. The pressure change δp_n at H is therefore equal to the weight of the air superadded to that which has been at and above H, less the weight of the air that has descended below H as a result of compression.

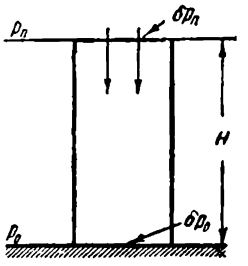

Figure 104. Pressure variations at the earth's surface, due to an influx of air mass

The mass increment $\frac{\delta p_n}{g}$ forms therefore only a part of the total contribution to the mass that has already lain above the H level. Thus, the pressure below should increase by more than δp_n, since the pressure variation underneath is caused by the total gain of mass of the air column. Thus equation (3) really gives, not the pressure variation δp_n at the level H, that results in a variation $\frac{p_0}{p_n}$ times greater in the ground level pressure, but the total pressure variation δp_0, connected with the mass influx above H, divided by $\frac{p_n}{p_0}$ owing to the compression of air.

We now assume, that the pressure does not change in the upper layer, but the temperature changes in one of the previously considered layers. Then

$$\frac{\partial p_0}{\partial t} = -\frac{gh}{R}\frac{p_0}{T_k^2}\frac{\partial T_k}{\partial t}. \tag{4}$$

This equation shows, that the same temperature changes at different altitudes cause ground level pressure variations which are greater the higher the level at which the temperature varies, since temperature decreases with altitude. However, the same temperature variation at higher altitude influences a smaller mass, (if we consider layers of equal thickness), owing to the decrease of density with altitude. Therefore the total mass variation in the column will be smaller, the higher lies the layer at which the temperature change takes place. In order to explain this paradox, we shall discuss pressure change at the lower boundary of the layer in which the temperature varies.

Since we have assumed the pressure at the upper boundary of the layer to remain constant, its variation at the lower boundary will be given by the formula

$$\frac{\partial p_{k-1}}{\partial t} = -\frac{gh}{R}\cdot\frac{p_{k-1}}{T_k^2}\frac{\partial T_k}{\partial t}. \tag{5}$$

According to (5), the pressure variation at the lower boundary of the layer will be smaller, the higher this layer is located, i.e., the smaller its mass. Consequently, for a higher-lying layer the pressure variation will be smaller, even though the temperature variation is the same. However, according to (3), ground level pressure variation is connected with the variation given by (5) by the formula

$$\frac{\partial p_0}{\partial t} = \frac{p_0}{p_{k-1}}\cdot\frac{\partial p_{k-1}}{\partial t}. \tag{6}$$

Combining (6) and (5) we get (4). This return to the initial equation (4) shows, that the weight variation of the air column, due to the temperature variation in the considered layer, is proportional to the pressure at the lower level of this layer. Thus, in a layer of given thickness the influence of a certain temperature change on the weight of the whole air column is smaller the higher this layer lies, since in that case a smaller mass of air acts upon it. But if equilibrium is maintained in the lower layers (by a mass influx), the mass of the air column must vary in such a way, that the ground level pressure would change according to (4).

§2. Influence of Air Advection at High Altitudes on Ground Level Pressure Variation

The actual relationships between pressure and temperature variations in the atmosphere are much more complicated than those suggested in §1. Essentially, the changes in the state of the lower atmospheric layers do not precisely counterbalance the variations in weight of the higher layer. The air in this layer does not remain immobile, but moves vertically, since the lower air layers are being compressed. Figure 105 gives a schematic representation of the displacement of a layer. We measure only the pressure variation at some fixed spatial position, for instance in the *ABCD* layer before, and in the *A'B'C'D'* layer after advection of air masses of a different temperature. These are the changes discussed in §1. In fact the processes occurring in an air mass during advection are influenced by those "individual" variations, which take place in the individual air particles, located in the layer *ABCD* before the advection and in layer *EFGH* after it. Therefore, the barometric formula cannot disclose the mechanism of the pressure variation at ground level which is due to advection at great altitudes. For the solution of this problem we must consider the individual variation of pressure $\delta_i\pi$ on a surface, consisting permanently of the same particles and moving up and down like a piston. Let us imagine an air column of unit cross section, laterally bounded by rigid walls, but open at the top. This means, that the pressure variation in the air column is possible only by air mass exchange through the upper open end, i.e., by high altitude advection of air masses, while at the lower layers there is no air flux at all. Let us single out in this column an air layer of thickness dz. We will discuss now the changes in the state of the air column as a result of air advection at a great altitude. Let us denote the temporal variation of any magnitude by δ; individual variations will be denoted by the subscript i and the local variations by the subscript e.

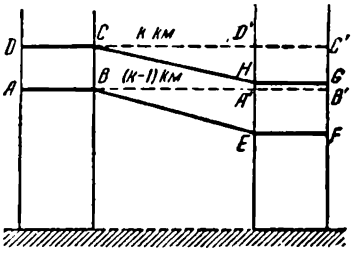

Figure 105. Influence of mass advection at high altitudes on ground level pressure variation

We consider individual displacements of particles, individual and local variations of temperature $\delta_i T$, $\delta_e T$, and pressure $\delta_i p$, $\delta_e p$. When these variations are determined by comparison with the results of actual observations we shall be able to ascertain to what extent the assumptions concerning advection at great altitudes are correct.

The individual pressure variation $\delta_i p$ at any level obviously equals the entire mass advection $\delta\pi$ at great altitudes. This mass advection is also equal to the local variation of ground level pressure δp_0. Therefore

$$\delta_i p = \delta p_0 = \delta\pi. \tag{1}$$

The local pressure variation at any point equals the sum of the individual variation and that resulting from vertical air displacement:

$$\delta_e p = \delta\pi + g\rho\delta z. \tag{2}$$

The same holds for the temperature:

$$\delta_e T = \delta_i T + \gamma\delta z, \tag{3}$$

where δz is the altitude variation of the surface composed of those physical particles whose individual changes of state are measured.

We shall assume that the state of the air column in question varies polytropically. Then we obtain from the polytropic equation

$$\frac{\delta_i p}{p} = k\frac{\delta_i \rho}{\rho} \quad \text{or} \quad \delta_i \rho = \frac{\rho}{p}\cdot\frac{\delta\pi}{k}. \tag{4}$$

Denoting the thickness of the considered elementary layer by dz, we find from geometrical considerations that

$$d(\delta z) = \delta(dz) \tag{5}$$

(since $d(\delta z)$ is the difference between the displacements of the upper and the lower boundaries of the layer, and $\delta(dz)$ is the variation of the thickness of the layer). From the law of mass conservation now follows

$$\delta(\rho dz) = 0$$

or

$$\frac{\delta_i \rho}{\rho} + \frac{\delta(dz)}{dz} = 0. \tag{6}$$

Combining (4), (5) and (6), we obtain

$$\frac{d(\delta z)}{dz} = -\frac{1}{k}\cdot\frac{\delta\pi}{p},$$

or, integrating and recalling that $(\delta z)_{z=0} = 0$,

$$\delta z = -\frac{\delta\pi}{k}\int_0^z \frac{dz}{\rho}. \tag{7}$$

Formula (7) gives the magnitude of the displacement of a plane permanently composed of the same particles and located at the level z. Let us assume, that the temperature varies linearly with altitude. Then instead of p we can put $p_0\left(\frac{T}{T_0}\right)^{\frac{g}{R\gamma}}$ after the integral sign. Integrating, we get:

$$\delta z = \frac{\delta\pi}{k}\frac{R}{R\gamma - g}\cdot\frac{T_0}{p_0}\left[\left(1 - \frac{\gamma z}{T_0}\right)^{\frac{R\gamma - g}{R\gamma}} - 1\right] \tag{8}$$

or, expanding in a series and rejecting the terms of higher than second order,

$$\delta z = -\frac{\delta\pi}{k}\cdot\frac{z}{p_0}\left(1+\frac{1}{2}\frac{gz}{RT_0}\right). \tag{9}$$

Formulas (8) and (9) show, that the individual particle descends ($\delta z < 0$), if mass is added to the upper atmospheric layers ($\delta\pi > 0$), and ascends ($\delta z > 0$), if air is expelled at high altitudes ($\delta\pi < 0$). These displacements of the individual particle are greater, the higher its altitude and the smaller the polytropic index. Thus, for example, at an adiabatic transmission of pressure variations (k = 1.4) the vertical displacement is 1.4 times smaller than for isothermic transmission (k = 1.0). This is explained by the fact that in adiabatic compression the whole air column is heated and expands, diminishing the vertical displacement of the individual air particles as compared with the isothermic compression, where no such effect exists.

Substituting (7), (8) and (9) in (2) we obtain for the local variation:

$$\begin{aligned}
\delta_e p &= \delta\pi\left[1-\frac{gp}{k}\int_0^z\frac{dz}{p}\right],\\
\delta_e p &= \delta\pi\left[1+\frac{1}{k}\frac{g}{R\gamma-g}\cdot\frac{\rho RT_0}{p_0}\left\{\left(1-\frac{\gamma z}{T_0}\right)^{\frac{R\gamma-g}{R\gamma}}-1\right\}\right],\\
\delta_e p &= \delta\pi\left[1-\frac{1}{k}\frac{gpz}{p_0}\left(1+\frac{1}{2}\frac{gz}{RT_0}\right)\right].
\end{aligned} \tag{10}$$

Formulas (10) show that the local pressure variation attains its maximum at ground level, corresponding to full advection, and decreases with altitude. The local pressure variations at one and the same altitude will differ, depending on the specific mechanism of transmission of pressure variations. Thus, for instance, for an adiabatic transmission the local pressure variation will be greater than for an isothermic transmission.

We now calculate the individual and local temperature variations. From the polytropic equation we obtain:

$$\delta_i T = \frac{k-1}{k}\cdot\frac{T}{p}\delta\pi. \tag{11}$$

Substituting (11) in (3) and taking into account (7), (8) and (9), we get

$$\begin{aligned}
\delta_e T &= \frac{k-1}{k}\cdot\frac{T}{p}\cdot\delta\pi\left[1-\frac{\gamma}{k-1}\cdot\frac{p}{T}\int_0^z\frac{dz}{p}\right],\\
\delta_e T &= \frac{k-1}{k}\cdot\frac{T}{p}\cdot\delta\pi\left[1+\frac{1}{k-1}\frac{R\gamma}{R\gamma-g}\left\{1-\left(\frac{T}{T_0}\right)^{\frac{g-R\gamma}{R\gamma}}\right\}\right],\\
\delta_e T &= \frac{k-1}{k}\cdot\frac{T}{p}\cdot\delta\pi\left[1-\frac{\gamma}{k-1}\cdot\frac{p}{T}\cdot\frac{z}{p_0}\left(1+\frac{1}{2}\frac{gz}{RT_0}\right)\right].
\end{aligned} \tag{12}$$

The derived formulas enable the calculation of the vertical displacement of the particle and the variations in pressure and temperature at the given level, due to advection at great altitudes, if the initial distributions of pressure and temperature are known. In practice it is more convenient to use equations (9), (10) and (11) in the following particular form, assuming the state of the air column to vary adiabatically and introducing the mean barometric temperature T_m:

$$\begin{aligned}
\delta z &= -\frac{\delta\pi}{\varkappa}\cdot\frac{RT_m}{g}\left(\frac{1}{p}-\frac{1}{p_0}\right),\\
\delta_e p &= \delta\pi\left[1-\frac{1}{\varkappa}\cdot\frac{T_m}{T}\left(1-\frac{p}{p_0}\right)\right],\\
\delta_e T &= \frac{R}{c_p}\cdot\frac{T}{p}\cdot\delta\pi\left[1-\frac{\gamma c_v}{g}\frac{T_m}{T}\left(1-\frac{p}{p_0}\right)\right].
\end{aligned} \tag{13}$$

§ 3. The Tendency Equation

The local variation of barometric pressure within 3 hours is known in meteorology as the barometric tendency.

The tendency equation is obtained by combining the equation of continuity with the third of the atmospheric flow equations taken, as before, in the form of a hydrostatic equation.

One may obtain the impression that the continuity equation in a nonstationary state furnishes a means for the determination of pressure variation with time from the spatial distribution of velocity and density. However, this is not easily carried out, since for this purpose one would have to know the distribution of the meteorological elements much more precisely than observation permits. In view of the importance of this question we will discuss it in greater detail.

The pressure p at a certain level h represents to a high degree of accuracy the weight per unit area of the air mass above this level

$$p = \int_h^\infty g\rho dz. \tag{1}$$

Hence, one can express the local pressure variation by the mass variation:

$$\left(\frac{\partial p}{\partial t}\right)_h = \int_h^\infty g \frac{\partial \rho}{\partial t} dz. \tag{2}$$

Replacing $\frac{\partial \rho}{\partial t}$ in (2) by its equivalent expression according to the continuity equation, we obtain

$$\left(\frac{\partial p}{\partial t}\right)_h = -\int_h^\infty g\rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz - \int_h^\infty g \left(\rho \frac{\partial w}{\partial z} + w \frac{\partial \rho}{\partial z}\right) dz - \\ - \int_h^\infty g \left(u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y}\right) dz. \tag{3}$$

Calculation of these improper integrals may cause misunderstanding due to the vertical velocity being undefined at infinity, and also in connection with the altitude changes of the force of gravity.

Therefore, it is physically more justified to consider not the entire atmosphere, but a certain layer between the levels h_0 and h_1. Ignoring the variation of g with altitude, we then have:

$$\frac{\partial p_0}{\partial t} = \frac{\partial p_1}{\partial t} - g \int_{h_0}^{h_1} \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz + g(\rho_0 w_0 - \rho_1 w_1) - g \int_{h_0}^{h_1} \left(u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y}\right) dz. \tag{4}$$

This equation, known as the tendency equation, shows that the pressure variation at some fixed point at the level h_0 is determined by five factors:

1) the change of pressure at the upper level, caused by processes occurring above that level and equal to

$$\Delta p_1 = \frac{\partial p_1}{\partial t}; \tag{5}$$

2) the horizontal velocity divergence in the $h_0 - h_1$ layer

$$\Delta p_2 = -g \int_{h_0}^{h_1} \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz, \tag{6}$$

whereby positive divergence corresponds to a pressure drop at the h_0 level, and a negative divergence (or convergence) corresponds to a rise in pressure;

3) the vertical transport of air through the lower boundary h_0 of the layer

$$\Delta p_3 = g\rho_0 w_0, \tag{7}$$

whereby an upward flow ($w_0 > 0$) causes a pressure increase at the h_0 level and a downward flow ($w_0 < 0$) causes a pressure decrease.

4) by the vertical transport of air through the upper boundary h_1 of the layer

$$\Delta p_4 = -g\rho_1 w_1, \tag{8}$$

whereby the upward flow ($w_1 > 0$) at this level causes a decrease in pressure at the h_0 level, and a downward flow ($w_1 < 0$) causes an increase of that pressure;

5) the horizontal transport or advection of masses in the $h_0 - h_1$ layer

$$\Delta p_5 = -g \int_{h_0}^{h_1} \left(u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} \right) dz, \tag{9}$$

whereby the substitution of a cold by a warm mass causes a pressure drop and vice versa.

We will now estimate the magnitude of every term of equation (4). Let us take 3 hours as our unit time interval; $\Delta p_1 \ldots \Delta p_5$ will then represent barometric tendencies, due to one of the above listed factors. If the mass of air in the $h_0 - h_1$ layer does not change, a pressure variation aloft causes exactly the same variation below, i.e., in that case the pressure variation is transmitted in the same manner as in an incompressible fluid.

We further introduce the mean values of the derivatives of the velocity components:

$$\frac{\partial u}{\partial x} = \frac{\int_{h_0}^{h_1} \rho \frac{\partial u}{\partial x} dz}{\int_{h_0}^{h_1} \rho dz} = \frac{g \int_{h_0}^{h_1} \rho \frac{\partial u}{\partial x} dz}{p}; \quad \frac{\partial v}{\partial y} = \frac{g \int_{h_0}^{h_1} \rho \frac{\partial v}{\partial y} dz}{p}, \tag{10}$$

where

$$p = g \int_{h_0}^{h_1} \rho \, dz = p_0 - p_1 \tag{11}$$

(in order to keep the number of symbols to a minimum we do not introduce new designations for the mean velocities). From (6) we obtain:

$$\Delta p_2 = -p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \tag{12}$$

The relation (12) shows, that the pressure variation due to velocity divergence is directly proportional to the divergence of the mean velocity and also to the pressure difference between the boundaries of the layer, i.e., to the mass of the layer. Let us express the horizontal divergence of the velocity by the derivatives of the velocity magnitude and direction.

Writing $u = c\cos\alpha$, $v = c\sin\alpha$, where c is the magnitude of the velocity and α is the angle between the velocity and the positive direction of the x-axis, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial c}{\partial x}\cos\alpha - c\sin\alpha\frac{\partial\alpha}{\partial x},$$
$$\frac{\partial v}{\partial y} = \frac{\partial c}{\partial y}\sin\alpha + c\cos\alpha\frac{\partial\alpha}{\partial y}.$$

We now direct the x-axis along the current s and the y-axis parallel to its normal n. Then the angle α vanishes and we obtain:

$$\frac{\partial u}{\partial x} = \frac{\partial c}{\partial s};\ \frac{\partial v}{\partial y} = c\frac{\partial\alpha}{\partial n}.$$

Consequently,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial c}{\partial s} + c\frac{\partial\alpha}{\partial n}. \tag{13}$$

Thus, the divergence of the velocity is expressed by the derivatives of the velocity magnitude and direction, i.e., the rotation of the velocity towards the normal to the current. Then we have

$$\Delta p_2 = -p\left(\frac{\partial c}{\partial s} + c\frac{\partial\alpha}{\partial n}\right) = \Delta p_2' + \Delta p_2'', \tag{14}$$

where

$$\Delta p_2' = -p\frac{\partial c}{\partial s};\ \Delta p_2'' = -pc\frac{\partial\alpha}{\partial n}. \tag{15}$$

Assuming that $p = 100$ mb, $\Delta s = 100$ km, we obtain:

$$\Delta p_2' = -10^2\cdot\frac{\Delta c}{10^5}\cdot 60\cdot 60\cdot 3 = -10\Delta c \quad \text{or} \quad \Delta c = -0{,}1\Delta p_2'.$$

Thus in order to obtain a pressure variation of 1 mb in 3 hours it suffices to have the mean velocity in the lower 1 km layer vary by 100 cm/sec per 100 km.

Similarly, assuming $p = 1000$ mb, $c = 10$ m/sec, $\Delta n = 100$ km, we obtain:

$$\Delta p_2'' = -10^2\cdot 10\,\frac{\Delta\alpha}{10^5}\cdot 60\cdot 60\cdot 3 \cong -100\,\Delta\alpha \quad \text{or} \quad \Delta\alpha = 0{,}01\Delta p_2''.$$

Consequently, a pressure variation of 1 mb/3 h obtains for a variation of the mean wind direction in the lower 1 km layer by 0.01 radian, or approximately 0.5°/100 km.

These calculations show, that even a very small velocity divergence causes palpable local pressure variations.

We will further estimate the influence of vertical flows on the magnitude of the barometric tendency.

Assuming $g = 10^3$ cm/sec^2, $\rho_0 = 10^{-3}$ g/cm^3, we obtain

$$\Delta p_3 = 10^3 \times 10^{-3}\cdot w\cdot 60\cdot 60\cdot 3\cdot 10^{-3} = 10\cdot w \quad \text{or} \quad w = 0{,}1\Delta p_3 \text{ cm/sec}.$$

Thus, for a pressure variation of 1 mb per 3 h it suffices to have a vertical velocity of the order of 1 mm/sec.

Finally we will estimate the influence of mass advection. Calculating the average value of the integral in (9), we obtain:

$$\Delta p_5 = -\left(u\frac{\partial \eta}{\partial x} + v\frac{\partial v}{\partial y}\right) = -c\frac{\partial n}{\partial n}\cos\left(c, \frac{\partial v}{\partial n}\right), \qquad (16)$$

where we assume

$$u = \frac{\int_{h_0}^{h_1} u\frac{\partial \rho}{\partial x}\,dz}{\int_{h_0}^{h_1}\frac{\partial \rho}{\partial x}\,dz} = \frac{g\int_{h_0}^{h_1} u\frac{\partial \rho}{\partial x}\,dz}{\frac{\partial p}{\partial x}},$$

$$v = \frac{g\int_{h_0}^{h_1} v\frac{\partial \rho}{\partial y}\,dz}{\frac{\partial p}{\partial y}}; \quad p = -g\int_{h_0}^{h_1}\rho\,dz = p_0 - p_1.$$

Assuming

$$\cos\left(c, \frac{\partial p}{\partial n}\right) = 1, \quad c = 10\,\text{m/sec} \quad \Delta n = 100\,\text{km}$$

We find:

$$\Delta p_5 = -\frac{10\cdot\Delta p\cdot 60\cdot 60\cdot 3}{10^5} \cong -2\cdot\Delta p,$$

i.e., for the pressure to vary by 1 mb/3 h suffices a wind velocity of 5 m/sec, parallel to pressure gradient of 1 mb/100 km in the lower 1 km layer.

The above calculations show, that it is almost impossible to use the tendency equation for quantitative estimation of pressure variation, since the prerequisite would be a measurement of the meteorological elements with greater accuracy than is practically attainable. Therefore, we are usually restricted to the drawing of qualitative conclusions which follow from the tendency equation.

The difficulties encountered in the application of the tendency equation are obviously due to the integrals in formula (4) being extended over a layer of a very great thickness, a circumstance which renders the determination of the mean velocities unduly difficult. In this thick layer conditions may vary with altitude so irregularly that the effects of some layers may cancel out completely. Therefore, the net effect constituting the observed pressure variation is much smaller than could be expected. Thus, the local pressure variation may be regarded as a small difference between very large magnitudes. Precisely this is the root of the great difficulties encountered in prediction of pressure variations.

In the Soviet weather service pressure contour maps are widely used. It is therefore worthwhile to give a formula for the local pressure variation based on such maps. Differentiating the individual derivative $\frac{dp}{dt}$ with respect to z:

$$\frac{\partial}{\partial z}\left(\frac{dp}{dt}\right) = \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) =$$

$$= \frac{\partial^2 p}{\partial z\partial t} + u\frac{\partial^2 p}{\partial z\partial x} + v\frac{\partial^2 p}{\partial z\partial y} + w\frac{\partial^2 p}{\partial z^2} + \frac{\partial u}{\partial z}\cdot\frac{\partial p}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial p}{\partial y} + \frac{\partial w}{\partial z}\cdot\frac{\partial p}{\partial z} =$$

$$= \frac{d}{dt}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial u}{\partial z}\cdot\frac{\partial p}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial p}{\partial y} + \frac{\partial w}{\partial z}\cdot\frac{\partial p}{\partial z}. \qquad (17)$$

Let us eliminate $\frac{\partial w}{\partial z}$ between (17) and the continuity equation, which can be written in the form

$$\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{d\rho}{dt} - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \qquad (18)$$

We then obtain:

$$\frac{\partial}{\partial z}\left(\frac{dp}{dt}\right)=\frac{d}{dt}\left(\frac{\partial p}{\partial z}\right)-\frac{1}{\rho}\frac{\partial p}{\partial z}\frac{d\rho}{dt}-\frac{\partial p}{\partial z}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial p}{\partial x}-\frac{\partial p}{\partial z}\cdot\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial p}{\partial y} \quad (19)$$

The first two terms on the right-hand side can be transformed as follows:

$$\frac{d}{dt}\left(\frac{\partial p}{\partial z}\right)-\frac{1}{\rho}\frac{\partial p}{\partial z}\cdot\frac{d\rho}{dt}=\rho\frac{d}{dt}\left[\frac{1}{\rho}\frac{\partial p}{\partial z}\right]=-\rho\frac{dg}{dt}=0, \quad (20)$$

disregarding the time variation of the acceleration of gravity. Putting $\frac{\partial p}{\partial z}$ before the brackets on the right side we obtain:

$$\frac{\partial}{\partial z}\left(\frac{dp}{dt}\right)=-\frac{\partial p}{\partial z}\left[\frac{\partial u}{\partial x}-\frac{\partial u}{\partial z}\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial z}}+\frac{\partial v}{\partial y}-\frac{\partial v}{\partial z}\frac{\frac{\partial p}{\partial y}}{\frac{\partial p}{\partial z}}\right]. \quad (21)$$

Let us recall that the ratios $\frac{\partial p}{\partial x}:\frac{\partial p}{\partial z}$ and $\frac{\partial p}{\partial y}:\frac{\partial p}{\partial z}$ are equal to the tangents of the angles β_x, β_y, formed by the isobaric surface with the x and y axes. Therefore, (21) will assume the form

$$\frac{\partial}{\partial z}\left(\frac{dp}{dt}\right)=-\frac{\partial p}{\partial z}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\left(\frac{dz}{dx}\right)_p+\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\left(\frac{dz}{dy}\right)_p\right]. \quad (22)$$

Let us further note, that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\left(\frac{dz}{dx}\right)_p=\left(\frac{\partial u}{\partial x}\right)_p;\quad \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\left(\frac{dz}{dy}\right)_p=\left(\frac{\partial v}{\partial y}\right)_p, \quad (23)$$

where the index p indicates that the values of the functions u and v are taken not on the surface $z=\text{const.}$, but on the isobaric surface $p=\text{const.}$

Equation (22) will thus assume the form

$$\frac{\partial}{\partial z}\left(\frac{dp}{dt}\right)=-\frac{\partial p}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_p. \quad (24)$$

Integrating (24) from h_1 to h_2, we obtain

$$\frac{dp_2}{dt}-\frac{dp_1}{dt}=-\int_{p_1}^{p_2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_p dp. \quad (25)$$

Solving this equation for the local derivative $\frac{\partial p_1}{\partial t}$ and replacing the integral by its mean value, we finally find

$$\frac{\partial p_1}{\partial t}=\frac{\partial p_2}{\partial t}-\left(u_1\frac{\partial p_1}{\partial x}+v_1\frac{\partial p_1}{\partial y}\right)+\left(u_2\frac{\partial p_2}{\partial x}+v_2\frac{\partial p_2}{\partial y}\right)+$$
$$+g(\rho_1 w_1-\rho_2 w_2)-p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)p, \quad (26)$$

where, as before, $p=p_1-p_2$, and u and v are determined by the formulas

$$\frac{\partial u}{\partial x}=\frac{\int_{p_0}^{p_1}\frac{\partial u}{\partial x}dp}{p};\quad \frac{\partial v}{\partial y}=\frac{\int_{p_0}^{p_1}\frac{\partial v}{\partial y}dp}{p}. \quad (27)$$

Thus, the local pressure variation at the lower level h_1 is determined by the following factors:

1) the variation $\Delta p_1 = \frac{\partial p_2}{\partial t}$ of the pressure at the upper level h_2, whereby this variation is transmitted downwards without any additional change in case it is not accompanied by inward or outward air flux either in a vertical or in a horizontal direction;

2) advection of pressure at the lower level h_1

$$\Delta p_2 = -\left(u_1 \frac{\partial p_1}{\partial x} + v_1 \frac{\partial p_1}{\partial y}\right) = -c_1 \frac{\partial p_1}{\partial u} \cos\left(c_1 \frac{\partial p}{\partial n}\right), \qquad (28)$$

whereby the pressure increases if the angle between the wind velocity $\boldsymbol{c}$ and the pressure gradient $\left(-\frac{\partial p}{\partial n}\right)$ is acute, and decreases if it is obtuse; if the angle is right the pressure remains constant;

3) advection of pressure at the upper level

$$\Delta p_3 = \left(u_2 \frac{\partial p_2}{\partial x} + v_2 \frac{\partial p_2}{\partial y}\right) = c_2 \frac{\partial p_2}{\partial n} \cos\left(c_2 \frac{\partial p_2}{\partial n}\right), \qquad (29)$$

whereby the pressure below increases, if the angle between the wind velocity $\boldsymbol{c}$ and the pressure gradient $\left(-\frac{\partial p}{\partial n}\right)$ is obtuse, and decreases if the angle is acute; the pressure will not change if the angle is right, i.e., if the wind blows along the isobar;

4) vertical transport of air masses across the horizontal h_1-level

$$\Delta p_4 = g \rho_1 w_1,$$

whereby an ascending current $(w_1 > 0)$ is accompanied by pressure rise at the h_1 level, and a descending current — by a pressure drop;

5) vertical transport of air masses across the horizontal h_2 level

$$\Delta p_5 = -g \rho_2 w_2,$$

whereby an ascending current of air at the h_2 level is accompanied by a pressure drop at the h_1 level, and a descending current — by a pressure rise;

6) horizontal divergence of the velocity at a certain intermediate isobaric surface, lying between the isobaric surfaces $p_1 = \text{const.}$ and $p_2 = \text{const.}$

$$\Delta p_6 = \int_{p_1}^{p_2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p dp = -p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p = -p \left(\frac{\partial c}{\partial s} + c \frac{\partial \alpha}{\partial n}\right)_p, \qquad (30)$$

whereby the velocity divergence causes a decrease of pressure, and its convergence (negative divergence) causes a pressure increase.

Chapter XIII

ATMOSPHERIC INTERFACES

§1. Surfaces of Discontinuity. Discontinuities of the Zero and First Orders

Careful analysis of synoptic charts and of vertical cross sections of the atmosphere leads to the conclusion that among the regions containing uniform air masses, characterized by slow and gradual variation of the meteorological elements, narrow transition zones often occur in the atmosphere, separating the uniform air masses, in which the majority of meteorological elements undergo rapid variations.

These transition zones are usually so narrow, that on small-scale maps they are represented schematically by lines at which the meteorological elements change abruptly, i.e., have a discontinuity. Such schematization, widely used in synoptic practice, is very convenient for the theoretical study of the transition zones.

Thus, instead of transition zones with rapid variations of the meteorological elements we shall discuss discontinuity surfaces of these elements, separating two air masses. The results obtained will be in closer agreement with reality the sharper the change of the meteorological elements in the transition zone, which we schematize as a surface of discontinuity.

Figure 106 shows a homogeneous scalar field, the field of temperature. At both boundaries of the layer of rapid temperature variation the isotherms break abruptly but are not severed. If the temperature is measured at two infinitesimally close points of this field, the difference between the values obtained will be infinitesimal too. However, if a discontinuity occurs in the temperature field, any isotherm approaching this surface will merge with it along a certain segment and then depart from the other side of the surface at a point not coinciding with the point of entry. The temperature difference between two infinitesimally close points, located on opposite sides of the surface of discontinuity, will be finite.

Figure 107 shows a homogeneous vector field, the field of velocity. Passing a layer of rapid velocity variation the streamlines bend, and passing a surface of discontinuity – they break.

As will be shown later the surface of discontinuity, so important in meteorology, is at the same time also an interface between two air masses. Examples of such surfaces are : frontal surfaces, inversions and the tropopause. All these surfaces have the common property of moving together with the air masses they separate, neither overtaking them nor lagging behind.

Anyhow, the particles of one of the air masses will always lie to one side of the surface of discontinuity and the particles located on the other side will always belong to the other air mass. It is natural to assume that the discontinuity surface remains composed of the original air particles for a considerably long period, and the particles move in space together with the surface.

Besides these discontinuities that separate air masses, other types of discontinuity surfaces might occur, which have quite different properties. As examples of such surfaces may serve the discontinuities following an artillery shell in its flight and blast or sound waves. These surfaces of discontinuity are characterized by their velocity of propagation which does not depend on the velocity of the air particles but surpasses it greatly; obviously the air particles composing these surfaces are constantly replaced by new particles.

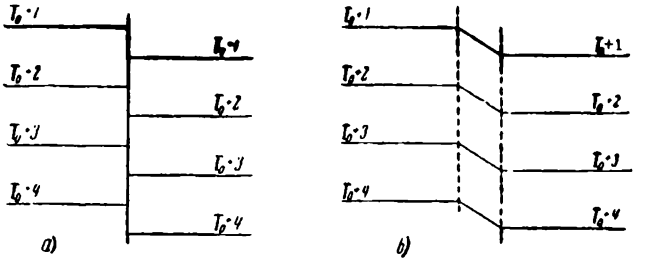

Figure 106. A homogeneous scalar field of temperature

a) temperature discontinuity; b) layer of rapid temperature variation.

From a mathematical point of view, the discontinuities may possess different orders, depending on whether the meteorological element itself or its derivatives are discontinuous.

Let Φ be some definite meteorological element. If on a certain surface the function Φ itself is discontinuous, this surface is called a discontinuity surface of zero order. If, however, Φ varies continuously right through the surface and only a first derivative of Φ with respect to any argument is discontinuous, we have a discontinuity surface of the first order. Discontinuities of the second and higher orders may be similarly defined. In the atmosphere we usually deal either with the meteorological elements themselves, or with their gradients. We shall therefore limit our discussion to zero and first order discontinuities.

We shall demonstrate in the following sections, that frontal surfaces are discontinuity surfaces of the first order with respect to the pressure, and of zero order with respect to the temperature, density, specific volume and wind velocity.

The tropopause, separating the air masses of the troposphere from the stratosphere, is a first order discontinuity surface for the temperature, density, specific volume and wind velocity and for the pressure it is no discontinuity surface at all, either of zero or of the first order.

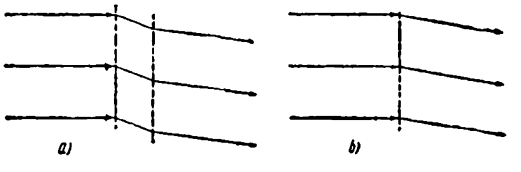

Figure 107. A homogeneous vector field of velocity

a) layer of rapid variations of velocity; b) velocity discontinuity.

§ 2. Dynamic and Kinematic Conditions

It is usually considered self-evident, that at the meteorologically significant surfaces of discontinuity the pressure and the normal component of the velocity are continuous. In view of the great importance of this point, we will explain it in greater detail.

For the sake of simplicity let us consider a one-dimensional flow of an inviscid

compressible fluid in a cylindrical tube of unit cross section. Let us direct the x-axis along the axis of the tube. We regard the velocity of any point of the fluid as parallel to the x-axis. Since the motion is one-dimensional by definition, all the quantities – the density ρ, pressure p, velocity u, depend only on x and t.

Generally, the discontinuity surface progresses in the tube with velocity c_F in relation to the stationary system of coordinates, and with velocity c_R in relation to the fluid and one should, generally speaking, assume that c_R has different values on opposite sides of the discontinuity surface.

In the absence of discontinuities, the one-dimensional fluid flow can be described by two equations – Euler's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + X \qquad (1)$$

and the equation of continuity

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0. \qquad (2)$$

In the presence of a discontinuity surface, the flow equations should contain its velocity of propagation, c_R or c_F, in addition to u, ρ, and p, which generally have different values on the opposite sides of the discontinuity surface (u_1, ρ_1, p_1 to its right and u_2, ρ_2, p_2 to its left).

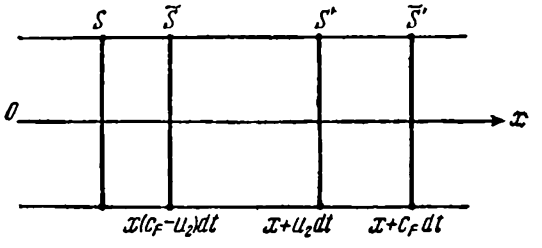

Figure 108. Derivation of the analogue of the equation of continuity

With the aid of Figure 108 it is easy to construct the following table, describing the motion of the discontinuity surface:

Time instant	t	$t+\Delta t$
Coordinate of discontinuity surface	x	$x+c_F dt$
Coordinate of particles composing the discontinuity surface S	x	$x+u_2\Delta t$
Coordinate of particles composing the discontinuity surface S'	$x+c_F\Delta t-u_1\Delta t$	$x+c_F\Delta t$
Mass of the fluid enclosed between the surfaces on which the considered particles lie	$\rho_1(c_F-u_1)\Delta t$	$\rho_2(c_F-u_2)\Delta t$
Linear momentum of the same fluid mass	$\rho_1(c_F-u_1)\Delta t\cdot u_2$	$\rho_2(c_F-u_2)\Delta t\cdot u_1$

Since the mass enclosed between surfaces remaining constituted of the same fluid particles does not change, we can write:

$$\rho_1(c_F-u_1)=\rho_2(c_F-u_2) \qquad (3)$$

or, introducing the velocity of propagation of the discontinuity surface c_2

$$\rho_1 c_{R1} = \rho_2 c_{R2}.$$

Equation (4), connecting the velocity of propagation of the discontinuity surface with the density of the fluid, is evidently an analogue of the equation of continuity.

In order to obtain the equation of motion of the fluid in the presence of a discontinuity surface, in analogy with Euler's equation, we shall use the law of conservation of momentum. The change of momentum equals (according to (3))

$$\rho_2 (c_F - u_2)\,\Delta t \cdot u_1 - \rho_1 (c_F - u_1)\,\Delta t \cdot u_2 = \rho_2 (c_F - u_2)(u_1 - u_2)\,\Delta t.$$

The acting forces are the pressure on the cross sections normal to the axis of the tube S and $\tilde{S}$ and the external forces.

Since the considered volume is small, the impulse of the external forces may be ignored, as their sum in this case is a quantity of the second order. Therefore, we may take the total impulse as equal to

$$(p_1 - p_2)\,\Delta t.$$

Consequently, from the conservation of momentum we get:

$$\rho_2 (c_F - u_2)(u_2 - u_1) = p_1 - p_2. \qquad (5)$$

This equation is the analogue of Euler's equation. Substituting

$$u_1 = c_F - c_{R1} = c_F - \frac{\rho_2}{\rho_1} c_{R2},$$
$$u_2 = c_F - c_{R2} \qquad (6)$$

in (5), we obtain:

$$c_{R2} = \sqrt{\frac{\rho_1}{\rho_2}\frac{p_1 - p_2}{\rho_1 - \rho_2}},$$
$$c_{R1} = \sqrt{\frac{\rho_2}{\rho_1}\frac{p_2 - p_1}{\rho_2 - \rho_1}}. \qquad (7)$$

Let us evaluate the order of magnitude of the velocity of propagation c_R of the discontinuity surface, i.e., its velocity relative to the fluid particles.

For the sake of simplicity, we assume that the fluid is barotropic, i.e., that

$$p = f(\rho).$$

By the mean value theorem we have:

$$\frac{p_1 - p_2}{\rho_1 - \rho_2} = \frac{f(\rho_1) - f(\rho_2)}{\rho_1 - \rho_2} = f'(\tilde{\rho}),$$

where $\tilde{\rho}$ is a certain value of ρ, lying between ρ_1 and ρ_2.

In an adiabatic process

$$p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\varkappa},$$

where $\varkappa = \frac{C_p}{C_v}$. Then

$$f'(\tilde{\rho}) = \varkappa \frac{\tilde{p}}{\tilde{\rho}}$$

$$c_{R2} \approx \sqrt{\frac{\varkappa p}{\rho}}. \qquad (8)$$

However, the expression on the right-hand side is the well known formula for the velocity of sound, which equals 332 m/sec under normal atmospheric conditions.

Thus, the velocity of propagation of the discontinuity surface is a magnitude of the order of the velocity of sound. But such velocities have no meteorological significance. Therefore, disregarding discontinuity surfaces which propagate with velocities comparable with the sound velocity, we should assume

$$c_{R2} = c_{R1} = 0,$$

i.e., we ought to consider only such discontinuity surfaces as move together with the fluid and therefore permanently consist of the same particles. Thus, the discontinuity surfaces considered in meteorology are at the same time interfaces, continually separating the same masses of air. These surfaces move considerably slower than sound.

From the condition that at the interface

$$c_R = 0,$$

follows

$$p_1 = p_2 = p.$$

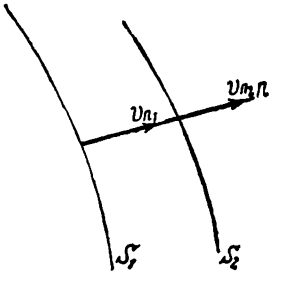

Figure 109. Heightening and moderation of the discontinuity in the transition zone

This dynamic condition is always fulfilled for interfaces.

If $c_R = 0$, then according to (6) we have

$$c_F = u_1; \quad c_F = u_2.$$

Consequently,

$$u_1 = u_2 \quad \text{or} \quad v_{n1} = v_{n2}, \tag{10}$$

i.e., the normal component of velocity changes continuously across the discontinuity surface.

Since the velocity v has a jump discontinuity at the interface, but its normal component remains continuous, it is obvious that the tangential component v_t must be discontinuous.

In conclusion, we will interpret the significance of equation (10) for a layer of rapid variation of meteorological elements.

Let us regard the transition zone as bounded by two surfaces S_1 and S_2 (Figure

109). Such boundaries are arbitrary of course, having no real existence. However, this circumstance does not interfere with our reasoning. It is obvious that if

$$v_{n1} = v_{n2},$$

the transition zone in which the meteorological elements vary rapidly moves without changing its breadth.

If $v_{n1} > v_{n2} > 0$, the boundary S_1 moves faster than S_2; the transition zone narrows and consequently the discontinuity becomes more acute.

If $v_{n1} < v_{n2}$, the transition zone broadens and the discontinuity is lessened.

§ 3. The Slope of an Interface

The most important synoptic applications of the theory of discontinuity surfaces involve the slope of the interface. We will first consider the simple case of a stationary interface. Let a warm air mass M_1 and a cold mass M_2 be in mutual contact. Assuming for the sake of simplicity that the interface between them is planar, we will show that the stationary slope of the surface in relation to the horizon cannot assume arbitrary values, since it bears a definite relationship to the velocity and density discontinuities.

We first assume that both masses do not move with respect to the earth. Obviously, the system consisting of these two masses M_1 and M_2 will be in stable equilibrium only when the interface is a horizontal plane and the cold mass is below the warm.

An interface between two stationary masses cannot be vertical. Although each one of the masses taken separately is in equilibrium, the whole system consisting of the two masses with a vertical interface will not be in equilibrium. Even if the pressures are initially equal, a pressure difference will appear below, at the interface of the neighboring air masses, as a result of the unequal altitude variations of the pressure, resulting in a flow of warm air above the cold mass and a flow of cold air underneath the warm mass. Therefore, the cold mass M_2 will finally be underneath the warm mass M_1.

Even if only one mass is in motion, the interface will not stay horizontal, but will be inclined at a certain very small angle to the horizon, which under certain special conditions will not change with time.

Let us now turn to the derivation of the formulas for the slope angle of the interface.

We have shown above, that the pressure should remain continuous on the interface. Therefore the equation of the interface may be written in the form

$$p_1 - p_2 = 0 \qquad (1)$$

or in differential form

$$\left(\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x}\right)dx + \left(\frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y}\right)dy + \left(\frac{\partial p_1}{\partial z} - \frac{\partial p_2}{\partial z}\right)dz = 0, \qquad (2)$$

hence, denoting by α_x and α_y the angles formed by the interface with the x and y axes, and by θ the angle formed between the x-axis and the tangent to the front, i.e., the line of intersection of the interface and the surface of the earth, we

obtain

$$\left.\begin{aligned} \operatorname{tg}\alpha_x &= \frac{dz}{dx} = -\frac{\frac{\partial p_1}{dx} - \frac{\partial p_2}{\partial x}}{\frac{\partial p_1}{\partial z} - \frac{\partial p_2}{\partial z}}, \\ \operatorname{tg}\alpha_y &= \frac{dz}{dy} = -\frac{\frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y}}{\frac{\partial p_1}{\partial z} - \frac{\partial p_2}{\partial z}}, \\ \operatorname{tg}\theta &= \frac{dy}{dx} = -\frac{\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x}}{\frac{\partial p_1}{dy} - \frac{\partial p_2}{dy}}. \end{aligned}\right\} \qquad (3)$$

We orient the coordinate axes so that the y-axis will be tangent to the front. Then $\operatorname{tg}\theta = \infty$ whence

$$\frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y} = 0 \qquad (4)$$

therefore, $\operatorname{tg}\alpha_y = 0$. We obtain the following formula for the angle α which the interface makes with the horizon

$$\operatorname{tg}\alpha = -\frac{\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x}}{\frac{\partial p_1}{\partial z} - \frac{\partial p_2}{\partial z}} \qquad (5)$$

Thus the slope of the interface is determined solely by the pressure distribution at the front, whereby the greater the discontinuity in the component of the pressure gradient normal to the front and the smaller the discontinuity in its vertical component—the greater is the slope angle.

Let us note, that equation (5) holds generally and rigorously. In the following we shall derive a number of formulas not strictly rigorous, but practically convenient.

Instead of the vertical component of the pressure gradient we will use the density; according to the basic hydrostatic equation, we have

$$\frac{\partial p_1}{\partial z} = -g\rho_1; \qquad \frac{\partial p_2}{\partial z} = -g\rho_2.$$

Instead of the horizontal component of the pressure gradient we will use the slope angle β of the isobaric surface. Since

$$\frac{\partial p_1}{\partial x} = -\frac{\partial p_1}{\partial z}\left(\frac{dz}{dx}\right)_{p_1} = g\rho_1 \operatorname{tg}\beta_1,$$
$$\frac{\partial p_2}{\partial x} = -\frac{\partial p_2}{\partial z}\left(\frac{dz}{dx}\right)_{p_2} = g\rho_2 \operatorname{tg}\beta_2,$$

therefore

$$\operatorname{tg}\alpha = \frac{\rho_2 \operatorname{tg}\beta_2 - \rho_1 \operatorname{tg}\beta_1}{\rho_2 - \rho_1} \qquad (6)$$

Usually $\operatorname{tg}\alpha \gg \operatorname{tg}\beta$, and consequently the interface rises much more steeply than the isobaric surfaces. If the air motion is geostrophic, it is convenient to use the temperature and wind velocity at the interface to express its slope. The coordinates will be again so chosen, that the y-axis lies parallel to the front. The geostrophic equations for the warm and cold masses will have the form

$$-2\omega \sin\varphi v_1 = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}, \qquad -2\omega \sin\varphi v_2 = -\frac{1}{\rho_2}\frac{\partial p_2}{\partial x},$$

$$2\omega \sin\varphi u_1 = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial y}, \qquad 2\omega \sin\varphi u_2 = -\frac{1}{\rho_2}\frac{\partial p_2}{\partial y}.$$

$$\frac{1}{\rho_1}\frac{\partial p_1}{\partial z} - g = 0, \qquad -\frac{1}{\rho_2}\frac{\partial p_2}{\partial z} - g = 0.$$

Substituting these equations in formula (5) we get:

$$\operatorname{tg}\alpha = \frac{2\omega \sin\varphi}{g}\,\frac{v_2\rho_2 - v_1\rho_1}{\rho_2 - \rho_1}. \qquad (7)$$

Formula (7) is not very convenient, since the density is not measured in meteorology. Eliminating the density through the equation of state $\rho = \frac{p}{RT}$ and availing ourselves of the dynamic condition at the discontinuity surface $p_1 = p_2$, we obtain the so called Margules' formula

$$\operatorname{tg}\alpha = \frac{2\omega \sin\varphi}{g}\,\frac{T_1 v_2 - T_2 v_1}{T_1 - T_2}. \qquad (8)$$

The air humidity can be accounted for by introducing the virtual temperature into formula (8).

We shall derive a formula for the approximate estimation of the slope of the interface, introducing the differences of the temperature and of the wind velocity

$$\Delta T = T_1 - T_2; \quad \Delta v = v_1 - v_2.$$

Then

$$\operatorname{tg}\alpha = \frac{2\omega \sin\varphi}{g}\left(v_2 - T_2\frac{\Delta v}{\Delta T}\right) \qquad (9)$$

or

$$\operatorname{tg}\alpha = \frac{2\omega \sin\varphi}{g}\left(v_1 - T_1\frac{\Delta v}{\Delta T}\right). \qquad (10)$$

Thus, the slope of the interface depends on the slope of the isobaric surface in any of the air masses considered and upon the discontinuities of wind velocity and temperature.* Since, as we shall later see, the slope of the isobaric surface is considerably smaller than that of the interface, the first term in the brackets in formulas (9) and (10) may be dropped. Then we have

$$\operatorname{tg}\alpha \cong -\frac{2\omega \sin\varphi}{g} \cdot T \cdot \frac{\Delta v}{\Delta T}, \qquad (11)$$

where T is the mean temperature.

Assuming $\omega \cong 7.3 \cdot 10^{-5}$, $T = 275^{\circ}$, $g \cong 10$ m/sec^2, we obtain the following formula for the slope angle α:

$$\operatorname{tg}\alpha = 4 \cdot 10^{-3} \sin\varphi \cdot \frac{\Delta v}{\Delta T}, \qquad (12)$$

where Δv is measured in m/sec and ΔT in degrees Celsius.

For a quick determination of the angle a nomogram can be used, calculated on the basis of formula (11). The magnitude of the slope angle of the interface is of the order of tens of minutes, rarely as high as 1 degree, while the slope of the isobaric surface is measured in tens of seconds, rarely reaching 2'–3'; therefore, under normal conditions the slope of the isobaric surface is about 30 times smaller than the slope of the interface. This justifies the assumption we made in deriving formula (11).

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* [Translator's note. The Russian text has "velocity and wind", an obvious misprint].

§ 4. Interfaces in the Pressure Field

We have shown above that if the y axis is parallel to the front, then

$$\frac{\partial p_1}{\partial y} = \frac{\partial p_2}{\partial y}. \qquad (1)$$

That means that the tangential component of the pressure gradient is continuous at the interface.

The normal component of the pressure gradient is discontinuous at the front,

$$\frac{\partial p_1}{\partial x} \gtrless \frac{\partial p_2}{\partial x}$$

depending on whether the angle α is acute or obtuse, i.e., whether the x-axis is directed from the (warm) mass of lesser density to the (cold) denser mass or vice versa.

If the angle α is acute, i.e., if the x-axis is directed from the warm to the cold air mass, we have

$$\frac{\partial p_1}{\partial x} < \frac{\partial p_2}{\partial x}, \qquad (2)$$

and if α is obtuse

$$\frac{\partial p_1}{\partial x} > \frac{\partial p_2}{\partial x}. \qquad (3)$$

This means, that the isobars at the front are refracted towards the higher pressure so that at the front the isobar has an infinitely large cyclonic curvature.

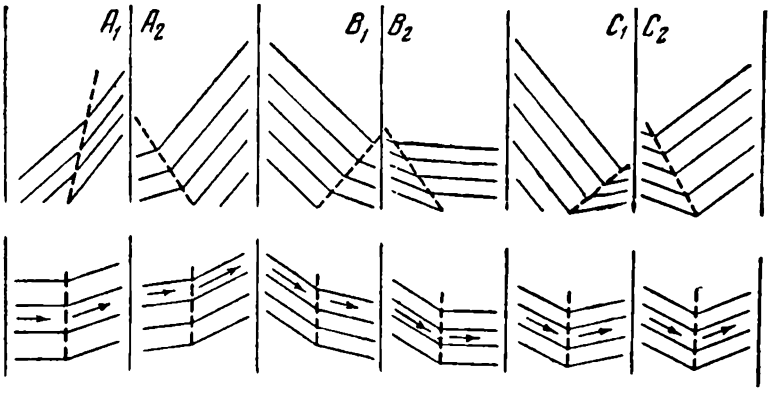

Figure 110. Location of the interface in the pressure field at a moving front
Upper sequence—vertical cross section;
Lower sequence—horizontal plan .

In the vicinity of the front one may find isobaric patterns as shown in Figures 110 and 111.

It also follows from (5) § 3, that the isobaric surfaces bend at the interface, forming a trough of an infinitely large curvature.

It should be pointed out that the slope of the isobaric surfaces is so small, that their refraction becomes noticeable only when the vertical scale used is many times larger than the horizontal one.

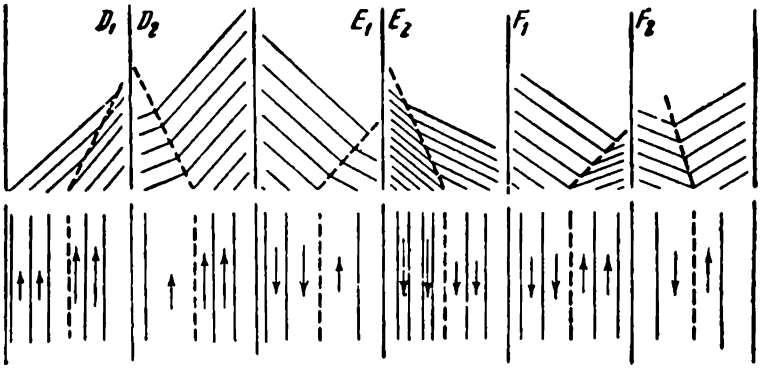

Figure 111. Location of the interface in a pressure field at a stationary front
Upper sequence—vertical cross section
Lower sequence—horizontal plan.

Different types of isobars and isobaric surfaces which may occur in the neighborhood of a front are listed in Table 45.

Table 45

Types of isobars and isobaric surfaces in the neighborhood of a front

Type	$\frac{\partial p}{\partial y}$	$\frac{\partial p}{\partial x}$	α and β
A_1	$\frac{\partial p_1}{\partial y}=\frac{\partial p_2}{\partial y}$	$\frac{\partial p_2}{\partial x}>\frac{\partial p_1}{\partial x}>0$	$0<\beta_1<\beta_2<\alpha<\frac{\pi}{2}$
A_2		$\frac{\partial p_1}{\partial x}>\frac{\partial p_2}{\partial x}>0$	$0<\beta_2<\beta_1<\frac{\pi}{2}<\alpha<\pi$
B_1		$\frac{\partial p_1}{\partial x}<\frac{\partial p_2}{\partial x}<0$	$0<\alpha<\frac{\pi}{2}<\beta_1<\beta_2<\pi$
B_2		$\frac{\partial p_2}{\partial x}<\frac{\partial p_1}{\partial x}<0$	$\frac{\pi}{2}<\alpha<\beta_2<\beta_1<\pi$
C_1		$\frac{\partial p_1}{\partial x}<0<\frac{\partial p_2}{\partial x}$	$0<\beta_2<\alpha<\frac{\pi}{2}<\beta_1<\pi$
C_2		$\frac{\partial p_1}{\partial x}>0>\frac{\partial p_2}{\partial x}$	$0<\beta_1<\frac{\pi}{2}<\alpha<\beta_2'<\pi$
D_1	$\frac{\partial p_1}{\partial y}=\frac{\partial p_2}{\partial y}=0$	$\frac{\partial p_2}{\partial x}>\frac{\partial p_1}{\partial x}>0$	$0<\beta_1<\beta_2<\alpha<\frac{\pi}{2}$
D_2		$\frac{\partial p_1}{\partial x}>\frac{\partial p_2}{\partial x}>0$	$0<\beta_2<\beta_1<\frac{\pi}{2}<\alpha<\pi$
E_1		$\frac{\partial p_1}{\partial x}<\frac{\partial p_2}{\partial x}<0$	$0<\alpha<\frac{\pi}{2}<\beta_1<\beta_2<\pi$
E_2		$\frac{\partial p_2}{\partial x}<\frac{\partial p_1}{\partial x}<0$	$\frac{\pi}{2}<\alpha<\beta_2<\beta_1<\pi$
F_1		$\frac{\partial p_1}{\partial x}<0<\frac{\partial p_2}{\partial x}$	$0<\beta_2<\alpha<\frac{\pi}{2}<\beta_1<\pi$
F_2		$\frac{\partial p_1}{\partial x}>0>\frac{\partial p_2}{\partial x}$	$0<\beta_1<\frac{\pi}{2}<\alpha<\beta_2<\pi$

We will consider the frontal equation, obtained from the condition of pressure continuity at the interface:

$$p_1(x, y, t)-p_2(x, y, t)=0.$$

Choosing the normal to the front as the x-axis for an individual particle moving together with the front, we have

$$\frac{d(p_1-p_2)}{dt}=\frac{\partial(p_1-p_2)}{\partial t}+u\frac{\partial(p_1-p_2)}{\partial x}+v\frac{\partial(p_1-p_2)}{\partial y}=0,$$

but

$$\frac{\partial p_1}{\partial y}=\frac{\partial p_2}{\partial y}.$$

consequently

$$u=-\frac{\frac{\partial(p_1-p_2)}{\partial t}}{\frac{\partial(p_1-p_2)}{\partial x}} \qquad (4)$$

However, as has been explained in §2, the normal component of the velocity of the individual particle contiguous to the front should equal the velocity of propagation of the front c_f, whence

$$c_f=-\frac{\frac{\partial p_1}{\partial t}-\frac{\partial p_2}{\partial t}}{\frac{\partial p_1}{\partial x}-\frac{\partial p_2}{\partial x}} \qquad (5)$$

It was shown above, that $\frac{\partial p_1}{\partial x}-\frac{\partial p_2}{\partial x}\neq 0$, therefore, if the front moves at a velocity $c_f\neq 0$, the numerator in formula (5) is different from zero. We will choose the coordinate axes so that c_f is positive, and explain the behavior of the barometric tendency during the passage of the front. For a warm front we have:

$$\frac{\partial p_1}{\partial t}-\frac{\partial p_2}{\partial t}=-c_f\left(\frac{\partial p_1}{\partial x}-\frac{\partial p_2}{\partial x}\right), \qquad (6)$$

and for a cold front

$$\frac{\partial p_2}{\partial t}-\frac{\partial p_1}{\partial t}=c_f\left(\frac{\partial p_1}{\partial x}-\frac{\partial p_2}{\partial x}\right). \qquad (7)$$

The approach of a type A front will be accompanied by a fall in pressure, which will continue at a slower rate or even cease altogether after the front has passed.

Figure 112. Behavior of the barometric tendency, due to a transition of the pressure field following the passage of a front

The approach of a B type front will be accompanied by a pressure increase, which will be speeded after the front has passed.

Finally, a type C front will cause a pressure drop before it, which will be followed by an increase of pressure in its wake.

The nature of pressure variations upon passage of fronts of the mentioned types is shown in Figure 112. The pressure would have thus varied during the passage of a front had its variation been due only to the transition of the pressure field following the front. But in addition to this transition, other factors also affect the pressure field. Therefore, the curves describing the barometric tendency are considerably different (Figure 113). However, in all cases of front passage the characteristic downward convex bend is evident on the barogram.

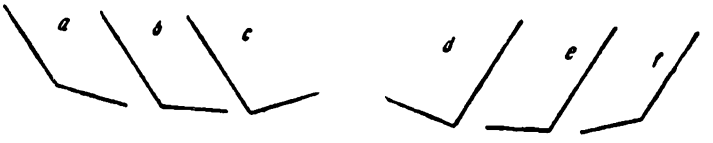

Figure 113. Types of barometric tendency upon front passage a,b,c,—for a warm front; d,e,f—for a cold front.

The following discussion relates only to a pressure discontinuity surface of the first order, i.e., to frontal surfaces of the troposphere. This does not apply to the tropopause. It is easy to show that the tropopause is a discontinuity surface of a higher order than the first. Aerologic investigations show that at the tropopause the temperature changes in a continuous manner and only its gradient suffers a discontinuity. From the equation of state $\rho = \frac{p}{RT}$ it follows that the density is also continuous at the tropopause; consequently $\rho_1 = \rho_2$ and $\frac{\partial p_1}{\partial z} = \frac{\partial p_2}{\partial z}$. Since $\operatorname{tg}\alpha$ is finite, the numerator in formula (5) § 3 must vanish. Consequently

$$\frac{\partial p_1}{\partial x} = \frac{\partial p_2}{\partial x}.$$

Thus, the isobars do not break at the line of intersection of the tropopause with a horizontal surface, but their curvature reaches there a finite cyclonic local maximum. At the tropopause the following relation must also hold:

$$\frac{\partial p_1}{\partial t} = \frac{\partial p_2}{\partial t}.$$

Only the second derivatives of the pressure will be discontinuous at the tropopause, i.e., the tropopause is a discontinuity surface of the second order with respect to the pressure.

§ 5. Interfaces in the Geostrophic Field of Flow

We will now explain the distribution of the geostrophic wind in the neighborhood of a front. According to (1) § 4, we have $\rho_1 u_1 = \rho_2 u_2$ or $\Delta u = u_1 - u_2 \approx 0$, i.e., the normal component (in relation to the front) of the velocity is continuous. From (11) § 3 follows

$$\Delta v = v_1 - v_2 = -\frac{g}{2\omega \sin\varphi}\frac{T_1 - T_2}{T}\cdot \operatorname{tg}\alpha, \qquad (1)$$

i.e., the tangential component of the wind velocity has a discontinuity, together with the temperature (or the density). Consequently, the frontal surface is a discontinuity surface of zero order with respect to the the wind velocity.

The quantities g, ω, $T_1 - T_2$, T, , on the right side of equation (1) are positive and therefore the sign of Δv depends only on the signs of sin φ and $\operatorname{tg}\alpha$. Choosing the x-axis so that $\operatorname{tg}\alpha > 0$ for a warm front, we obtain the following relationships:

Northern hemisphere $\varphi > 0$	$\operatorname{tg}\alpha > 0$	$v_1 - v_2 < 0$
	$\operatorname{tg}\alpha < 0$	$v_1 - v_2 > 0.$
Southern hemisphere $\varphi < 0$	$\operatorname{tg}\alpha > 0$	$v_1 - v_2 > 0$
	$\operatorname{tg}\alpha < 0$	$v_1 - v_2 < 0.$

The geostrophic wind at the front is shown schematically in Figure 110. We see that the discontinuity of the wind is cyclonic for both warm and cold fronts. We

thus obtain results in agreement with the behavior of the isobars in the vicinity of the front. One should remember, however, that the rule we derived for the isobars is rigorous and hence unexceptionally valid for any front, but the corresponding rule for the wind has been based on a number of simplifying assumptions and is therefore less rigorous.

In some particular cases the horizontal acceleration may be considerable, even to the extent of altering the cyclonic curvature of the wind at the front to an anticyclonic one. However, such phenomena are very rare and of short duration.

Thus, all fronts are characterized by the cyclonic curvature of the wind, independently of the nature of the air masses on the two sides of the front. In the determination of the relative motion of air masses at the front one can also be guided by the following rule: "If the observer faces the front, moving together with one of the air masses, the second air mass located beyond the front will move to his left". This statement is true for all air masses and for an arbitrary front.

We finally note that upon passage of a front, the direction of the wind at any point changes in such a fashion, that "the wind veers" (Figure 114).

However, the fact that a front is accompanied by a wind rotation does not imply that every line of wind rotation is a front. A line of wind rotation is a front only in the presence of a discontinuity in density or in the temperature, which amounts to the same.

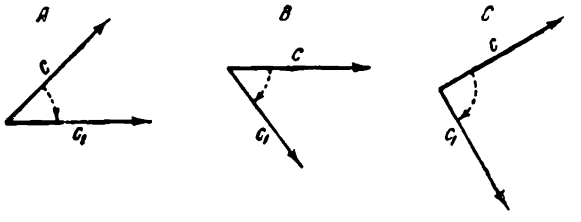

Figure 114. The wind veers upon passage of a front

§6. Altitude Variation of Gradient Wind in the Frontal Zone

Formula (1) §5 can also be used for the analysis of the vertical distribution of wind velocity. This formula shows that the gradient wind changes discontinuously at the frontal surface, whereby the wind velocity vector increment at the passage from a cold to a warm mass is directly proportional to the front slope $\operatorname{tg}\alpha$ and to the temperature jump at the front and is directed parallel to the front, so that the wedge of cold air remains to the left, and the warm air to the right. This rule follows from the discussion of the problem of gradient wind rotation with altitude in a homogeneous air mass (Chapter XI).

In case of a warm front $\operatorname{tg}\alpha > 0$ (Figure 115, A) and therefore $\Delta v = v_1 - v_2 < 0$, i.e., the gradient wind bends before the warm front, veering with increasing altitude in the frontal region. In case of a cold front $\operatorname{tg}\alpha < 0$, and therefore

$$\Delta v = v_1 - v_2 > 0.$$

Therefore behind a cold front the wind backs with increasing altitude (Figure 115, C).

If the front is stationary, the gradient wind may increase or decrease abruptly with increasing altitude, thereby preserving the direction of the cold stream, or it may abruptly reverse its direction.

Formula (1) §5 shows, that sharply defined fronts with considerable temperature discontinuities and a steep slope are accompanied by strong altitude variations of the gradient wind, while those which are ill defined in the temperature field and have small slopes cause but minor spontaneous variation of the wind with altitude.

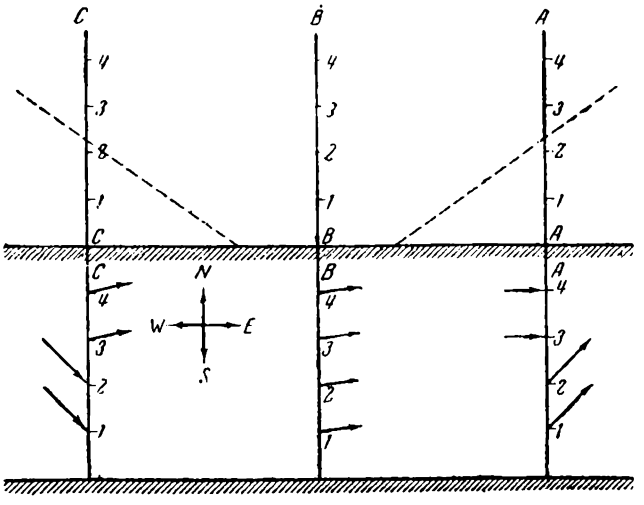

Figure 115. Wind variation with altitude in the vicinity of a front
A) warm front; B) warm mass; C) cold front.

Radiosonde observations can be used to determine the location of the front in the free atmosphere. However, as we have already mentioned before, a wind velocity discontinuity indicates the presence of a front only where the density (or the temperature) is discontinuous. Therefore, the radiosonde observations can be used only to supplement the temperature sounding.

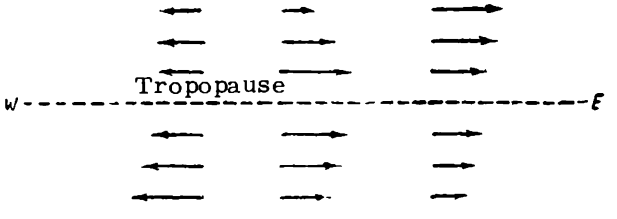

Figure 116. Wind distributions at the tropopause

We should note that if the wind veers with increasing altitude, the altitude of the frontal surface increases in the direction of the wind (Figure 115, A); if the wind backs with increasing altitude, the altitude of the frontal surface decreases in the direction of the wind (Figure 115, C).

We have seen earlier that the tropopause is a discontinuity surface of the first order with respect to the wind velocity, i.e., the wind velocity varies continuously at the tropopause.

It is known that under normal conditions the tropopause dips towards the pole. Figure 116 shows the types of wind distribution with altitude that are realizable in the vicinity of the tropopause.

§7. Location of the Interface in a Nonstationary Velocity Field

Let us assume that at the initial moment a certain interface is steeper than the stationary interface, whose slope is determined by formula (8) § 3, i.e., that (Figure 117)

$$\operatorname{tg}\alpha > \operatorname{tg}\alpha_0.$$

We suppose that the flow is unperturbed. Elementary but tedious analysis shows, that in that case the normal (in relation to the front) components of the acceleration are discontinuous, whereby

$$\dot{u}_1 - \dot{u}_2 = \frac{\Delta T}{T} \cdot g \cdot (\operatorname{tg}\alpha - \operatorname{tg}\alpha_0) > 0.$$

In addition, the vertical component of the acceleration is also discontinuous near the frontal surface

$$\dot{w}_1 - \dot{w}_2 = \frac{\Delta T}{T} \cdot g (\operatorname{tg}\alpha - \operatorname{tg}\alpha_0) > 0.$$

This means that air masses moving parallel to the interface are nonuniformly accelerated in the direction normal to the front, whereby the magnitude of this acceleration will be greater in a warm mass than in a cold one. The same applies to the vertical acceleration, which will also be greater in the warm than in the cold air mass.

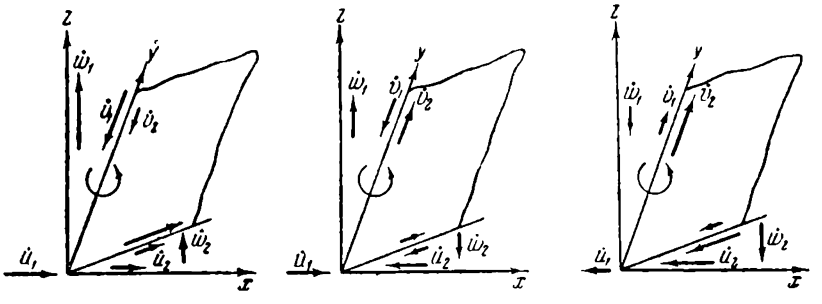

Figure 117. Development of acceleration near a nonstationary interface when $\alpha > \alpha_0$

The following three cases may be discerned:

1) $\dot{u}_1 > \dot{u}_2 > 0$; $\dot{w}_1 > \dot{w}_2 > 0$; $\dot{v}_1 < \dot{v}_2 < 0$;
2) $\dot{u}_1 > 0 > \dot{u}_2$; $\dot{w}_1 > 0 > \dot{w}_2$; $\dot{v}_1 < 0 < \dot{v}_2$;
3) $0 > \dot{u}_1 > \dot{u}_2$; $0 > \dot{w}_1 > \dot{w}_2$; $0 < \dot{v}_1 < \dot{v}_2$.

In the first case both masses move towards the cold mass, the velocity of the warm mass increasing whereby faster than that of the cold mass.

This motion is accompanied by the emergence and build up of vertical velocities whereby the vertical accelerations are greater for the warm than for the cold mass. Therefore, in both warm and cold masses vertical velocities will develop, the flow velocity of the warm air exceeding that of the cold air. From this argument follows that in the first case we deal with a warm front. The approaching warm air replaces the receding cold air, and in both cold and warm masses develop upward currents (the rising warm air overtakes the rising cold air).

The relative normal velocity components cause tangential accelerations, due to the rotation of the earth. In the first case, these additional accelerations have the same direction, whereby the acceleration of the warm mass exceeds that of the cold mass in absolute value, and the warm air will remain to the right of the cold.

In the second case the two air masses approach each other, i.e., both masses move in the direction of the front. The front itself will not be shifted only when the accelerations of the two masses in question are equal in magnitude. However, if the accelerations differ, the front will shift either in the direction of the cold mass (when $|\dot{u}_1| > |\dot{u}_2|$), or in the direction of the warm mass (when $|\dot{u}_1| < |\dot{u}_2|$). The front is correspondingly stationary ($|\dot{u}_1| = |\dot{u}_2|$), warm ($|\dot{u}_1| > |\dot{u}_2|$), or cold ($|\dot{u}_1| < |\dot{u}_2|$).

In that case, regardless of the nature of the front, an upward flow will develop in the warm mass and a downward flow in the cold.

Accelerations tangential to the front are here opposite in direction and contribute to the development of a cyclonic motion of the front.

Finally in the third case both masses move in the direction of the warm mass, whereby the absolute acceleration of the cold mass will be greater than that of the warm mass. This motion will be accompanied by downward flows developing in both air masses, whereby the cold mass will descend faster than the warm.

The accelerations tangential to the front will again have the same direction, whereby the absolute value of the acceleration of the warm mass will be smaller than that of the cold mass.

The aforementioned accelerations will bring about one of the following alternatives 1) the advancing warm mass will overtake and override the (also ascending) receding cold mass, owing to its higher vertical velocity. At that, the existing velocity difference at the front will be increased, and the slope angle of the stationary interface will increase too; 2) the warm air will flow along the wedge of downwards moving cold air it has encountered; in that case the existing velocity difference will grow, and in turn increase the slope angle of the stationary interface;

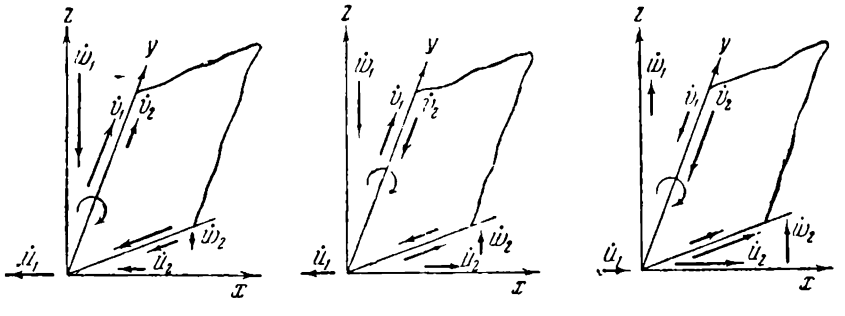

Figure 118. Development of acceleration near a nonstationary interface when $\alpha < \alpha_0$.

3) the receding warm mass will vertically lag behind the rapidly descending cold one which will consequently flow under it. The difference in the tangential velocity will also increase, which would necessitate a steeper slope of the interface in order to preserve equilibrium.

In all these cases the appearance of acceleration (due to normal velocities) is followed by a decrease in the interface slope, whereas the increased difference in the tangential component of the velocity increases the slope, so that the very

gentle slope of the interface (Figure 118) no longer corresponds to equilibrium conditions. Accelerations will then gradually develop at the interface in the opposite direction leading to one of the following alternatives: 1) both masses will move in the direction of the warm mass, simultaneously descending, whereby the warm will overtake the cold mass, or 2) both masses will move in different directions, whereby the warm mass will descend along the wedge of the ascending cold air mass, or 3) both the warm and the cold mass will move in the direction of the cold mass, simultaneously ascending, whereby the cold mass will recede faster than the warm one, forerunning it along the vertical as well.

At the same time, in all cases discussed, the tangential components of the emergent accelerations have an anticyclonic character; therefore the velocity difference at the front gradually diminishes with time and may even reverse its sign. The slope angle of the interface will increase while the stationary frontal slope angle will decrease. Thus, we have returned to the initial situation of our analysis.

A more detailed study shows that the interface slightly oscillates around the position of equilibrium, which does not coincide with the position of the stationary surface, but is displaced from it in the same direction as the initial position of the interface.

§ 8 Classification of Fronts

The above described properties of fronts, concerning their position in the pressure, velocity, and temperature fields, are applicable not only to stationary (immobile) but to moving fronts as well. On the base of these properties one can settle the question of the presence or absence of a front on a synoptic chart or in a vertical cross section.

However, a number of additional properties can be found which are characteristic of the different types of front, related to the stability of fronts and their motion with respect to the position of the warm and the cold air masses.

Thus, fronts may be classified according to their properties. Such a classification facilitates the determination of a front as belonging to a certain type.

Fronts may be classified according to their geographical position. This usage is adopted to distinguish between four basic types of air masses: arctic, polar, tropical and equatorial.

Polar air is often separated from arctic air by an arctic front. Tropical air is separated from polar air by a polar front.

Tropical air approaching the equator is gradually converted into equatorial air. Therefore no front exists between the tropical and the equatorial air. However, when equatorial air of the summer hemisphere meets the equatorial air of the winter hemisphere an intertropical front is formed.

Thus, on the basis of the geographical classification, one can establish three main categories of fronts, which are constantly in existence: the arctic, polar and intertropical fronts.

Besides these main types, from time to time there may appear secondary fronts within a major air mass, if the distributions of temperature and velocity are favorable to frontogenesis (the formation of fronts). Arctic, tropical and equatorial

air masses are usually characterized by a highly uniform horizontal temperature distribution, and hence secondary fronts do not appear within these air masses. In polar air, on the contrary, secondary fronts very often appear between "mature" and "fresh" polar air, and also between continental and maritime polar air.

Everyday synoptic practice shows, that most of the fronts detected on synoptic charts are connected with the major fronts. A synoptic chart showing numerous secondary fronts is usually indicative of an incomplete analysis.

Besides the general frontal characteristics(discontinuity of the temperature, of the tangential component of the wind velocity, etc), which have been discussed in the foregoing sections and which apply to all fronts, no special features exist which might determine the geographical type of fronts. Nevertheless, the usefulness of such classification is obvious. Since the major fronts predominate over the secondary in the frequency of their occurence, the vastness of territory they overlie and the duration of their existence, the adopted synoptic procedure in the analysis of any weather characteristic related to fronts is to attribute it first of all to the major fronts. Only when the studied weather feature cannot be explained by the major front, one turns to the secondary fronts. Thereby, it is advisable to pay special attention to the explanation of the processes which have led to the build-up and dissolution of these secondary fronts. The most important classification of fronts is based on the nature of frontal migration and the motion of air masses. In this classification one distinguishes between the following four front types:

1) warm front, i.e., a front moving so that warm air replaces the cold;

2) cold front, i.e., a front moving so that cold air replaces the warm air;

3) stationary front, i.e., a front that does not move at all: in this case there is no replacement of one air mass by another;

4) occluded front, i.e., a front due to air masses of a cold front overtaking the warm masses of the slower warm front. An occluded front forms on condition that the temperature is discontinuous between the cold air retreating before the warm front, and the cold air following in the wake of the cold front.

It can be shown that the velocity of the front is determined by the formula

$$c_f - u_1 - w_1 \operatorname{ctg} \alpha = u_2 - w_2 \operatorname{ctg} \alpha \qquad (1)$$

If the vertical velocities w are very small, one can assume as an approximation, that the frontal velocity equals the normal component of the wind velocity. The velocity in question is, of course, not the wind velocity at the earth's surface, but the velocity of the geostrophic wind, observed above the friction level. Consequently a fast moving front should intersect the isobars almost at a right angle and the isobars themselves must be quite densely spaced.

Stationary fronts must parallel the isobars. Strictly stationary fronts are rarely encountered in the atmosphere. Very slow fronts are much more common; these, called quasi-stationary fronts, greatly complicate the prognostic task, since their residual motion is determined by accelerations which are extremely difficult to evaluate.

Warm and cold fronts are easily distinguished from each other on synoptic

charts, even when the temperature difference according to ground based observations is not sharply defined. The systems of clouds and hydrometeors are different for the two classes. The difference between the warm and the cold fronts can be noticed also in the field of the barometric tendency and in the pressure field.

An occluded front has a more complicated structure, due to the superposition of the warm front on the cold one. If the air is colder behind the cold front than before the warm front, the occlusion resembles a cold front, and under opposite temperature conditions, the occlusion is similar to a warm front. Usually an occluded front results in an unstable cyclonic wave.

Finally, fronts may be classified according to the nature of the vertical distribution of the velocity and the conditions of stability of the air masses adjacent to the front.

Observations have shown that frontal cloud systems usually develop in warm air, probably owing to the fact that the velocities of ascent are greater in the warm than in the cold mass. This is due to the cold air having more rigid kinematic boundaries than the warm air, being limited from below by the surface of the earth and from above by the frontal surface.

Therefore fronts or atmospheric interfaces are classified with respect to the behavior of the warm air masses.

We distinguish between surfaces of sliding ascent or anafronts, where $w_1 > 0$, and surfaces of sliding descent, or catafronts, where $w_1 < 0$. Discerning also the warm ($c_f < 0$), cold ($c_f > 0$) and stationary ($c_f = 0$) fronts and taking into consideration the relationship (1), we obtain a great number of theoretically possible fronts.

We have seen above, that if the actual interface ascends more steeply than the stationary surface, the cold air starts to flow beneath the warm air, displacing the latter. We call such a surface, a surface of passive sliding ascent. It is natural to relate this surface to the cold front, since in this case the cold air actively flows beneath the warm air, lifting the latter upwards.

In case of a warm front the warm air actively accumulates on top of the cold, dislodging and replacing the latter in its ascent along the interface. Therefore, a warm front is a surface of active sliding ascent.

We have also seen above, that if the slope of the interface is smaller than the slope of the corresponding stationary surface, the cold air starts flowing out from under the warm air which descends and replaces it, while the receding cold air descends. This interface is called a surface of passive sliding descent. Obviously, the surface of passive sliding descent may be associated with the warm front.

Finally, if the surface of sliding descent is related to a cold front, the warm air moves in the same direction as the cold, overtaking it and descending along the interface. This surface is called a surface of active sliding descent.

The mentioned four main types of sliding surfaces differ fundamentally from each other in certain properties, above all by their slope angles and also by the system of clouds and precipitations associated with the front.

It follows from the foregoing argument that a surface of a passive sliding ascent should have the largest slope angle, and that of passive sliding ascent — the smallest. We arrive at this conclusion not only by considering accelerations developing at

the interface as a result of its deviation from the position corresponding to stationary conditions; the influence of friction with the earth's surface, which we have ignored till now, has the same effect. In fact, in case of a cold front, the friction between air and ground surface will retard the motion of the cold air advancing in contact with the earth's surface; consequently the interface will have a slope steeper than in the absence of friction. Occasionally, the slope of the cold front on the lower layers of the atmosphere may be greater than 90° owing to the effect of friction, and hence the invasion of cold air in that case starts aloft and only later reaches the earth's surface. In such cases the wedge of cold air is blunted. In case of a warm front friction reduces the slope of the interface. A thin layer of cold air, several tens or hundreds of meters high, passes in this case at a considerable altitude behind the warm front.

From the preceding examination of the slopes of sliding surfaces one may conclude that the surface of passive sliding ascent should possess the largest slope. Aerologic soundings show that these slopes may reach values of the order of 1/25—1/80. Next come surfaces of active sliding ascent with a slope usually between 1/50--1/200, and after them the active surfaces of sliding descent. The smallest slope should be found in surfaces of passive sliding descent, whose slope is usually very small. It is in the range of 1/400 -1/1000.

A warm front is in most cases an anafront, i.e., a surface of sliding ascent. Only in rare cases is the warm front a catafront, i.e., a surface of sliding descent.

There are two kinds of cold fronts. The first kind are anafronts up to high altitudes. The cold fronts of the second kind are anafronts in their lower part and from the altitude of 1—2 km upwards turn into catafronts. The anticyclonic inversions of compression, not reaching down to the earth's surface, are catafronts too. As a rule catafronts do not reach down to the earth's surface. Usually the lower parts of the catafronts are either ill-defined at the earth's surface as a result of flow divergence, or they become anafronts, owing to the increase of the slope.

In Table 46 are given the different air motions realizable at sliding surfaces. One can see from this table, that at surfaces of sliding ascent and descent both ascending and descending air motions are possible.

§ 9. Frontogenesis and Frontolysis

Frontogenesis is the process of formation of a discontinuity or the increase of an already existing one. Frontolysis is a process of breakdown or weakening of a discontinuity.

The main feature of discontinuity surfaces in the free atmosphere is the temperature discontinuity. We shall now discuss the potential temperature or equivalent-potential temperature (in case of saturation).

Not every temperature discontinuity testifies to the presence of a surface separating two different air masses. A temperature discontinuity may sometimes be due to local transient condition. A mere temperature discontinuity is thus insufficient for the formation of an interface significant for atmospheric processes.

Helmholtz has already pointed out in his treatment of the general circulation of the atmosphere, that an interface is formed when two air masses of different origin and temperature meet.

Table 46

Possible motions of air at a sliding surface

Surface		Sliding ascent ($w_1 > 0$)	Sliding descent ($w_1 < 0$)
Warm front	u_1	$u_1 > c_f > 0$	1) $0 < u_1 < c_f$, if $\lvert w_1 \rvert < c_f \operatorname{tg} \alpha$ 2) $u_1 < 0$, if $\lvert w_1 \rvert > c_f \operatorname{tg} \alpha$
$c_f > 0$	u_2, w_2	1) $u_2 > c_f > 0$, $w_2 > 0$ 2) $c_f > u_2 > 0$, $w_2 < 0$, if $\lvert w_2 \rvert < c_f \operatorname{tg} \alpha$ 3) $u_2 < 0$, $w_2 < 0$, if $\lvert w_2 \rvert > c_f \operatorname{tg} \alpha$	
	α	$\frac{1}{50} > \operatorname{tg}\alpha > \frac{1}{200}$ Active	$\frac{1}{400} > \operatorname{tg}\alpha > \frac{1}{1000}$ Passive
Cold front	u_1	$u_1 > 0 > c_f$, if $w_1 > \lvert c_f \rvert \operatorname{tg} \alpha$ $u_2 < 0$, if $w_1 < \lvert c_f \rvert \operatorname{tg} \alpha$	
$c_f < 0$	u_2, w_2	1) $u_2 > 0, w_2 > 0$, $\lvert w_2 \rvert > \lvert c_f \rvert \operatorname{tg} \alpha$ 2) $u_2 < 0$, $w_2 > 0$, $\lvert w_2 \rvert < \lvert c_f \rvert \operatorname{tg} \alpha$ 3) $u_2 < 0$, $w_2 < 0$	
	α	$\frac{1}{25} > \operatorname{tg}\alpha > \frac{1}{80}$ Passive	Active
Quasi-stationary front $c_f = 0$	u_1	$u_1 = w_1 \operatorname{ctg} \alpha > 0$	$u_1 = w_1 \operatorname{ctg} \alpha < 0$
	u_2, w_2	1) $u_2 > 0$, $w_2 > 0$, $u_2 = w_2 \operatorname{ctg} \alpha$ 2) $u_2 < 0$, $w_2 < 0$, $u_2 = w_2 \operatorname{ctg} \alpha$	$u_2 > 0, w_2 > 0$ $u_3 < 0, w_2 < 0$

In order to isolate the major frontogenetic factor, we consider the equation of heat balance $\frac{dq}{dt} = c_p \frac{T}{\theta} \frac{d\theta}{dt}$, solving it with respect to the partial local derivative $\frac{\partial \theta}{\partial t}$:

$$\frac{\partial \theta}{\partial t} = -u \frac{\partial \theta}{\partial x} - v \frac{\partial \theta}{\partial y} + \frac{\theta}{T} \cdot \frac{1}{c_p} \cdot \frac{dq}{dt}$$

(we restrict ourselves to the discussion of two-dimensional motion).

Let us assume for the sake of simplicity that the isotherms of the potential (or equivalent-potential) temperature are rectilinear. Orienting the x-axis parallel to the isotherms, we obtain:

$$\frac{\partial \theta}{\partial t} = -v \frac{\partial \theta}{\partial y} + \frac{\theta}{T} c_p \frac{dq}{dt}.$$

The variation in the temperature difference between two neighboring points will be determined by the formula

$$\frac{\partial (\theta_1 - \theta_2)}{\partial t} = -\left(v_1 \frac{\partial \theta_1}{\partial y} - v_2 \frac{\partial \theta_2}{\partial y}\right) + \frac{\theta}{T} c_p \left(\frac{dq_1}{dt} - \frac{dq_2}{dt}\right). \quad (1)$$

Equation (1) shows that the variation in temperature difference between two neighboring points is determined by two factors — advection and heat influx. Let us assume that advection is absent or is so very weak that it may be disregarded. Then the variation in the temperature difference between these points will be determined only by the difference in heat influx to the surrounding air masses. Let us consider all three forms of heat transfer: radiation, eddy transfer of heat, and latent heat conversions. In a free atmosphere the radiation conditions at two neighboring points cannot differ appreciably. Therefore, in a free atmosphere fronts cannot form or break down by the action of radiation. On the contrary, when there are sharp contrasts in the underlying surfaces, the radiation conditions near ground level may vary considerably (at the sea shore, for instance).

However, such differences are short-lived and in the course of the day they change their sign. Besides, they are limited to the lowest few hundred meters of the atmosphere. The corresponding temperature discontinuity is not accompanied by a discontinuity in the velocity, the pressure gradient and other frontal characteristics.

Since a temperature discontinuity caused by difference in the radiation conditions is locally limited and is of no general importance, the heat influx by radiation cannot lead to frontogenesis or frontolysis.

Turbulent heat exchange results, as is well known, in equilibration of temperature; consequently it cannot lead to frontogenesis; frontolysis, on the other hand, greatly depends on turbulent heat exchange.

Temperature difference caused by evaporation or condensation in a free atmosphere cannot attain considerable magnitudes. Only at the earth's surface might obtain conditions favorable to the appearance of a considerable temperature difference. This difference, however, will also not be representative, being only of local significance.

Thus, frontogenesis can be attributed only to advection of air.

Let us assume for the sake of simplicity, that at the initial moment the temperature is a linear function of the ordinate y, whereby

$$\frac{\partial \theta_1}{\partial y} = \frac{\partial \theta_2}{\partial y} = \frac{\partial \theta}{\partial y} > 0.$$

If the y-axis is directed towards the high temperatures, we have

$$\frac{\partial (\theta_1 - \theta_2)}{\partial t} = -\frac{\partial \theta}{\partial y}(v_1 - v_2),$$

i.e., in the linear temperature field the time variation of the temperature difference between two neighboring points entails a discontinuity of the normal(to the isotherm) component of velocity which is directly proportional to this variation.

Thus, frontogenesis will develop only in a flow field in which the isotherms converge. In the opposite case, frontolysis will develop.

Therefore,the most vigorous frontogenesis and frontolysis will take place in a deformation field, when two air masses move towards each other, spreading out laterally.

In the simplest case of a plane purely deformational field the streamlines are equilateral hyperbolas (Figure 119). Such a region corresponds to a saddle in the pressure field. The point in the center of the hyperbolic field is called the neutral

or hyperbolic point. The wind velocity at this point is zero. The axis, along which the two air masses approach each other is called the compression axis, and the second axis—the expansion axis.

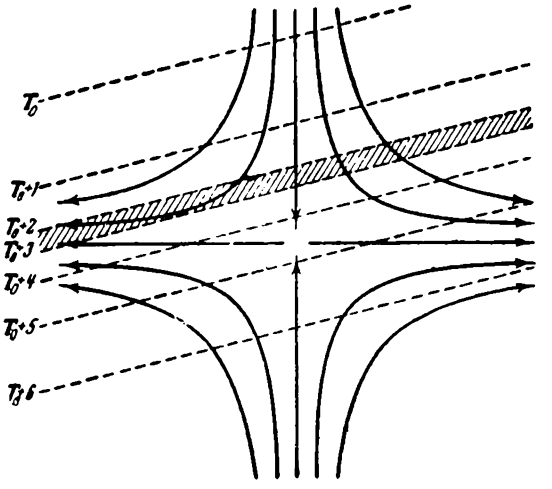

Figure 119. Frontogenesis in a deformation field

Figure 119 shows that the most favorable conditions for frontogenesis obtain when the isotherms are parallel to the expansion axis. Then they must approach each other, remaining parallel to their previous direction. If the isotherms are parallel to the compression axis they must move apart, remaining parallel to their original direction.

Let us evaluate the order of magnitude of the velocity of frontogenesis and frontolysis for these simple cases. Assuming $v_1 = -v_2 = 10$ m/sec, we obtain the following variation per hour of the temperature difference between two air particles belonging to different streams:

$\left\lvert \frac{\partial \theta}{\partial y} \right\rvert_{av}$	1°/1000 km	1°/500 km	1°/10 km
$\frac{\partial (\theta_1 - \theta_2)}{\partial t}$	0°.7	1°.4	7°

This table shows, that the velocity of frontogenesis and frontolysis is especially high in those cases, where a frontal zone already exists. Consequently, a front may become considerably more pronounced or diffuse in a very short time — of the order of several hours.

If the isotherms form with the expansion axis an angle α, smaller than 45°, the isotherms will approach each other under the influence of the deformation field, i.e., frontogenesis will set in, though at a slower rate, than in the case when α equals 0°, whereby the isotherms will turn in such a manner, that $\alpha \rightarrow 0$.

If the isotherms form with the expansion axis an angle α, greater than 45° (Figure 120) they first move apart, i.e., frontolysis develops, and turn so that $\alpha \rightarrow$ 45°. Upon further rotation the isotherms will again move closer to each other, i.e., frontogenesis will recommence. Thus, only isotherms perpendicular to the expansion axis will consistently move apart. All the other isotherms in the deformation field will end by getting closer to each other.

As already indicated above, a saddle in the pressure field corresponds to such a field of deformation, the saddle being formed by a crosswise position of two cyclones and two anticyclones. Therefore, frontogenesis takes place mainly in those regions, where such a pressure field is most frequent.

It is obvious that atmospheric deformation fields are by no means two-dimensional. Besides the two horizontal axes of deformation, usually a third deformation axis also exists, located in the vertical plane and inclined towards the horizon. We will consider the deformation field in the vertical plane.

If the horizontal axis is the compression axis, the inclined one will be the expansion axis (Figure 121). Repeating the above considerations, one can easily see that in this case the isentropic surfaces will approach each other, which corresponds to frontogenesis. If however, the horizontal axis is the axis of expansion (Figure 122), the isentropic surfaces will move apart, which corresponds to frontolysis.

The kinematic conditions for the formation of inversions of precipitation differ from the frontolytic conditions in that in case of frontolysis the compression axis is almost horizontal, while in case of precipitative inversion it is nearly vertical.

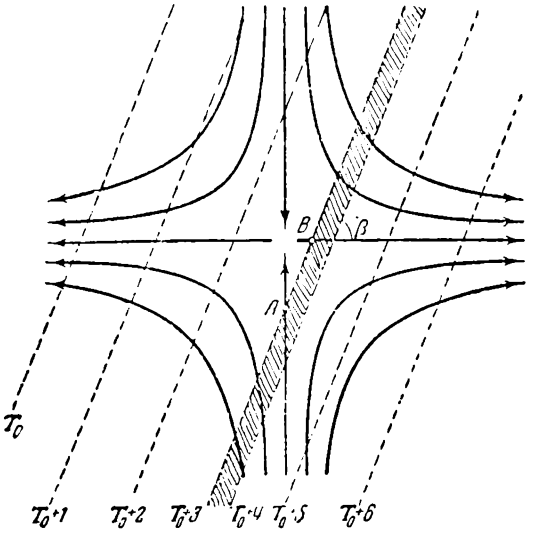

Figure 120. Frontolysis in a deformation field

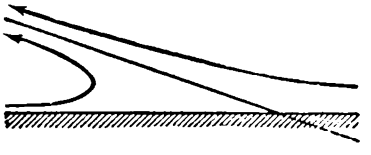

Figure 121. Frontogenesis in a vertical plane

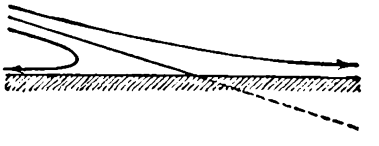

Figure 122. Frontolysis in a vertical plane

Therefore in case of precipitative inversion the isentropic surfaces, drawing closer and closer together, form an almost horizontal inversion layer.

The above consideration involves simple schemes, which nevertheless illustrate sufficiently well the frontogenetic role of the mutual position of the temperature field and the flow field. These schemes will be brought somewhat closer to reality, if the effect of simple transfer will be excluded, also recalling that all the aforesaid relates to a coordinate system that moves together with the current.

Chapter XIV

KINEMATICS OF THE PRESSURE FIELD

§1. Characteristic Curves and Singular Points of the Pressure Field

In the present chapter we shall discuss the very important practical problem of the regularity of variations in the pressure field, the explanation of which does not require the introduction of physical and dynamical concepts, but is entirely based on a given instantaneous pressure distribution and its temporal variation.

Atmospheric pressure in the only meteorological element which is unconditionally representable and whose measurement is performed with great precision in the meteorological stations. Air pressure representing the state (weight) of the entire air column at the point of observation, is subject to accidental variation less than any other meteorological element. On the other hand, the pressure and its horizontal distribution are decisive for the distribution and variation of a number of meteorological elements. The singular formations of the pressure field and their displacements are related to the characteristic features of the weather and its variations.

Thus, for example, the wind speed and direction depend to a great extent on the pressure gradient; in layers located above the friction level, where the effect of friction with the earth's surface becomes negligible, the pressure field corresponds quite closely with the velocity field. The cyclonic and anticyclonic nature of the isobaric curvature and the divergence of the isobaric gradient affect the vertical flow in the atmosphere. Frontogenesis is connected with the generation and growth of discontinuities of the pressure gradient, etc.

The extensive experience accumulated by synopticians over the last decades shows, that all major features of the synoptic chart are reflected in the characteristic curves of the pressure field corresponding to the same chart.

It is therefore natural that the study of the pressure field, its singularities and evolution should occupy a central position in meteorology.

The aim of this chapter is to derive the principal relations determining from a given pressure distribution its subsequent variation, with an eye to their prognostic value.

The pressure field on a synoptic chart drawn for any instant of time t, can be best represented by a family of isobars. In the next moment the isobars, generally speaking, will shift, deform, and assume new positions. The isobar shift can be quantitatively determined, if one knows their distribution and the pressure variation at some instant.

The equation of the isobar family may be written thus

$$p(x, y, t) = a,$$

where a is the parameter characterizing a particular isobar. The isobars are curves characterizing the distribution of pressure; they may therefore be called characteristic curves. All points located on some isobar are ordinary points of the pressure field. But beside the infinity of ordinary points, which lie on any of the isobars, the

pressure field generally also possesses a finite number of singular points, isolated among the other points of the pressure field and characterized by not lying on any isobar. These so-called isolated singularities of the pressure field may be centers of high or low pressure. In addition, the pressure field there may include also a finite number of singular points, at which the isobar intersects itself. These are the so-called neutral, or hyperbolic points.

The variations in the pressure field necessitate the introduction of characteristic curves of a higher order. Let

$$\frac{\partial p}{\partial t} = T(x, y, t)$$

represent the pressure variations (or barometric tendencies). The lines of equal barometric tendency

$$\frac{\partial p}{\partial t} = T_0, T_1, T_2, \ldots$$

(the so-called isallobars) are also characteristic curves of the pressure field.

In the pressure field may be observed also more complicated characteristic curves, on which certain derivatives of the pressure functions are constant. Instances of such characteristic curves are the axes of pressure troughs and highs, being geometric loci of points of maximal isobaric curvature, the lines of maximal curvature of the pressure profile, fronts, i.e., lines characterized by a discontinuity of the pressure gradient, etc.

In the following section we shall derive the general formulas, determining the velocities and accelerations of the characteristic curves.

§2. The Motion of Curves

Let us consider in the (x, y) plane, a family of curves

$$f(x, y, t) = a, \qquad (1)$$

depending on two parameters t and a. The first parameter stands for the time and the second indicates the particular curve of the family in question.

After a time t elapses, the curve corresponding to a certain value of the parameter a will shift and deform. In order to determine the new position of the characteristic curve in the (x, y) plane it suffices to substitute the corresponding time in equation (1). Let us point out that we are not interested in following the change in position of a particular physical particle moving along our curve since the correspondence of the points on two curves belonging to the same value of the parameter a is not governed by any physical law.

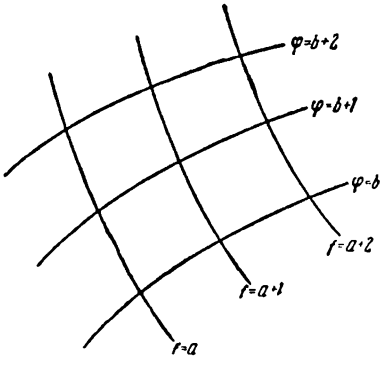

Figure 123. Characteristic curves (family a) and directing curves (family b)

The characteristic curves in question are not material lines at all. They are not permanently constituted of the same physical particles, being merely geometrical manifolds joining geometrical points at which the pressure or the tendency, etc, are equal. But even in case the lines are material (for instance, vortex filaments) they will not consist continually of the same particles. Only in case of a barotropic fluid is it possible to determine a well-defined relationship between the physical particles for two positions of the vortex filament.

However, for the determination of the displacement of the characteristic curves we must follow their line elements on synoptic charts in such a manner as to afford the identification of these curves in passing from one chart to another.

Therefore we must label the points on the curve so as to be able to trace the motion of its separate line elements. To do this we may use the following method. We introduce another family of curves

$$\varphi(x, y, t) = b, \qquad (2)$$

where b is constant for any particular curve. Let us call the new family of curves the "directing" family (Figure 123).

Thus, there are two families of moving curves undergoing deformation in the (x, y) plane. Giving definite values to the parameters a, b, t, we obtain two curves: a characteristic curve of family (1) and a directing curve of family (2). These curves will generally intersect each other at one or several points.

Let us follow the motion of one of the points of intersection. The point we pick out on the curve a is "individualized". In contrast to all other points on the curve a it possesses the additional property, that during the entire motion it stays also on the given curve b of the directing family (2).

In order to determine the coordinates of the intersection point of the curves a and b we have to solve the system of equations

$$\left.\begin{aligned} f(x, y, t) &= a, \\ \varphi(x, y, t) &= b. \end{aligned}\right\} \qquad (3)$$

This system (3) has a solution only if the functions f and φ satisfy the following conditions: 1) these functions should be continuous in the neighborhood of (x_0, y_0, t_0); 2) they should have continuous partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}$ in the same neighborhood; 3) the functional determinant

$$D = \begin{vmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y} \\ \frac{\partial \varphi}{\partial x}, & \frac{\partial \varphi}{\partial y} \end{vmatrix} \qquad (4)$$

should differ from zero at (x_0, y_0, t_0).

These conditions are uniquely fulfilled by one pair of functions

$$\left.\begin{aligned} x &= x(a, b, t), \\ y &= y(a, b, t), \end{aligned}\right\} \qquad (5)$$

corresponding to the system of equations (3) and equal to x_0 and y_0, for $t = t_0$. This result has a definite kinematic meaning.

We shall regard equations (5) as the "laws of motion" of the individual point. The derivatives $\frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ may be regarded as components of the "velocity"

and "acceleration" respectively of this "individualized" point of the curve. We use these terms with the proviso that the "individualized" point of the characteristic curve is still not a physical particle and that the above analogy with a physical particle is only necessary in order to enable us to use the appropriate mathematical apparatus: this analogy may be lacking in some cases, especially since the "individualization" can be accomplished by a great variety of methods.

For the calculation of the components of velocity and acceleration of the "individualized" point, which we shall denote by c_x, c_y, A_x, A_y, respectively, we use the rule of differentiation of implicit functions.

Differentiating (3) with respect to t, we obtain

$$\left.\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + c_x \frac{\partial f}{\partial x} + c_y \frac{\partial f}{\partial y} = 0, \\ \frac{d\varphi}{dt} &= \frac{\partial \varphi}{\partial t} + c_x \frac{\partial \varphi}{\partial x} + c_y \frac{\partial \varphi}{\partial y} = 0, \end{aligned}\right\} \qquad (6)$$

whence

$$\left.\begin{aligned} c_x &= \frac{1}{D}\left(\frac{\partial f}{\partial y} \cdot \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial y} \cdot \frac{\partial f}{\partial t}\right), \\ c_y &= \frac{1}{D}\left(\frac{\partial \varphi}{\partial x} \cdot \frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} \cdot \frac{\partial \varphi}{\partial t}\right). \end{aligned}\right\} \qquad (7)$$

Differentiating (6) again with respect to t, we get

$$\begin{aligned} \frac{d^2 f}{dt^2} &= \frac{\partial^2 f}{\partial t^2} + 2c_x \frac{\partial^2 f}{\partial x\, \partial t} + 2c_y \frac{\partial^2 f}{\partial y\, \partial t} + 2c_x c_y \frac{\partial^2 f}{\partial x\, \partial y} + \\ &\quad + c_x^2 \frac{\partial^2 f}{\partial x^2} + c_y^2 \frac{\partial^2 f}{\partial y^2} + A_x \frac{\partial f}{\partial x} + A_y \frac{\partial f}{\partial y} = 0, \\ \frac{d^2 \varphi}{dt^2} &= \frac{\partial^2 \varphi}{\partial t^2} + 2c_x \frac{\partial^2 \varphi}{\partial x\, \partial t} + 2c_y \frac{\partial^2 \varphi}{\partial y\, \partial t} + 2c_x c_y \frac{\partial^2 \varphi}{\partial x\, \partial y} + \\ &\quad + c_x^2 \frac{\partial^2 \varphi}{\partial x^2} + c_y^2 \frac{\partial^2 \varphi}{\partial y^2} + A_x \frac{\partial \varphi}{\partial x} + A_y \frac{\partial \varphi}{\partial y} = 0, \end{aligned} \qquad (8)$$

whence

$$\begin{aligned} A_x &= \frac{1}{D}\left[\left(\frac{\partial^2 \varphi}{\partial t^2} + 2c_x \frac{\partial^2 \varphi}{\partial x\, \partial t} + 2c_y \frac{\partial^2 \varphi}{\partial y\, \partial t} + 2c_x c_y \frac{\partial^2 \varphi}{\partial x\, \partial y} + c_x^2 \frac{\partial^2 \varphi}{\partial x^2} + \right.\right. \\ &\quad \left. + c_y^2 \frac{\partial^2 \varphi}{\partial y^2}\right)\frac{\partial f}{\partial y} - \left(\frac{\partial^2 f}{\partial t^2} + 2c_x \frac{\partial^2 f}{\partial x\, \partial t} + 2c_y \frac{\partial^2 f}{\partial y\, \partial t} + \right. \\ &\quad \left.\left. + 2c_x c_y \frac{\partial^2 f}{\partial x\, \partial y} + c_x^2 \frac{\partial^2 f}{\partial x^2} + c_y^2 \frac{\partial^2 f}{\partial y^2}\right)\frac{\partial \varphi}{\partial y}\right], \\ A_y &= \frac{1}{D}\left[\left(\frac{\partial^2 f}{\partial t^2} + 2c_x \frac{\partial^2 f}{\partial x\, \partial t} + 2c_y \frac{\partial^2 f}{\partial y\, \partial t} + 2c_x c_y \frac{\partial^2 f}{\partial x\, \partial y} + \right.\right. \\ &\quad \left. + c_x^2 \frac{\partial^2 f}{\partial x^2} + c_y^2 \frac{\partial^2 f}{\partial y^2}\right)\frac{\partial \varphi}{\partial x} - \left(\frac{\partial^2 \varphi}{\partial t^2} + 2c_x \frac{\partial^2 \varphi}{\partial x\, \partial t} + 2c_y \frac{\partial^2 \varphi}{\partial y\, \partial t} + \right. \\ &\quad \left.\left. + 2c_x c_y \frac{\partial^2 \varphi}{\partial x\, \partial y} + c_x^2 \frac{\partial^2 \varphi}{\partial x^2} + c_y^2 \frac{\partial^2 \varphi}{\partial y^2}\right)\frac{\partial f}{\partial x}\right]. \end{aligned} \qquad (9)$$

We have now computed the first and second time-derivatives of the coordinates of the "individualized" point (regarded as "velocity" and "acceleration" components of an individual particle); to ensure the existence and continuity of these derivatives, the functions f and φ must fulfil the following two conditions, in addition to those listed above:

4) The existence of continuous partial time-derivatives $\frac{\partial f}{\partial t}$ and $\frac{\partial \varphi}{\partial t}$; 5) the existence of continuous partial derivatives of the second order of the functions f and φ with respect to all the independent variables in the neighborhood of (x_0, y_0, t_0).

We have thus obtained formulas determining the components of the instantaneous acceleration of the individualized points of the characteristic curves. These formulas seem rather complicated. However, in certain cases they can be considerably

simplified. The formulas can be put in a particularly simple form when the given family of curves is the family (1), characterizing the distribution of some meteorological element or its derivatives, and the second "directing" family can be chosen at will, to fit the conditions of the specific problem.

In that case the "directing" family generally has a purely geometrical meaning, but nevertheless, a case may occur where this family is associated with certain physical properties of the field as well. Thus, for example, considering the axis of a pressure trough as a characteristic curve, we can take another such axis (e.g., the pressure high axis) for the directing curve. The point thus individualized will have a definite physical significance as a node, i.e., a netural or hyperbolic point.

We shall consider only the first case, where the choice of the directing family is arbitrary.

We assume that the directing family is a set of fixed straight lines, running parallel to the x-axis.

Then we have

$$f(x, y, t) = a; \quad \varphi = y = b, \tag{10}$$

and all the partial derivatives of φ, except

$$\frac{\partial \varphi}{\partial y} = 1,$$

must vanish:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial x\, \partial y} = \frac{\partial^2 \varphi}{\partial y^2} = \frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial t^2} = 0.$$

consequently $D = \frac{\partial f}{\partial x}$. Equations (7) give

$$c_x = -\frac{\frac{\partial f}{\partial t}}{\frac{\partial f}{\partial x}}; \quad c_y = 0, \tag{11}$$

and from equation (9) we obtain:

$$A_x = -\frac{\frac{\partial^2 f}{\partial t^2} + 2c_x \frac{\partial^2 f}{\partial x\, \partial t} + c_x^2 \frac{\partial^2 f}{\partial x^2}}{\frac{\partial f}{\partial x}}; \quad A_y = 0. \tag{12}$$

By the foregoing, we clearly do not impose any restrictions on the function f except for continuity, differentiability and the condition $D \neq 0$, and therefore formulas (11), (12), (14) and (15) are applicable to a very wide class of functions.

We shall now apply the formulas obtained for the velocity and acceleration to examine the motion of the more important characteristic curves of the pressure field.

§ 3. Velocity and Acceleration of the Isobars

Let the isobar

$$p(x, y, t) = a, \tag{1}$$

be the characteristic curve, whose motion we study. Orienting the x-axis along the isobaric ascendent (Figure 124) at a given point, we obtain:

$$c_l = -\frac{\frac{\partial p}{\partial t}}{\frac{\partial p}{\partial x}}, \tag{2}$$

$$A_l = -\frac{\frac{\partial^2 p}{\partial t^2} + 2c_l \frac{\partial^2 p}{\partial x\, \partial t} + c_l^2 \frac{\partial^2 p}{\partial x^2}}{\frac{\partial p}{\partial x}}. \tag{3}$$

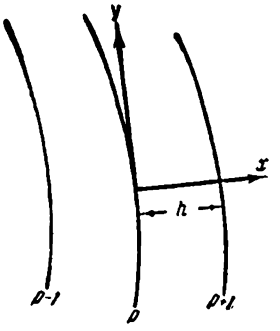

Figure 124. Computation of the velocity of an isobar

Formulas (2) and (3) determine exactly the instantaneous values of the velocity and acceleration of the isobar at an arbitrarily chosen point, i.e., these formulas establish the instantaneous kinematic state of the pressure field.

But formulas (2) and (3) can also be used for the extrapolation of the pressure distribution over a small time interval t. The displacement S_t of the isobar in the direction of the normal will be determined by the following approximate extrapolation formula:

$$S_t = c_i t + A_t \frac{t^2}{2}, \tag{4}$$

where S_t is the displacement of the isobar along the original normal, and t is the prognostic period.

Let us write formula (2) in another form. We recall that we have chosen the coordinate axes so that $\frac{\partial p}{\partial x}$ is a pressure ascendent. Working with individual isobars we may write

$$\frac{\partial p}{\partial x} = |\nabla p| = \frac{1}{h},$$

where h is the distance between two neighboring isobars. If we also assume that $\frac{\partial p}{\partial t}$ = T, where T is the barometric tendency, formula (2) becomes

$$c_i = -Th. \tag{5}$$

Thus, the velocity of the isobar along its normal is equal to minus the tendency, multiplied by the distance between the individual isobars. From (5) it follows, that in the region of positive tendencies (T>0) the isobars move along the pressure gradient ($c_i < 0$), and in the region of negative tendencies (T<0), they move along the pressure ascendent ($c_i > 0$). If the acceleration determined by formula (3), is small, then for a not excessively long period of prognosis the interpolation formula (4) has the form

$$S_i = c_i t.$$

However, if the isobar is accelerated we have

$$h = h_0 + (c_2 - c_1)\, t. \tag{6}$$

$$h = h_0 + (c_2 - c_1)\, t + (A_2 - A_1)\frac{t^2}{2}, \tag{7}$$

and the pressure gradient equals $\frac{1}{h}$.

Since the wind velocity is usually proportional to the pressure gradient, formulas (6) and (7) allow to forecast it.

However, temporal variations involve not only the density of the isobars and consequently the magnitude of wind velocity, but also their orientation and therefore the wind direction. Restricting ourselves to the case of uniform isobaric motion, we obtain the rotation velocity in the form

$$\frac{\partial c_l}{\partial y} = -\left(I_y \cdot h + T \cdot \frac{\partial h}{\partial y}\right), \qquad (8)$$

where $I_y = \frac{\partial T}{\partial y} = \frac{\partial^2 p}{\partial y\, \partial t}$ is the component of the isallobaric ascendent tangent to the isobar. Thus, the instantaneous velocity of rotation of the uniformly moving isobar depends upon 1) the isobaric density h, 2) the isobaric divergence $\frac{\partial h}{\partial y}$, 3) the barometric tendency, and 4) the relative position of isobars and isallobars. If the isobars are all parallel to each other, $\frac{\partial h}{\partial y} = 0$, whence

$$\frac{\partial c_l}{\partial y} = -I_y h.$$

As can be seen from formula (8), the sign of $\frac{\partial c_l}{\partial y}$ is opposite to that of I_y. Therefore, in the region where the component of the isallobaric ascendent is positive ($I_y > 0$) and the isobars rotate to the left, and vice versa. Thus, in the region of the isallobaric minimum, the isobaric curvature becomes cyclonic and at the isallobaric maximum, it becomes anticyclonic.

If the isobars are parallel to the isallobars, $I_y = 0$ and consequently,

$$\frac{\partial c_l}{\partial y} = -T \frac{\partial h}{\partial y}. \qquad (9)$$

In case of isobaric divergence the isobars deviate towards the depression and in case of convergence — in the direction of the elevation.

The foregoing argument demonstrates the importance of the relative position of the isallobars and isobars for the determination of the changes in the pressure distribution.

We will now discuss formula (3), determining the acceleration of the isobars. With the chosen coordinate axes, the terms entering in formula (3) have the following meaning: $\frac{\partial p}{\partial x}$ is the presure ascendent, encountered already in formula (2); $\frac{\partial^2 p}{\partial t^2} = \frac{\partial T}{\partial t}$ is the curvature of the barogram, or the local change of the barometric tendency; $\frac{\partial^2 p}{\partial x\, \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial t}\right) = I_x$ is the component of the isallobaric ascendent along the normal to the isobar; $\frac{\partial^2 p}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right)$ is the normal variation of the pressure ascendent, or the curvature of the pressure profile.

All the quantities entering in formula (3), with the exception of $\frac{\partial^2 p}{\partial t^2}$, can be taken from a simple synoptic chart. Only for the determination of $\frac{\partial^2 p}{\partial t^2}$ is it necessary to have either a barogram, or two adjoining synoptic charts taken at a 3 hour interval, with the values of the barometric tendency marked on them. We should note, that the barogram curvature greatly influences the value of the acceleration of the isobar. Expressing the pressure gradient, as before, by the distance between the individual isobars, we obtain:

$$\frac{\partial p}{\partial x} = \frac{1}{h}; \quad \frac{\partial^2 p}{\partial x^2} = -\frac{1}{h^2}\frac{\partial h}{\partial x};$$

recalling that $\frac{\partial^2 p}{\partial x \partial t} = I_x$, we write formula (3) in the form

$$A_i = -\frac{\partial^2 p}{\partial t^2} \cdot h - 2c_i I_x h + \frac{c_i^2}{h} \frac{\partial h}{\partial x}. \tag{10}$$

Thus, one can consider the acceleration of the isobar to consist of three parts

$$A_i = A' + A'' + A''', \tag{11}$$

where

$$A' = -h \frac{\partial^2 p}{\partial t^2}, \tag{12}$$

$$A'' = -2c_i I_x h, \tag{13}$$

$$A''' = \frac{c_i^2}{h} \cdot \frac{\partial h}{\partial x}. \tag{14}$$

Most difficult is the determination of the first component A' of the acceleration, because it is expressed by a magnitude neither measured directly, nor calculated by the use of a single chart. This magnitude $\frac{\partial^2 p}{\partial t^2}$ is the barogram curvature. Typical barograms are shown in Figure 125.

Figure 125 shows, that if the barogram is a straight line, i.e., when the pressure varies gradually and has therefore a constant tendency, or when a point of inflection occurs in the recording curve, we have

$$\frac{\partial^2 p}{\partial t^2} = 0 \text{ and } A' = 0. \tag{15}$$

If the barogram is curved cyclonically (Figure 125 *a*, *d*, *g*), which is possible at an accelerated pressure rise (*a*), at a decelerated pressure drop (*d*), or when the pressure starts rising after having fallen (*g*), we obtain

$$\frac{\partial^2 p}{\partial t^2} > 0 \text{ and } A' < 0. \tag{16}$$

If the barogram is curved anticyclonically (Figure 125, *b*, *e*, *h*), which is possible at a decelerated pressure rise (*b*), at an accelerated pressure drop (*e*), or when the pressure starts falling after having risen (*h*), we have

$$\frac{\partial^2 p}{\partial t^2} < 0 \text{ and } A' > 0. \tag{17}$$

Anyhow, the barogram facilitates the determination of the sign of the derivative $\frac{\partial^2 p}{\partial t^2}$ (by the nature of its curvature). The magnitude of this derivative cannot be obtained from the barogram, and for its determination one can use any of the following three methods.

1) This magnitude can be obtained as the difference between two consecutive three-hour barometric tendencies, if we adopt a 3 hour interval as the time unit, as is usually done in practice; we have

$$\frac{\partial^2 p}{\partial t^2} = T_1 - T_0, \tag{18}$$

Whereby these tendencies should be taken for consecutive times, 3 hours apart.

2) Since the tendency is in itself a difference between two consecutive pressure values

$$\frac{\partial p}{\partial t} = T = p_1 - p_0,$$

the value of $\frac{\partial^2 p}{\partial t^2}$ can be obtained from three consecutive pressure observations by the formula

$$\frac{\partial^2 p}{\partial t^2} = (p_2 - p_1) - (p_1 - p_0) = p_2 - 2p_1 + p_0. \qquad (19)$$

For greater accuracy of the determination of $\frac{\partial^2 p}{\partial t^2}$ one should use barometer readings and not barograph recordings, since the former are more reliable, being much less subject to all kinds of random fluctuations, which distort the value of the barometric tendency as recorded by the barograph.

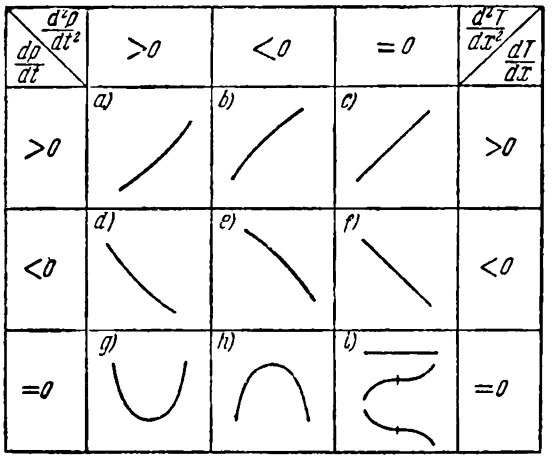

Figure 125. Different types of barograms

The difficulty in the determination of the derivative $\frac{\partial^2 p}{\partial t^2}$ consists also in the necessity of knowing the tendencies at two consecutive moments, not more than 3 hours apart. Incidentally, the precise sympiezometers widely used by the Soviet weather service eliminate the difficulty in the determination of $\frac{\partial^2 p}{\partial t^2}$. By the use of these instruments one can obtain with sufficient precision the pressure differences necessary for the calculation of $\frac{\partial^2 p}{\partial t^2}$.

3) Finally, we would like to point out, that $\frac{\partial^2 p}{\partial t^2}$ can be calculated from the isallobar velocity, if this has been determined beforehand with satisfactory accuracy.

In fact, applying formula (11) § 2 to the isallobar, we find

$$c_T = -\frac{\frac{\partial T}{\partial t}}{\frac{\partial T}{\partial x}} = -\frac{\partial^2 p}{\partial t^2} \cdot H, \qquad (20)$$

where c_T is the velocity of the isallobar in the direction of the pressure ascendent, and H is the distance between adjacent individual isallobars.

Formula (20) is completely analogous to the formula (2) for the velocity of isobars. The isallobar velocity is positive, if the barogram is curved anticyclonically (Figure 125, *b*, *e*, *h*), and negative if the barogram curves cyclonically (Figure 125, *a*, *d*, *g*). If the tendency is uniform $\frac{\partial T}{\partial t} = 0$, the isallobar is static ($c_T = 0$), except in the center of the isallobaric region, where $\frac{\partial T}{\partial x} = 0$, being a singular point, for which c_T is of the indeterminate type $\frac{0}{0}$.

By the aid of two consecutive synoptic charts drawn at 3 hour intervals, the isallobar velocity of the c_T can be determined. Knowing the distance H between adjacent individual isallobars, we obtain

$$\frac{\partial^2 p}{\partial t^2} = -\frac{c_T}{H}, \qquad (21)$$

and the corresponding component of acceleration will be determined by the formula

$$A' = c_T \cdot \frac{h}{H}. \qquad (22)$$

Thus, the first component A' of the acceleration can be expressed by any of the following formulas

$$A' = \begin{cases} -(T_1 - T_0)\,h, \\ -(p_2 - 2p_1 + p_0)\,h, \\ c_T \cdot \dfrac{h}{H}. \end{cases} \qquad (23)$$

Formulas (23) show that other conditions being equal, the acceleration of the isobar will be greater, the farther apart are the isobars spaced (large h), the denser are the isallobars (small H) and the greater their velocity (large c_T).

Thus, the component A' of the acceleration will be greatest where the region of rapidly moving isallobars overlaps the region of dense isallobars. A' vanishes, or approaches zero only for a uniformly or almost uniformly varying pressure, which corresponds on the synoptic chart to zero or very small isallobaric gradients, i.e., to stationary isallobaric regions. This situation is encountered in the centers of isallobaric systems, and also in the warm sector of the cyclone.

Let us also note, that A' and c_T have the same sign. The isobars are accelerated if the isallobars move along the isallobaric ascendent, and decelerate if the isallobars move along the isallobaric gradient.

We shall now turn to the study of the second component A'' of the acceleration, determined by the relationship (13).

Writing $I_x = \frac{1}{H_x}$, where H_x is the distance between adjacent individual isallobars, measured along the normal to the isobar, we obtain:

$$A''_i = -2c_i I_x h = -2c_i \frac{h}{H_x} = 2T \frac{h^2}{H_x}. \qquad (24)$$

All the quantities in formula (24) are obtainable from one synoptic chart.

The sign of the acceleration component A'' is determined by the sign of the product of the tendency and the component of the isallobaric ascendent, according to the scheme presented in Figure 126. This is positive, if the pressure and isallobaric ascendents form an acute angle in the region of positive tendencies or an obtuse angle in the region of negative tendencies.

The acceleration component A'' vanishes in each of the three following cases:

1) When the tendency vanishes. This happens in the vicinity of the axes of symmetry of troughs and highs. In these regions the velocity of the isobars also vanishes and the motion of the isobars is determined exclusively by the value of A'_i.

2) When the isallobaric gradient $I = 0$, i.e., in the centers of isallobaric regions.

3) When the isallobars are orthogonal to the isobars, i.e., when $\cos(I, x) = 0$.

Finally, the third component A'''_i can also be calculated by the use of data obtainable from a single synoptic chart:

$$A'''_i = \frac{c_i^2}{h} \cdot \frac{\partial h}{\partial x} = T^2 h \frac{\partial h}{\partial x}. \qquad (25)$$

J_x \ T	>0	<0
>0	>0	<0
<0	<0	>0

Figure 126. Scheme of the signs of acceleration component A''

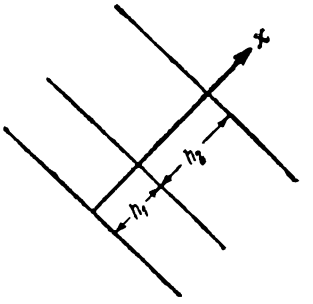

Figure 127. Calculation of $\frac{\partial h}{\partial x}$

For the calculation of $\frac{\partial h}{\partial x}$ we denote the distance between the isobars $(p_0 - 1)$ and p_0 by h_1 and the distance between the isobars p_0 and $(p_0 + 1)$ by h_2 (Figure 127).

For the middle isobar p_0 we may put $h = \frac{1}{2}(h_1 + h_2)$,

$$\frac{\partial h}{\partial x} = \frac{h_2 - h_1}{\frac{1}{2}(h_1 + h_2)}; \qquad (26)$$

Substituting (26) in (25) we obtain after reduction:

$$A_l''' = T^2 (h_2 - h_1). \qquad (27)$$

Consequently, the acceleration will be positive, if the isobars move apart in the direction of the pressure ascendent and vice versa. The acceleration A''' vanishes:

1) in regions, where the isobars are uniformly spaced, which happens in the anterior and posterior parts of moving cyclones and anticyclones, and also in the warm parts of cyclones;

2) in regions, of vanishing tendency.

On the synoptic chart we often find regions, for which the motion of the isobars can be calculated without acceleration, or without taking into account certain terms in the expression of acceleration. Therefore, in order to avoid complicated and cumbersome calculations, one should look for such regions and perform the calculations for them.

§ 4. Velocity and Acceleration of Troughs and Highs

In the preceding section we have shown, that the motion of the isobar greatly depends on the curvature of the barogram* $\frac{\partial^2 p}{\partial t^2}$, i. e., on the local change of tendency, since this magnitude enters in the equation of the acceleration of the isobar. At points where the barogram curvature reaches high values, its determination turns out to be technically rather difficult. The barogram curvature is observed to be especially great at highs, troughs or pressure centers. Postponing the discussion of the motion of pressure centers until the next section, we will deal here with the motion of troughs and highs. It should be stressed again that in principle one could predict the position of troughs and highs by precalculating the motion of the isobars, and only the practical difficulties in the determination of $\frac{\partial^2 p}{\partial t^2}$ force us to look for some other way to precalculate the velocity and the acceleration of troughs and highs. For the sake of brevity we shall apply the term "baric axis", since the formulas for troughs and for highs are similar. The task in which we are engaged naturally consists of two parts:

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*[Translator's note: The Russian text reads barometric tendency , an obvious misprint.]

1) to precalculate the motion of the baric axis, and

2) to precalculate the pressure changes on the baric axis.

We shall treat only the first part of the problem. The baric axis is the geometric locus of points of maximal isobaric curvature. We now introduce the equations of this axis.

If the equation of the curve is given in the explicit form

$$y = y(x, t),$$

with t as parameter, its curvature σ is determined by the formula

$$\sigma = \frac{y''}{[1 + y'^2]^{3/2}}. \tag{1}$$

If, however, the equation of the curve is given in the implicit form

$$f(x, y, t) = \text{const.},$$

then, since

$$y' = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}; \quad y'' = -\frac{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2 - 2\frac{\partial^2 f}{\partial x \partial y}\cdot\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left(\frac{\partial f}{\partial y}\right)^3},$$

its curvature is determined by

$$\sigma(x, y, t) = -\frac{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2 - 2\frac{\partial^2 f}{\partial x \partial y}\cdot\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]^{3/2}}. \tag{2}$$

The curvature of the isobar

$$p(x, y, t) = \text{const.}$$

will therefore be

$$\sigma(x, y, t) = -\frac{\frac{\partial^2 p}{\partial x^2}\left(\frac{\partial p}{\partial y}\right)^2 - 2\frac{\partial^2 p}{\partial x \partial y}\cdot\frac{\partial p}{\partial x}\cdot\frac{\partial p}{\partial y} + \frac{\partial^2 p}{\partial y^2}\left(\frac{\partial p}{\partial x}\right)^2}{\left[\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2\right]^{3/2}}. \tag{3}$$

We look for the points, where the curvature of the isobars has a maximum; but these points also lie on the isobars. Therefore, their coordinates should also satisfy the equations of the corresponding isobars. To be definite, we take x as the argument of the function y, satisfying the isobaric equation

$$p = \text{const.} \tag{4}$$

The necessary condition for the magnitude $\sigma(x, y)$ to be maximal, is given by the equality

$$\left.\begin{aligned} \frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial y}\cdot\frac{dy}{dx} &= 0, \\ \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx} &= 0. \end{aligned}\right\} \tag{5}$$

And this is possible only if

$$\begin{vmatrix} \frac{\partial \sigma}{\partial x}, & \frac{\partial \sigma}{\partial y} \\ \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \end{vmatrix} = 0. \tag{6}$$

This relationship is satisfied by points of maximum isobaric curvature which lie on the corresponding isobars. Relationship (6) is the equation of the baric axis.

Designating the left-hand side of (6) by F, we obtain the equation of the baric axis in the form

$$F=\begin{vmatrix} \frac{\partial \sigma}{\partial x}, & \frac{\partial \sigma}{\partial y} \\ \frac{\partial p}{\partial x}, & \frac{\partial p}{\partial y} \end{vmatrix}=0 \tag{7}$$

According to (11) § 2, the velocity of the baric axis is:

$$c_L=-\frac{\frac{\partial F}{\partial t}}{\frac{\partial F}{\partial x}}, \tag{8}$$

where

$$\left.\begin{aligned} \frac{\partial F}{\partial t}&=\begin{vmatrix} \frac{\partial^2 \sigma}{\partial x\,\partial t}, & \frac{\partial \sigma}{\partial y} \\ \frac{\partial^2 p}{\partial x\,\partial t}, & \frac{\partial p}{\partial y} \end{vmatrix}+\begin{vmatrix} \frac{\partial \sigma}{\partial x}, & \frac{\partial^2 \sigma}{\partial y\,\partial t} \\ \frac{\partial p}{\partial x}, & \frac{\partial^2 p}{\partial y\,\partial t} \end{vmatrix}, \\ \frac{\partial F}{\partial x}&=\begin{vmatrix} \frac{\partial^2 \sigma}{\partial x^2}, & \frac{\partial \sigma}{\partial y} \\ \frac{\partial^2 p}{\partial x^2}, & \frac{\partial p}{\partial y} \end{vmatrix}+\begin{vmatrix} \frac{\partial \sigma}{\partial x}, & \frac{\partial^2 \sigma}{\partial x\,\partial y} \\ \frac{\partial p}{\partial x}, & \frac{\partial^2 p}{\partial x\,\partial y} \end{vmatrix}. \end{aligned}\right\} \tag{9}$$

Formula (8) may be somewhat simplified by a special choice of the coordinate axes. Namely, we direct the x-axis along a tangent to the isobar so that initially $\frac{\partial p}{\partial x}=0$; obviously, the x-axis will not necessarily be normal to the baric axis (Figure 128). The instantaneous velocity of the baric axis will then be expressed by the formula

$$c_L=-\frac{\frac{\partial^2 \sigma}{\partial x\,\partial t}\cdot\frac{\partial p}{\partial y}-\frac{\partial^2 p}{\partial y\,\partial t}\cdot\frac{\partial \sigma}{\partial y}}{\frac{\partial^2 \sigma}{\partial x^2}\cdot\frac{\partial p}{\partial y}-\frac{\partial^2 p}{\partial x^2}\cdot\frac{\partial \sigma}{\partial y}}, \tag{10}$$

where σ is determined by the relationship (3).

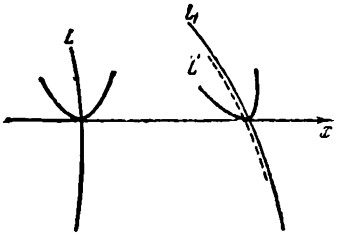

Figure 128. The baric axis and the characteristic line

Differentiating (3), we get a correct but practically unapplicable formula, determining the instantaneous velocity of the trough in the direction of the tangent to the isobar

$$c_L=\frac{1}{A}\left\{3\frac{\partial^2 p}{\partial x\,\partial t}\left[\frac{\partial^3 p}{\partial x^2\,\partial t}\frac{\partial p}{\partial y}-2\left(\frac{\partial^2 p}{\partial x\,\partial y}\right)^2+\frac{\partial^2 p}{\partial x^2}\left(\frac{\partial^2 p}{\partial x^2}-\frac{\partial^2 p}{\partial y^2}\right)\right]-\right.$$
$$\left.-\frac{\partial p}{\partial y}\left[\frac{\partial^4 p}{\partial x^3\,\partial t}\frac{\partial p}{\partial y}+\frac{\partial^3 p}{\partial x^3}\cdot\frac{\partial^2 p}{\partial y\,\partial t}-3\frac{\partial^3 p}{\partial x^2\,\partial t}\cdot\frac{\partial^2 p}{\partial x\,\partial y}-3\frac{\partial^2 p}{\partial x^2}\frac{\partial^3 p}{\partial x\,\partial y\,\partial t}\right]\right\}, \tag{11}$$

where

$$A=\frac{\partial p}{\partial y}\left(\frac{\partial^4 p}{\partial y^4}-2\frac{\partial^3 p}{\partial x^2}\cdot\frac{\partial^3 p}{\partial x^2\partial y}\right)+3\left(\frac{\partial^2 p}{\partial x^2}\right)^2\cdot\left(\frac{\partial p}{\partial y}-\frac{\partial p}{\partial x}\right).$$

The formula for acceleration of the baric axis will be even more complicated. Thus, precise formulas for the extrapolation of the motion of troughs and highs

cannot be used for practical purposes. We shall therefore try to derive approximative formulas.

Let us consider the case of a progressive movement of baric axis. By the previous choice of coordinates, all the points lying on the baric axis will in this case move parallel to the x-axis, the equation of the baric axis will have the form

$$\frac{\partial p}{\partial x} = 0 \tag{12}$$

throughout the whole motion and not only at the initial moment, as in the general case.

Considering (12) as an equation of a characteristic curve, we get

$$c_L = -\frac{\frac{\partial^2 p}{\partial x\, \partial t}}{\frac{\partial^2 p}{\partial x^2}}, \tag{13}$$

$$A_l = \frac{\frac{\partial^3 p}{\partial x\, \partial t^2} + 2C_l \frac{\partial^3 p}{\partial x^2\, \partial t} + C_L^2 \frac{\partial^3 p}{\partial x^3}}{\frac{\partial^2 p}{\partial x^2}}. \tag{14}$$

However, also in case of rotation of the baric axis there will be in the neighborhood of the trough such a characteristic line, for which the condition $\frac{\partial p}{\partial x} = 0$ will be constantly fulfilled. But this line will coincide with the baric axis only at the initial moment, and will gradually depart from it.

Figure 128 indicates the position L of the baric axis at the initial moment and its position L' after time t. L' describes the position of the characteristic line determined by the condition $\frac{\partial p}{\partial x} = 0$ and coinciding initially with the trough. If the baric axis is rather pronounced, the $\frac{\partial p}{\partial x} = 0$ line lies very close to it. A considerable difference appears in case the baric axis is not clearly defined. But this case is not of practical importance.

§ 5. Motion of Pressure Centers. Other Formulas for Extrapolation of the Pressure Field

At a center the pressure has a relative maximum or minimum. Generally speaking, one can draw two baric axes through the pressure center, one of them through the points of maximal curvature of the isobars and the second through the points of minimal curvature. Therefore, the problem of the motion of the baric center reduces to the determination of the motion of the point of intersection of the two baric axes.

We shall limit the discussion of the motion of pressure centers to their representation by points, remaining for a certain period of time stationary points of the pressure field, i.e., points for which the following relationships continually hold

$$\frac{\partial p(x, y, t)}{\partial x} = 0, \quad \frac{\partial p(x, y, t)}{\partial y} = 0, \tag{1}$$

although the pressure at these points changes with time.

It is well known that a function of two variables which satisfies condition (1) has a minimum if

$$\frac{\partial^2 p}{\partial x^2} > 0; \quad \frac{\partial^2 p}{\partial y^2} > 0; \quad \Delta = \left(\frac{\partial^2 p}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 p}{\partial x^2} \cdot \frac{\partial^2 p}{\partial y^2} < 0, \tag{2}$$

and a maximum, if

$$\frac{\partial^2 p}{\partial x^2} < 0; \quad \frac{\partial^2 p}{\partial y^2} < 0; \quad \Delta = \left(\frac{\partial^2 p}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 p}{\partial x^2} \cdot \frac{\partial^2 p}{\partial y^2} < 0. \tag{3}$$

Let us assume, that the function $p(x, y, t)$ has continuous derivatives of the first and second orders with respect to all arguments. Applying the rule of differentiation of implicit functions, we obtain from (1) the following expressions for the components of velocity of the pressure center:

$$\left.\begin{aligned} c_x &= \frac{dx}{dt} = \frac{1}{\Delta}\left(\frac{\partial^2 p}{\partial x\, \partial t} \cdot \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 p}{\partial y\, \partial t} \cdot \frac{\partial^2 p}{\partial x\, \partial y}\right), \\ c_y &= \frac{dy}{dt} = \frac{1}{\Delta}\left(\frac{\partial^2 p}{\partial y\, \partial t} \cdot \frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial x\, \partial t} \cdot \frac{\partial^2 p}{\partial x\, \partial y}\right). \end{aligned}\right\} \tag{4}$$

If the baric axes which pass through the pressure center do not rotate, then $\frac{\partial^2 p}{\partial x\, \partial y} = 0$, and formulas (4) are simplified:

$$c_x = -\frac{\frac{\partial^2 p}{\partial x\, \partial t}}{\frac{\partial^2 p}{\partial x^2}}, \qquad c_y = -\frac{\frac{\partial^2 p}{\partial y\, \partial t}}{\frac{\partial^2 p}{\partial y^2}}. \tag{5}$$

In this simple case the components of the acceleration of the pressure center also have a relatively simple form:

$$\left.\begin{aligned} A_x &= -\frac{\frac{\partial^3 p}{\partial x\, \partial t^2} \cdot \frac{\partial^2 p}{\partial x^2} - 2\frac{\partial^2 p}{\partial x\, \partial t} \cdot \frac{\partial^3 p}{\partial x^2 \partial t}}{\left(\frac{\partial^2 p}{\partial x^2}\right)^2}, \\ A_y &= -\frac{\frac{\partial^3 p}{\partial y\, \partial t^2} \cdot \frac{\partial^2 p}{\partial y^2} - 2\frac{\partial^2 p}{\partial y\, \partial t} \cdot \frac{\partial^2 p}{\partial y\, \partial t}}{\left(\frac{\partial^2 p}{\partial y^2}\right)^2}. \end{aligned}\right\} \tag{6}$$

Analogous formulas can be derived for the motion of a front, for pressure variations on the axis of the pressure system and at the center, for frontogenesis and frontolysis and also for the generation and dissolution of pressure centers.

So, for example, by the use of general formulas, it is easy to obtain the velocity and the acceleration of the front, the equation of which may be written in the following form:

$$p_1(x, y, t) - p_2(x, y, t) = 0. \tag{7}$$

If the y-axis is parallel to the front and the x-axis orthogonal to it, then according to the preceding chapter

$$\frac{\partial p_1(x, y, t)}{\partial y} - \frac{\partial p_2(x, y, t)}{\partial y} = 0 \tag{8}$$

and

$$\frac{\partial p_1(x, y, t)}{\partial x} - \frac{\partial p_2(x, y, t)}{\partial x} \neq 0. \tag{9}$$

Therefore, the front velocity c_f will be given by:

$$c_f = -\frac{\frac{\partial p_1}{\partial t} - \frac{\partial p_2}{\partial t}}{\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x}}, \tag{10}$$

and its acceleration by the formula

$$A_f = -\frac{\left(\frac{\partial^2 p_1}{\partial t^2} - \frac{\partial^2 p_2}{\partial t^2}\right) + 2c_f\left(\frac{\partial^2 p_1}{\partial x\,\partial t} - \frac{\partial^2 p_2}{\partial x\,\partial t}\right) + c_f\left(\frac{\partial^2 p_1}{\partial x^2} - \frac{\partial^2 p_2}{\partial x^2}\right)}{\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x}}. \quad (11)$$

In case of occlusion of a cyclone the cold front overtakes the warm one, displacing the warm air mass. Therefore, the occlusion process may be quantitatively estimated by the shortening of the distance between the cold and the warm fronts along the x-axis:

$$S_{kf} - S_{wf} = (c_{kf} - c_{wf})\,t + (A_{kf} - A_{wf})\,\frac{t^2}{2}, \quad (12)$$

where the index k denotes the magnitudes referring to the cold front and the index w those which refer to the warm front.

For a more detailed study of the kinematics of the pressure field we refer the reader to the handbook "Methods of Pressure Field Extrapolation" (Hydrometeorological Publishing House, 1940) and especially to the detailed papers by A.F. Dyubyuk and E.S. Kuznetsov.

The extrapolation formulas derived in this chapter and others of the same type are purely kinematic. Therefore, on the basis of these formulas one can deduce nothing as to the causal connections between the different variations in the pressure field. The synoptician, of course, has often recourse to extrapolation, for inferring future variations of the pressure field from those already observed. The extrapolation formulas given and similar ones, allow to base the extrapolation on a better theoretical foundation. They permit the forecaster to take into account the structure of the pressure field in his evaluation of future variations.

However, the essence of extrapolation is namely this, that on the basis of infinitesimal variations occurring at a certain moment, we try to predict the value of some function in the more or less distant future. But is this infinitesimally small variation more accessible than the finite variation in the remote future? The answer is, no. First, because the instruments we use are not perfect, and second, which is the main reason, because this statistical regularity, which we ascribed to the extrapolated function (continuity and differentiability), is true as long as we consider its broad features, and is invalidated for the finer details. Therefore, formal extrapolation can never enable us to penetrate the future.

Chapter XV

TURBULENT EXCHANGE. SPECTRAL STRUCTURE OF TURBULENCE

§ 1. Fundamental Formulas of Turbulent Exchange

The first experiments in the field of turbulence have already shown that as soon as the Reynolds number exceeds a critical value, the flow ceases to be laminar. The trajectories of the individual particles are no longer regular and become increasingly tangled and mixed up. Hereby only the mean velocity of the flow preserves a well-defined value and direction, while each air particle acquires its own additional velocity.

In Euler's representation turbulence is expressed by temporal fluctuations of the velocity and other quantities. In the Lagrange representation, turbulence is expressed by chaotic motion of particles along very complicated trajectories, whereby the particles are constantly mixed. This macroscopic motion recalls the chaotic motion of molecules.

The most characteristic feature of atmospheric turbulence is the turbulent mass exchange and the corresponding mixing of air. In the macroscopic transfer, the "turbulence element", or the individual eddy does not simply penetrate the surrounding air mass, but is carried along with it. Therefore, in the course of time the element of turbulence dissolves, so to speak, in the surrounding air. This is the principal difference between a vortex in the atmosphere, being a real viscous fluid, and a vortex in a perfect fluid. A vortex formed in a perfect fluid moves without mixing with the surrounding fluid and persists indefinitely, hence the notion of "mixing" is out of place as regards perfect fluids, whereas for the real atmosphere it is a basic and most characteristic property of motion.

Thus, exactly as the chaotic motion of molecules causes the development of relaxation phenomena—viscosity, heat conduction and diffusion, so the chaotic motion of the elements of turbulence—moles [sic] or eddies—results in analogous relaxation processes. These, in distinction from molecular processes, are called the "eddy viscosity" (or "exchange coefficient"), "eddy conduction of heat" and "eddy diffusion".

The effect of turbulent mass exchange, or mixing in the atmosphere, is considerably greater than the effect of molecular exchange, and hence the latter can very often be disregarded.

The mathematical theory of turbulent exchange has been developed over the last 30 years. We shall discuss the theory of turbulent exchange on the basis of a very strict and systematic presentation, published in 1931 by the Soviet scientist L. V. Keller.

For the sake of simplicity we shall consider a flow of air over a horizontal surface, caused by the pressure gradient. We assume that the motion of the air is likewise horizontal, but only on the average. We direct the x-axis along the mean current. The components of the mean velocity along the y-and z-axes vanish in this case.

As regards the components of the additional local velocity, they all differ from zero. We will limit our discussion to plane motion.

We consider certain properties of air, which we would call substances. This is an arbitrary term, since it covers not only substances in the proper sense of the word, (dust, water vapor, etc), but also quantitative characteristics of air.

We assume that a substance displays the following three features, distinguishing it from other characteristics:

1) Substance is not destroyed during its transference together with the elementary particles of air, i.e., it remains constant as long as the particle in question does not mix with the surrounding air.

2) The total quantity of substance is preserved when two masses of air are mixed [additivity].

3) A substance is "passive". This means that its admixture with the air does not greatly affect the development of turbulence.

Substance is a quantitative characteristic of air, as opposed to other characteristics which are qualitative. The difference between them is based on the behavior of these properties upon mixing. If we mix two air masses M' and M'' possessing a certain characteristic to the amounts Q' and Q'' respectively, the substance of the mixture will be:

$$MQ = M'Q' + M''Q'', \tag{1}$$

and the qualitative characteristic of the mixture will be:

$$Q = \frac{M'Q' + M''Q''}{M' + M''} \tag{2}$$

Examples of quantitative characteristics or substances, are: momentum, enthalpy, electric space charge, and kinetic energy.

Among the qualitative attributes we mention: density, velocity, and temperature.

One should especially emphasize that upon mass exchange of air, not the temperature and velocity are exchanged, but the corresponding enthalpy and momentum. Only in case of an incompressible fluid may one speak of exchange or mixing of temperature and velocity.

Let us denote the specific substance, i.e., the amount of substance referred to unit mass, by s, and the density of the substance, i.e., the amount of substance per unit volume, by σ. Between s and σ exists the following obvious relationship: $\sigma = \rho \cdot s$. We shall refer in our discussion to the density of the substance, and write the results for the specific substance by analogy.

Let us consider a sufficiently large horizontal surface f, moving together with the mean air current (Figure 129). Each element of turbulence, moving on the average together with this surface, will cross it, ascending or descending with its own vertical velocity. Together with this vertical mass exchange of air, the substance will also be transferred along the vertical.

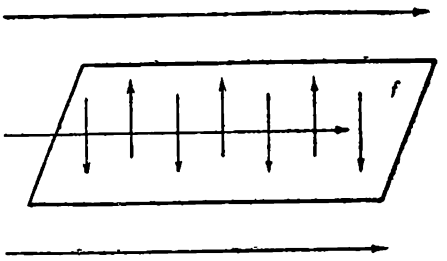

Figure 129. Determination of the eddy flux of a quantitative characteristic

The quantity of substance transferred across a surface element per unit time will equal $\sigma \cdot w \cdot df$, and the total amount of substance Q passing through a unit area per unit time equals

$$Q = \frac{1}{f} \int_{(f)} \sigma \cdot w \cdot df. \tag{3}$$

We shall consider the substance of a particle as a function of its position and the time: $\sigma(x, y, z, t)$. However, the fluctuations in the distribution of the substance over the surface f are smoothed out in averaging, so that $\sigma = \sigma(z, t)$.

The vertical velocity of a particle is also a function of its position and the time w(x, y, z, t).

It is important to point out that the surface f over which the product $\sigma \cdot w$, i.e., the vertical flux of the substance is averaged, should be very large. At any rate, it should be large enough to ensure a sufficiently comprehensive totality of the values of σ and w to afford the selection (with respect to some auxiliary characteristic) of a subset large enough to possess a stable average.

Let us make this selection with respect to two characteristics: the initial altitude z_0 at which the substance was located before its transfer to the surface f and the initial value of the velocity w_0. It is obvious that at the initial altitude z_0 the velocities can possess any value whatsoever. Therefore, to each value of w may correspond a large number of different values of $\sigma(x_0, y_0, z_0, t_0)$. The integral (3) will then split into a large number of integrals, each extended over a surface belonging to the corresponding values of z_0 and w_0. Integrating over such a surface and dividing by the surface area we obtain a certain average value of $\sigma(z_0, t_0)$, whereby we retain the previous designation for brevity. Thus we obtain a certain averaged field of the substance σ whereby, in contrast to the previously formed time average, we in this case average over the area.

We further assume that the distance $(z-z_0)$ covered by the particle transferring the substance and the time interval $(t-t_0)$ are sufficiently small. We expand the area average of the substance in a Taylor series and retain the first order terms only:

$$\sigma(z, t) = \sigma(z_0, t_0) + \frac{\partial \sigma}{\partial z}(z - z_0) + \frac{\partial \sigma}{\partial t}(t - t_0), \tag{4}$$

whence:

$$\sigma(z_0, t_0) = \sigma(z, t) - \frac{\partial \sigma}{\partial z}(z - z_0) - \frac{\partial \sigma}{\partial t}(t - t_0). \tag{5}$$

Multiplying (5) by w and integrating over the surface f, we obtain:

$$Q = \frac{1}{f} \int_{(f)} \sigma(z_0, t_0) \cdot w \cdot df,$$

or, taking (5) into account,

$$Q = \frac{1}{f} \int_{(f)} \left[\sigma(z, t) - \frac{\partial \sigma}{\partial z}(z - z_0) - \frac{\partial \sigma}{\partial t}(t - t_0) \right] w \cdot df. \tag{6}$$

In the integral (6) the following terms are constant: σ, $\frac{\partial \sigma}{\partial z}$, $\frac{\partial \sigma}{\partial t}$ being mean values for the whole area, t_0 being a parameter independent of the position of the surface element df on the horizontal plane f, and t, since the integration over the area is performed for a fixed instant of time.

For each element of f the variables are: the vertical velocity w, with which the air particle moves across the surface, and the distance $(z-z_0)$ from the level on which the particle was initially situated.

Putting the constants in front of the integral sign, we obtain:

$$Q = \left[\sigma(z, t) - \frac{\partial \sigma}{\partial t}(t - t_0) \right] \frac{1}{f} \cdot \int_f w \, df - \frac{\partial \sigma}{\partial z} \cdot \frac{1}{f} \int_{(f)} w \cdot (z - z_0) \, df.$$

Assuming that the volume of air crossing the surface f downwards equals the volume of air crossing it upwards, the first integral vanishes. Then we obtain for the flux of the substance:

$$Q = -\frac{\partial \sigma}{\partial z} \cdot \frac{1}{f} \int_{(f)} w (z - z_0) \, df. \tag{7}$$

Introducing the notation:

$$K = \frac{1}{f} \int_{(f)} w (z - z_0) \, df. \tag{8}$$

we have the eddy flux of the substance expressed by the formula:

$$Q = -K \cdot \frac{\partial \sigma}{\partial z}. \tag{9}$$

We shall call the magnitude K the coefficient of turbulence. As can be seen from (8), K is the mean value of the product of the values w and $(z-z_0)$:

$$K = \overline{w (z - z_0)}. \tag{10}$$

Let us remember that z and z_0 are coordinates of one and the same particle at the times t and t_0 and that $w = \frac{dx}{dt}$; the averaging is carried out at one instant of time over a sufficiently large area. We usually assume that such averaging may be replaced by averaging over a sufficiently long time interval at a certain fixed point z_0.

Further we have:

$$K = \overline{w (z - z_0)} = \overline{(z - z_0) \frac{d}{dt}(z - z_0)} = \frac{d}{dt} \overline{\left(\frac{z - z_0}{2} \right)^2}. \tag{11}$$

If K is a time-independent, the differential equation (11) can be easily integrated:

$$2K (t - t_0) = \overline{(z - z_0)^2}. \tag{12}$$

On the right-hand side of (12) stands the mean square of the distance of the particle from its initial level $\Sigma^2_{z-z_0}$. For constant K this mean square is directly proportional to the time $t-t_0$. Thus, the magnitude:

$$K = \frac{\overline{(z - z_0)^2}}{2(t - t_0)} = \frac{\Sigma^2_{z-z_0}}{2(t - t_0)} \tag{13}$$

is half the mean square of the distance, referred to unit time.

The coefficient of turbulence K may be yet differently expressed:

$$K = \overline{w(z - z_0)} = r\Sigma_w\Sigma_{z-z_0}, \tag{14}$$

where r is the coefficient of correlation between the vertical velocity w and the distance $(z-z_0)$, Σ_w and Σ_{z-z_0} being the mean squares of w and $z-z_0$ respectively.

We will term the product $l' = r\,\Sigma_{z_0-z}$ having the dimension of length, the "mixing length". Evidently,

$$K = l' \cdot \Sigma_w. \tag{15}$$

This somewhat arbitrary length is probably connected with the mean distance covered by the transferred particle in the exchange process. However, the notion of mean distance of transfer, as we shall see in the following, is also quite arbitrary. Therefore, the introduction of the "mean mixing length" is necessitated chiefly by the impossibility of measuring the mixing length itself.

Repeating the reasoning which has led to the derivation of formula (9) for the specific substance s, we obtain the following result:

$$Q = -A\frac{\partial s}{\partial z}, \tag{16}$$

where

$$A = \overline{\rho w(z - z_0)} \tag{17}$$

is the so-called exchange coefficient, first obtained by Schmidt, but in a somewhat different form. If we ignore the density variations which are extremely small in comparison with the variations in w and $(z-z_0)$, the following simple relationship between the coefficient of turbulence K and the exchange coefficient A will obtain:

$$A = \rho K. \tag{18}$$

It follows from the foregoing, that the values of the coefficients of exchange and of turbulence are completely independent of the nature of the particular substance and its distribution, and are exclusively determined by the extent of turbulence in the atmosphere. Therefore, the flux determined by equations (9) or (16) is also completely independent of the particular substance, being determined solely by its gradient and by the extent of atmospheric turbulence.

That means that having somehow determined the exchange coefficient, we can calculate the flux of any substance. For this purpose it is sufficient to know only the mean value of the vertical component of the gradient of this substance.

This circumstance greatly increases the usefulness of the exchange formula. Incidentally, we should not overestimate the importance of the obtained result. In the derivation of formulas (9) and (16) we have imposed a number of restrictions, some of which are very severe. We mention the most important: 1. Formulas (9) and (16) are applicable only to indestructible substances which are conserved;

therefore its application to the transfer of momentum, for instance, is not entirely justified, since momentum is gradually destroyed by the action of inner friction, and also varies under the influence of other forces (pressure gradient). 2. Formulas (9) and (16) are obtained by averaging over a very large area, which may be replaced by averaging over a very long time interval. They are therefore applicable only to very slowly varying processes.

Numerous calculations of A and K from observations on the transfer of different substances in the atmosphere have shown that A (and consequently K too) greatly varies with altitude and time. Therefore, the assumption that A = const., K = const., is too crude and sometimes completely distorts even the broadest features of the phenomenon.

Vertical transfer of substance leads to a temporal variation of the surface average of the substance in a turbulent air flow. We will derive formulas determining this variation.

Consider an elementary parallelepiped of a unit base and height Δz, moving without rotation with the mean air current. Due to turbulent exchange the substance will flow into the parallelepiped through its bottom and flow out through the top. The flux through the the lower and upper bases will respectively be:

$$Q(z) = -K(z)\left(\frac{\partial \sigma}{\partial z}\right)_z \text{ and } Q(z+\Delta z) = -K(z+\Delta z)\left(\frac{\partial \sigma}{\partial z}\right)_{z+\Delta z}.$$

The difference between the amount of substance flowing in and out per unit time will be:

$$\left(K\frac{\partial^2 \sigma}{\partial z^2} + \frac{\partial K}{\partial z}\frac{\partial \sigma}{\partial z}\right)\Delta z = \frac{\partial}{\partial z}\left(K\frac{\partial \sigma}{\partial z}\right)\Delta z,$$

whereby we have neglected the terms of second order. This difference equals the change of substance per unit time within a moving volume element $\frac{d\sigma}{dt}\cdot\Delta z\cdot 1$.

Equating these expressions and cancelling Δz, we obtain:

$$\frac{d\sigma}{dt} = \frac{\partial}{\partial z}\left(K\frac{\partial \sigma}{\partial z}\right). \tag{19}$$

If K does not depend on altitude, we obtain:

$$\frac{d\sigma}{dt} = K\frac{d^2\sigma}{dz^2}. \tag{20}$$

Repeating these considerations for the specific substance s, we obtain:

$$\frac{ds}{dt} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(A\frac{\partial s}{\partial z}\right), \tag{21}$$

and when A is independent of z,

$$\frac{ds}{dt} = \frac{A}{\rho}\frac{\partial^2 s}{\partial z^2}. \tag{22}$$

If we neglect the variations in ρ, equation (21) for the specific quantity of substance coincides with equation (19) for its density.

Equations (20) and (22) are called the equations of eddy diffusion or Fick's equations, because they are completely analogous to Fick's equations for molecular diffusion.

Let us note that the individual differentiation $\frac{d}{dt}$ in the left-hand side of the obtained equations can be replaced by local differentiation $\frac{\partial}{\partial t}$, since we are interested in the surface average of the substance, which by assumption varies only with altitude, but not horizontally. In most cases this assumption may be taken to hold in the atmosphere.

Concluding this section we may say that in case of horizontal motion the exchange coefficient essentially expresses the atmospheric turbulence. The value of the exchange coefficient includes the average intensity, as well as the average size of local motions in the atmosphere. It is remarkable that the value A can be found from observations on almost any meteorological element.

§ 2. Momentum Transfer

In the previous section we have derived exchange formulas for an arbitrary substance. We shall now apply these formulas to real substances, starting with the momentum.

Strictly speaking, momentum is not a passively transferred substance. It also does not possess the basic property of substance—its indestructibility. In reality momentum is destroyed by forces of internal friction and also destroyed or produced by its transfer into layers having other values of the pressure gradient.

We may assume, however, that in a not very thick layer, in which the process of momentum transfer occurs, the pressure gradient does not change with altitude and the variation of momentum under the influence of friction is so small that to a first approximation it may be neglected. We turn first of all to the analogy between turbulent and molecular transfer of momentum. Molecular transfer of momentum is expressed by Newton's law of viscosity:

$$\tau = \mu \frac{du}{dz},$$

where $\mu = \frac{1}{3} \rho l c$ is the coefficient of viscosity, ρ —the density, l—the molecular mean free path, c— the mean molecular velocity.

During turbulent flow, stresses caused by momentum transfer develop in the fluid, in addition to those due to the gradient of the mean current velocity. For plane-parallel flow, the additional stresses will equal $\rho \overline{u'w'}$.

The molecular stress does not disappear at that, and the additional turbulent stress is superimposed upon it, so that:

$$\tau = \mu \frac{du}{dz} - \rho \overline{u'w'}. \qquad (1)$$

Since the additional turbulent stresses is considerably greater than the molecular stress, one can always ignore the latter. The additional stress can only increase the total stress, but not diminish it; therefore:

$$\tau = -\rho \overline{u'w'} = -r \Sigma_{u'} \Sigma_{w'} > 0, \qquad (2)$$

and the coefficient of correlation between u' and w' should be negative.

For precise calculation of the additional stress, it is necessary to find all the separate values of u' and w' and then determine the mean value of their product $\overline{u'w'}$. However, the values of u' and w' cannot be directly measured. Therefore, the determination of the additional stress reduces to the discovery of the relationship

between u' and w' on the one hand, and determining the (measurable) mean flow velocity on the other hand.

Prandtl postulated a number of fruitful though not very rigorous hypotheses on the nature of these relationships, and in 1925 elaborated a theory of the transfer of momentum.

Since each fluid particle possesses, together with the horizontal velocity components, also a vertical component w', the particle trajectories in plane-parallel flow are sloping lines. According to Prandtl, a certain additional momentum $\rho u'$ is transferred from layer to layer with velocity w' along these trajectories.

Let us assume that a particle brings from a certain level z_1 a momentum consistent with the mean flow velocity u_1 at that level, and carries it over to another level z_2, where the particle mixes with the surrounding fluid, acquiring a velocity u_2. The segment of the vertical projection of the particle's trajectory enclosed between these two levels is called the "mixing length" after Prandtl. We shall denote the mixing length by l'.

Thus, at the levels z_1 and z_2 the individual air particle possesses a momentum equal to that of other particles moving with the mean flow velocity, and consequently its motion does not differ from the average motion of other particles. Only upon passage from the z_1 to the z_2 level does the particle exhibit its specific character, which it acquires at the z_1 level and loses at the z_2 level.

If the mixing length is short in comparison with the dimension of the flow system, we can assume:

$$\frac{u_2 - u_1}{l'} \approx \frac{du}{dz}. \tag{3}$$

Prandtl puts:

$$u_2 - u_1 = u',$$

i.e., he identifies the difference between the mean velocities at the different levels with the pulsation velocity at a fixed point. Consequently, following Prandtl,

$$u' = l' \frac{du}{dz}. \tag{4}$$

if one substitutes u' in (2), the additional stress will be expressed by the formula:

$$\tau = -\overline{\rho w' u'} = -\overline{\rho w' l'} \frac{\partial u}{\partial z}. \tag{5}$$

On the other hand, the momentum flux density is expressed by the formula:

$$Q = -\rho \overline{w' u'} = -A \frac{d\bar{u}}{dz}. \tag{6}$$

We therefore obtain the following expression for the exchange coefficient A:

$$A = \rho \overline{w' l'}. \tag{7}$$

Evidently the exchange coefficient is obtained similarly to the viscosity coefficient μ. However, the numerical factor 1/3 is missing in the exchange coefficient, since we introduced l' only for one direction-perpendicular to the flow, while in the consideration of three dimensional molecular exchange, the molecular mean free path was used. Instead of the average molecular velocity, the exchange

coefficient contains the vertical component of the pulsation velocity, perpendicular to the mean flow velocity Instead of the molecular mean free path, the mixing length, i.e., the segment of the vertical projection of the trajectory, traveled by the particle up to its complete mixing with the surrounding particles, is introduced in formula (7).

But formula (7), determining the exchange coefficient, still contains the additional instantaneous pulsation velocity w', which can be expressed by the mean velocity in the following manner. It is natural to assume that w' should be connected with the rotation of the fluid particle, or, which amounts to the same, with the vorticity, which after averaging can be expressed by $\frac{du}{dz}$.

Prandtl applies a proportionality law also in this case:

$$w' \approx l'' \frac{du}{dz}. \tag{8}$$

Combining (8) and (5), we finally get:

$$|\overline{u'w'}| = l^2 \left(\frac{du}{dz}\right)^2, \tag{9}$$

where $l^2 = l'l''$. The exchange coefficient will now be expressed by the formula:

$$A = \rho l^2 \cdot \left|\frac{du}{dz}\right|. \tag{10}$$

Prandtl's theory is not rigorous, for, as we have seen in the process of its derivation, Prandtl makes use of a number of assumptions which are not obvious at all, some of which even contradict experimental data.

1. Assuming $u_2 - u_1 \equiv u'$, Prandtl identifies the difference between the mean velocities at points located at different distances from a rigid wall, with the pulsation velocity at a given point. The pulsation of velocity is thus considered as a variation in the mean velocities, transferred between different points of the flow. This assumption is unfounded and contradicts the experimental results obtained in wind-tunnels, which have shown, that while the mean velocities away from the walls do not differ greatly among themselves, the pulsation velocity may reach 20 % of the mean velocity.

2. Prandtl assumes that mixing is a discontinuous process. At the beginning the velocity is simply transferred without any variation at all, and then, after the particle has covered a distance equal to the mixing length, it spontaneously mixes with the surrounding particles and acquires a velocity equal to the mean flow velocity at this point. It is not clear how the process of mixing can be discontinuous, how a particle can preserve the velocity it transfers without being affected by viscosity. We are even more baffled, noting that, according to Prandtl,the particle transfers not the instantaneous velocity it possessed at the outstart, but a mean velocity.

3. Prandtl assumes that the vertical pulsation velocity w' serves only for the transfer of the horizontal momentum, while in reality this velocity itself is being transferred, whereby the order of magnitude of w' and u' is the same as confirmed by measurements.

4. Prandtl introduces the length through a differential expression referring to the final point of the path, although the mixing length is determined by a known integral expression including the initial as well as to the final points of the path.

Regardless of a number of inaccuracies and obscurities, Prandtl's theory is widely applied in hydrodynamics and aerodynamics. This can be explained by the fact that the magnitudes l' and l'' entering in Prandtl's formulas can be considered not the proper mixing lengths, but only proportional to them. By such a generalization the theory loses its geometrical visuality but becomes more rigorous, whereby it becomes possible to draw certain other conclusions from Prandtl's formulas. We mention the simplest of these conclusions, based on the method of dimensions. We look for relations of the type:

$$|\overline{u'w'}| = f\left(\frac{du}{dz}\right). \tag{11}$$

The left-hand side of (11) has the dimension L^2T^{-2} and on the right-hand side stands a function of a quality whose dimension is T^{-1}. Evidently one has to introduce into the right-hand side some length l. Let us search for a function f in the form of a mononomial:

$$|\overline{u'w'}| = l^m \left(\frac{du}{dz}\right)^n, \tag{12}$$

where the exponents m and n are still undetermined. Equating the exponents of L and T on the right and left sides of equation (12), we obtain m = 2; n = 2, hence:

$$\overline{u'w'} = l^2\left(\frac{du}{dz}\right)^2. \tag{13}$$

This is Prandtl's formula, but the coefficient l is yet undetermined. It follows from the foregoing that the introduction of the mixing length is not formally obligatory, but it is still very tempting due to its physical (or rather geometrical) visuality.

Expressing the shear stresses and velocities at different points of the turbulent flow by the formula:

$$\tau = \rho l^2 \left(\frac{d\bar{u}}{dz}\right)^2,$$

we find the value of Prandtl's mixing length:

$$l = \sqrt{\frac{\tau}{\rho}} \left(\frac{du}{dz}\right)^{-1}. \tag{14}$$

Measurements performed in cylindrical tubes have shown that the magnitude l varies with the distance from the wall, within the limits of 0 – 14 % of the tube radius. It is usually considered that l is a linear function of the distance from the wall l = kz, whereby $k \cong 0.4$.

The above results are of special importance for the study of air currents in the lowermost atmospheric layer, adjacent to the earth's surface.

However it is not always possible to apply these results to the free atmosphere. Thus, for example, in the free atmosphere the mean velocity very often does not vary with altitude, i. e., u = const. In that case according to (10), the exchange coefficient vanishes, i. e., there should be no turbulent exchange at all in the free atmosphere. The observations on temperature and humidity variation in the free atmosphere show that when $\frac{du}{dz} = 0$ the turbulent exchange by no means ceases.

The notion of the mixing length has the advantage of direct geometrical visualization. The notion of the exchange coefficient is not so obvious, since it depends not only on the purely geometrical quantity l, but also on the velocity and the density.

However, only the foreknowledge of the exchange coefficient permits to use direct calculations in the numerous problems of atmospheric turbulence, or rather in the problems of turbulent exchange.

Incidentally, the foregoing argument is valid only for conditions obtainable in a free atmosphere, far from the earth's surface. For the friction layer adjacent to the earth's surface, the concept of mixing length as a measure of turbulence is of immense importance for meterology, owing to the simple geometrical relationship the mixing length bears to the altitude and also to the roughness of the underlying surface.

§ 3. Vorticity Transfer

Almost simultaneously with Prandtl, Taylor developed a theory of vorticity transfer. Taylor thinks that the erratic nature of turbulent fluid motion is connected with the irregularities in the distribution of hydrostatic pressure. Therefore, upon transfer of a finite fluid mass from one layer to another, one should not ignore the fluctuation of pressure and the associated formation of pressure gradients. One cannot assume that the momentum of a fluid mass is preserved during the transfer of this mass.

Restricting himself to the consideration of plane-parallel motion, Taylor argues that upon transfer of a fluid particle, not the momentum is preserved, but the vorticity.

Taylor's reasoning is quite convincing. In fact, the vorticity in a turbulent stream is preserved for a longer time than the momentum of the transferred mass, although the vorticity does vary eventually, owing to the influence of viscosity.

Let us write the equation of motion of a perfect incompressible fluid, taking place in the plane XOZ, in the form:

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}(u^2 + w^2) - w\eta, \tag{1}$$

where

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}.$$

Averaging equation (1) and assuming that the mean values $\bar{u}^2$, u'^2 and w'^2 are constant with respect to x, we obtain:

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \overline{w'\eta'}. \tag{2}$$

Comparing equation (2) with the averaged Reynolds equation, previously obtained (cf. § 12 Chapter VIII), which in our case has the form:

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \frac{d}{dz}(\overline{u'w'}), \tag{3}$$

we see that the variation of momentum can be expressed either by $-\overline{w'\eta'}$, or by $\frac{d}{dz}(\overline{u'w'})$, consequently,

$$-\overline{w'\eta'} = -\frac{d}{dz}(\overline{u'w'}). \tag{4}$$

Since, following Prandtl,

$$|\overline{u'w'}| = l^2\left(\frac{d\bar{u}}{dz}\right)^2,$$

we have,

$$\overline{w'\eta'} = \frac{d}{dz}\left\{l^2\left(\frac{d\bar{u}}{dz}\right)^2\right\}. \tag{5}$$

On the other hand, recalling that the averaged vorticity is $\frac{d\bar{u}}{dz}$ and applying the hypothesis of Prandtl to the eddy velocity $w' = l\frac{d\bar{u}}{dz}$ and to the eddy vorticity:

$$\eta' = l\frac{d\bar{\eta}}{dz} = l\frac{d^2\bar{u}}{dz^2},$$

we obtain:

$$\overline{w'\eta'} = l\cdot\frac{d\bar{u}}{dz}\cdot l\cdot\frac{d\bar{\eta}}{dz} = l^2\cdot\frac{du}{dz}\cdot\frac{d^2u}{dz^2}. \tag{6}$$

Thus, for the transfer of vorticity we get two different results. Starting from equation (3), drawn from the theory of momentum transfer, we obtain (assuming that $\frac{\partial p}{\partial x} = 0$)

$$\frac{\partial\bar{u}}{\partial t} = \overline{w'\eta'} = \frac{d}{dz}\left\{l^2\left(\frac{d\bar{u}}{dz}\right)^2\right\}, \tag{7}$$

and starting from equation (2), drawn from the theory of vorticity transfer, we obtain:

$$\frac{\partial\bar{u}}{\partial t} = \overline{w'\eta'} = l^2\cdot\frac{d\bar{u}}{dz}\cdot\frac{d^2\bar{u}}{dz^2}. \tag{8}$$

Therefore, introducing the exchange coefficient $A = \rho l^2\overline{\frac{du}{dz}}$ or rather the coefficient of turbulence $K = \frac{A}{\rho} = l^2\frac{d\bar{u}}{dz}$, we obtain:

$$\frac{\partial\bar{u}}{\partial t} = \frac{d}{dz}\left(K\frac{d\bar{u}}{dz}\right), \tag{9}$$

$$\frac{\partial\bar{u}}{\partial t} = K\overline{\frac{d^2u}{dz^2}}, \tag{10}$$

where, for the sake of simplicity, we omit the absolute value in the expressions for A and K, since in the cases discussed the velocity always increases upwards, so that $\frac{d\bar{u}}{dz} > 0$.

We will compare equations (9) and (10), writing equation (9) in full:

$$\frac{\partial\bar{u}}{\partial t} = K\overline{\frac{d^2u}{dz^2}} + \frac{\partial K}{\partial z}\cdot\frac{\partial\bar{u}}{\partial z}. \tag{11}$$

The right-hand side of (11), by the considerations of § 2, stands for the net influx of momentum per unit volume. The first term on the right-hand side, $K\overline{\frac{d^2u}{dz^2}}$, according to (10) represents the actual increment of momentum. Therefore, the second term on the right-hand side of (11) $\frac{\partial K}{\partial z}\frac{d\bar{u}}{dz}$ is the momentum lost as a result of pressure pulsation.

Equation (9) and (10) coincide only in one case: when $\frac{\partial K}{\partial z} = 0$, and consequently:

$$K = l^2\overline{\frac{du}{dz}} = \text{const}.$$

Since in general K depends on Z, the considerations of momentum and vorticity transfer should generally lead to different results. Since neither of these theories is rigorous, the final choice between them can be made only as a result of comparison with experimental results. However, experiments carried out so far do not favor any of these theories.

§ 4. Transfer of Other Characteristics of Air

We obtained the general expression for the flux of substance in the form:

$$Q_s = -A_s \frac{\partial s}{\partial z}. \tag{1}$$

This expression is, strictly speaking, valid only for substances which are indestructible, preserved upon mixing, and "passive" (see § 1). These conditions hold for the material admixtures suspended in the atmosphere (water vapor, smoke, fine dust, etc). However, this equation is usually applied also to such substances as momentum, heat, and vorticity which, strictly speaking, do not fulfil all these three conditions, particularly the condition of passivity.

As we have seen above, the exchange coefficient A does not depend on the substance transferred, but is a merely statistical magnitude, characteristic of the turbulent atmosphere. However, the nature of turbulence changes from point to point within a wide range. Turbulence depends upon lapse rates, distribution of wind with altitude and, in the friction-layer, also upon the distance from the underlying surface and upon its roughness.

In eddy transfer of momentum, the momentum current density is identified with the shearing stress, and the coefficient of eddy viscosity ν should coincide with the exchange coefficient A.

In eddy transfer of heat the enthalpy per unit mass, i.e.,

$$s = c_p \theta, \tag{2}$$

where θ is the potential temperature, is considered as the unit of transferred substance. Consequently, the eddy flux of heat will be determined by the equation:

$$Q_\theta = -c_p A_\theta \frac{\partial \theta}{\partial z}. \tag{3}$$

V. Schmidt postulated the validity of the general equation of turbulent exchange also for heat transfer. Accordingly we have:

$$A_\theta = A_s = A. \tag{4}$$

However, the assumption that the coefficients corresponding to momentum, heat, and material admixtures are all equal is not well founded. Momentum and heat are undoubtedly conserved when two air masses are mixed. But momentum is not indestructible. Masses of air, passing from one layer to another, may move under the action of other forces, such as local pressure gradients. Under the action of these forces momentum may be produced or destroyed. Heat, however, is not a passively transferable substance, since its transfer leads to variations in the lapse rates and this factor plays a major role in the development of turbulence. There is therefore no basis for the assumption that A_θ, A_s and A are all equal. One can only claim that the orders of magnitude of these constants cannot

differ greatly, but their numerical values are generally not identical.

The coefficient of eddy viscosity A as well as the coefficients A_{-} and A_{θ} have the same dimension [L^2T^{-1}] and in the absence of turbulence all of them vanish. Therefore the ratio:

$$\frac{A_s}{A} = \alpha_s \tag{5}$$

is a dimensionless parameter, being a function of dimensionless constants of the turbulent flow. If the fluid is incompressible (or only slightly compressible), the only characteristic constant of the turbulent flow (in the absence of a free boundary) is the Reynolds number. Therefore, the parameter α_s can only be a function of Re.

As was shown before, atmospheric Re numbers are very high. The above-developed semi-empirical theory of turbulence can accordingly be applied to a region with very high values of the Re number. In this case all functions of the Re number equal approximately their limiting values as $Re \to \infty$. In that case α_s = const., $A_s = \alpha_s A$ and the basic equation of turbulent exchange, according to Obukhov, assumes the form:

$$Q_s = -\alpha_s A \frac{\partial s}{\partial z} \tag{6}$$

or, substituting for A its values $\rho l^2 \left|\frac{d\bar{u}}{dz}\right|$,

$$Q_s = -\alpha_s \rho l^2 \left|\frac{d\bar{u}}{dz}\right| \cdot \frac{\partial s}{\partial z}, \tag{7}$$

where α_s is a dimensionless parameter, depending on the nature of the substance transferred. For momentum $\alpha_s = 1$. For other substance α_s, though of the order of unity, is not actually equal to it and has different values for different substances.

The expression of the eddy flux of heat will be written in the following form, if we take (3) into account:

$$Q_\theta = -\alpha_\theta \rho c_p l^2 \left|\frac{d\bar{u}}{dz}\right| \frac{\partial \theta}{\partial z}. \tag{8}$$

§ 5. Influence of Temperature Stratification on the Development of Turbulence. Richardson's Number

So far we have studied the development of turbulence in a homogeneous incompressible fluid and taken the dimensionless Reynolds number Re = ul/ν as a criterion of turbulence. However, the Reynolds number is not very suitable for the study of atmospheric turbulence.

In establishing his criterion of turbulence, Reynolds started from the premise that the flow is physically homogeneous and that only its boundaries and velocity change. In the atmosphere, however, the development of turbulence depends in the most fundamental way upon the lapse rate. Large lapse rates enhance the turbulence while small lapse rates and inversions hinder its development.

Experiments conducted in wind tunnels, where the lapse rate is zero, are therefore comparable with each other, but not with atmospheric conditions.

Recently, special tunnels of rectangular cross section, in which the top and bottom sides can be heated or cooled, were introduced in aerodynamic laboratories. This allows to create different temperature gradients in the air stream. The first experiments conducted in these tunnels have already shown, that when the isothermal flow pattern is replaced by one with a stable stratification, the critical velocity of transition from laminar to turbulent flow changes from 20 cm/sec to 100 cm/sec. For a stable stratification it was also observed that the flow is turbulent in the lower part of the tunnel and laminer in the upper part.

Thus, the Reynolds number is insufficient for the characterization of turbulence in the presence of nonhomogeneous temperature distribution.

Prandtl has shown that a new dimensionless parameter, which was subsequently called the Richardson number Ri, is a criterion for turbulence, in the presence of a nonhomogeneous density distribution.

We shall derive the Richardson number, starting from dimensional considerations.

Let us first explain what factors determine the development of turbulence. These are: 1. the vertical component of the gradient of the current velocity $\frac{\partial u}{\partial z}$, 2. the fluid density ρ 3. the vertical component of the density gradient $\frac{\partial \rho}{\partial z}$ and 4. the acceleration of gravity g.

We compose from these terms an expression:

$$\mathrm{Ri} = \rho^{\alpha}\left(\frac{\partial \rho}{\partial z}\right)^{\beta} \cdot \left(\frac{\partial u}{\partial z}\right)^{\gamma} g^{\delta}$$

and stipulate that the expression be dimensionless:

$$M^0 \cdot L^0 \cdot T^0 = \left(\frac{M}{L^3}\right)^{\alpha} \cdot \left(\frac{M}{L^4}\right)^{\beta} \cdot \left(\frac{1}{T}\right)^{\gamma} \cdot \left(\frac{L}{T^2}\right)^{\delta}.$$

Comparing the exponents, we find that they must satisfy the following conditions:

$$\begin{aligned} \alpha + \beta &= 0, \\ -3\alpha - 4\beta + \delta &= 0, \\ -\gamma - 2\delta &= 0. \end{aligned}$$

Expressing all undetermined exponents by δ:

$$\alpha = -\delta; \quad \beta = \delta; \quad \gamma = -2\delta,$$

we get:

$$\mathrm{Ri} = \left[g \frac{\frac{1}{\rho}\frac{\partial \rho}{\partial z}}{\left(\frac{\partial u}{\partial z}\right)^2} \right]^{\delta}; \qquad (1)$$

δ can be any number. It is simplest to equate δ to unity. Then the Richardson number will have the form:

$$\mathrm{Ri} = g \frac{\frac{1}{\rho}\frac{\partial \rho}{\partial z}}{\left(\frac{\partial u}{\partial z}\right)^2}. \qquad (2)$$

In the atmosphere the development of turbulence is influenced by: 1. the vertical component of the current velocity gradient, 2. the lapse rate of the potential temperature, 3. the potential temperature θ itself, and 4. the force of gravity.

Repeating the foregoing argument, we obtain the turbulence criterion for the atmosphere in the same form as established by Richardson in 1920, proceeding from energy considerations:

$$\mathrm{Ri} = \frac{g}{\theta}\frac{\frac{\partial \theta}{\partial z}}{\left(\frac{\partial u}{\partial z}\right)^2} \tag{3}$$

or

$$\mathrm{Ri} = \frac{g}{T}\cdot\frac{\left(\frac{\partial T}{\partial z}+\gamma_a\right)}{\left(\frac{\partial u}{\partial z}\right)^2}. \tag{4}$$

The number Ri also determines the dimensions of atmospheric turbulence.

One should distinguish three cases:

1. $\mathrm{Ri} > 0$; then $\frac{\partial \theta}{\partial z} > 0$. The stratification of the atmosphere is stable. Development of turbulence is impeded. The greater Ri, the more stable the stratification of the atmosphere. Consequently, at Ri above a certain critical value, atmospheric turbulence dies out completely. The critical Richardson number Ri_{crit} was repeatedly and variously evaluated by various authors, who reached different results. Thus, according to Richardson, $\mathrm{R}_{crit} = 1$, according to Prandtl, $\mathrm{Ri}_{crit} = \frac{1}{2}$, according to Taylor, $\mathrm{Ri}_{crit} = \frac{1}{4}$, and according to Tollmien, $\mathrm{Ri}_{crit} = 1/24$. Reliable values of Ri_{crit} can be obtained only by a careful interpretation of specially prepared and selected observations. Thereby, as can be seen from the formula for Ri_{crit}, it is necessary to determine simultaneously the distributions of temperature and wind with altitude. According to Sverdrup, interpretation of such observations leads to the estimate $\mathrm{Ri}_{crit} = 1/11$.

2. $\mathrm{Ri} < 0$, $\frac{\partial \theta}{\partial z} < 0$. In this case, the stratification of atmosphere is unstable. The energy of the turbulence increases at the expense of the diminishing energy of instability.

3. $\mathrm{Ri} = 0$, $\frac{\partial \theta}{\partial z} = 0$. The stratification of the atmosphere is indifferent. In that case turbulence develops in the same manner as in a homogeneous fluid, that is to say, the above-derived semi-empirical equations of turbulence are applicable here.

A. M. Obukhov generalized this theory to include the cases of a nonhomogeneous atmospheric temperature distribution. Starting from considerations of dimensional similitude, Obukhov assumes that all the dimensionless constants of a variable-density-fluid flow are functions of the number Ri. In addition, Obukhov assumes that all the exchange coefficients of the different substances are proportional to each other as in the case of a homogeneous fluid. Then, for the description of the various processes of turbulent exchange in a nonhomogeneous atmosphere it is sufficient, according to Obukhov, to know only the "universal" function of the Ri number, which enters as a corrective term in the equation of exchange.

Let A_0 be the exchange coefficient in an adiabatically stratified atmosphere, when Ri = 0. Then, at the same distribution of wind velocity the exchange coefficient for a nonadiabatic atmosphere will be expressed by the formula

$$A = \varphi(\mathrm{Ri}) \cdot A_0.$$

As to the nature of the function $\varphi(\mathrm{Ri})$, Obukhov argues as follows. By definition, the function $\varphi(\mathrm{Ri})$ fulfils the condition $\mathrm{Ri}=0$ $\varphi(0)=1$. As Ri increases, the stratification of the atmosphere becomes more stable and the intensity of turbulence decreases. Consequently, $\varphi(\mathrm{Ri})$ is a monotonously decreasing function. At a certain critical value Ri_{crit} turbulence ceases altogether. Consequently

$$\varphi(\mathrm{Ri}) = 0 \text{ for } \mathrm{Ri} < \mathrm{Ri}_{crit}$$

Introducting the universal function into the equations of eddy flux of momentum and heat, we obtain:

$$\tau = -\varphi(\mathrm{Ri}) \cdot \rho l^2 \left| \frac{d\bar{u}}{dz} \right| \frac{d\bar{u}}{dz}, \tag{6}$$

$$Q_\theta = -\alpha\varphi(\mathrm{Ri}) \cdot c_p \cdot \rho l^2 \left| \frac{d\bar{u}}{dz} \right| \cdot \frac{d\theta}{dz}. \tag{7}$$

§ 6. Surface Roughness

Experiments in wind tunnels have shown that the critical value of the Reynolds number Re is a function of the temperature distribution, or, in other words, of the Richardson number Ri.

However, observations show that atmospheric currents are always turbulent, even when the atmosphere is comparatively stable. Therefore, in that case, the decisive factor for the development of turbulence is the local unevenness of the particular underlying terrain which can be united under the statistical notion of roughness.

In the preceding sections we discussed flow patterns which can be reproduced in the laboratory, in smooth-walled tunnels. The unevenness of the walls essentially distorts the flow pattern, leading to development of turbulence at velocities for which the flow would have been laminar, had the walls been even.

Two types of roughness are distinguished in aerodynamics and hydrodynamics: 1. angular roughness characterized by sharp angular protuberances of a very irregular form; 2. undulatory roughness, characterized by gentler and more regular wave-like protuberances.

The roughness of the natural earth surface has different forms and it cannot be included in any of the above-mentioned types.

Dense grass is one of the extreme examples of such roughness. While the stream of air penetrates quite freely into the upper part of the vegetation growth, the densely growing stems are impenetrable to wind, which therefore cannot reach the ground.

In the forest, on the other hand, the wind quite freely penetrates into the space between the tree trunks, while there is almost no air movement among the tree tops.

The surface of the sea is wavy, but this is a mobile undulation.

Not being able to study separately the effect of each type of flow obstruction, we adopt a statistical approach to roughness, assuming that the individual protuberances are very numerous and have a great diversity of forms.

We will characterize a surface by the degree of its roughness, evaluating its total effect by a certain number which we shall call the coefficient of roughness or measure of roughness.

The main effect of roughness is that even in the immediate neighborhood of the wall the mean mixing length has a certain finite value different from zero.

This fact, according to Prandtl, is taken into account by the following procedure. We shift the origin of the coordinates underneath the surface in question to such a depth, that the observed mixing length will obtain at the surface. Thus,

$$l = \varkappa (z + z_0), \qquad (1)$$

where z is the distance from the point to the rough surface, the constant $\varkappa = 0.4$, and z_0, is a certain length, depending upon the linear measure of the roughness. From experiments in tunnels having even walls on which sand particles have been glued, it was found that $z_0 = \frac{1}{30} h_0$, where h_0 is the mean diameter of the particles. However the coefficient $\frac{1}{30}$ is not constant even for tunnels.

For natural surfaces, whose roughness differs essentially from the roughness obtained by the use of sand grains, such as that of a grass cover, we may take z_0 to be $\frac{1}{7}$ or $\frac{1}{8}$ of h_0, where h_0 is the mean height of the grass.

The influence of roughness and altitude on the atmospheric mixing length is noticeable only at low altitudes, whereas in the free atmosphere this effect is absent. Therefore the mixing length cannot increase indefinitely with altitude. This law is valid only for the friction layer characterized by the absence of variation in the direction of the wind. The distribution of wind velocity in this layer will be discussed in § 1 Chapter XVI.

Thus, the generation of turbulence in the atmosphere is favored by the non-homogeneity of the temperature field and the extremely diversified boundary conditions at the earth's surface, i.e., its roughness.

The heterogeneity of the temperature field engenders "thermal" or "static" turbulence, due to the variability of heating of the different parts of the underlying surface. Thermal turbulence has been earlier described as the "flotation" of isolated, more or less overheated masses of air in the form of "bubbles", whereby the space they evacuated was filled by air, flowing under these "bubbles" from all directions.

In recent years another scheme has been suggested to explain thermal turbulence. Thermal turbulence is being connected with the development of bounded closed cyclical air circulations in the atmosphere. In this case a cumulus will develop in the ascending branch of this closed cycle, while in the descending branch the clouds will evaporate, and a rift will form in the overcast.

The "dynamic" turbulence is generated mainly by the flow past obstacles at the earth's surface. Since the dimensions and form of these obstacles vary widely, so should the vortexes "obstructing" the atmosphere also vary in form and strength.

However, the nature, dimensions and form of these vortexes are still unknown.

§ 7. Horizontal Eddy Exchange of Mass

We have seen above that the flux of any substance is proportional to the gradient of this substance and to the exchange coefficient. If the eddy exchange of mass is equal in all directions, the flux of substance takes place chiefly in the vertical direction, since all the meteorological elements change vertically much faster than horizontally. The exchange coefficient in the vertical direction is of the order of $10-10^2$ g cm^{-1} sec^{-1}.

However, the horizontal coefficient can be considerably greater than the vertical exchange coefficient. Therefore, in spite of the small values of horizontal gradients, the horizontal flux may play an important role in the consideration of large-scale atmospheric processes.

Owing to the great gravitational stability of the atmosphere, manifest by increase of potential temperature with altitude, the turbulent exchange will take place along isentropic surfaces, i.e., surfaces of equal entropy or surfaces of equal potential temperature.

For the study of turbulent exchange along isentropic surfaces, which we shall call, for brevity, "lateral mixing" meteorologists have devised a method, the so-called "isentropic analysis".

In order to follow the trajectory of an air particle, the distribution of specific humidity is plotted on the charts of the isentropic surfaces. The values of the potential temperature and specific humidity, being conservative characteristics, can be regarded as Lagrangean coordinates of the air particle. For complete determination of the air particle one has to introduce a third generalized coordinate, i.e., a third conservative parameter. In the capacity of such a parameter we introduce the so-called "potential vorticity".

Let us write the equations of motion in the following simplified form:

$$\left.\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-2\omega\sin\varphi v&=-\frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+2\omega\sin\varphi u&=-\frac{1}{\rho}\frac{\partial p}{\partial y},\\ g&=-\frac{1}{\rho}\frac{\partial p}{\partial z}. \end{aligned}\right\} \qquad (1)$$

We select in the atmosphere some layer of air and assume for the sake of simplicity that this layer is barotropic

$$\rho=\rho(p).$$

Noting that:

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)&=-\frac{1}{\rho^2}\frac{d\rho}{dp}\cdot\frac{\partial p}{\partial y},\\ \frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)&=-\frac{1}{\rho^2}\frac{d\rho}{dp}\cdot\frac{\partial p}{\partial x}, \end{aligned}\right\} \qquad (2)$$

we differentiate the first of equations (1) with respect to y and the second equation with respect to x and subtract the second from the first. Then taking into account the relationship (2), we obtain

$$\frac{d\zeta}{dt}+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)2\omega\sin\varphi+(2\omega\sin\varphi+\zeta)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0, \qquad (3)$$

where:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}.$$

Since:

$$\frac{\partial}{\partial t}(2\omega \sin\varphi) = 0,$$

we have:

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) 2\omega \sin\varphi = \frac{d}{dt}(2\omega \sin\varphi),$$

and the relationship (3) will assume the form:

$$\frac{d}{dt}(\zeta + 2\omega \sin\varphi) + (\zeta + 2\omega \sin\varphi)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \qquad (4)$$

We write the equation of continuity in the form:

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial \rho w}{\partial z} = 0. \qquad (5)$$

Multiplying equation (5) by gdz and integrating between the limits from z_0 to z_1, we find:

$$\int_{z_0}^{z_1} g\frac{d\rho}{dt}dz + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\int_{z_0}^{z_1} g\rho\, dz + \int_{z_0}^{z_1} g\frac{\partial \rho w}{\partial z}\, dz = 0, \qquad (6)$$

where z_0 and z_1 are regarded as functions of x, y, t. Consequently,

$$\left.\begin{aligned} &\frac{d}{dt}\int_{z_0}^{z_1} g\rho\, dz = \int_{z_0}^{z_1} g\frac{d\rho}{dt}\, dz + g\left(\rho_1\frac{dz_1}{dt} - \rho_0\frac{dz_0}{dt}\right), \\ &\int_{z_0}^{z_1} g\rho\, dz = p_1 - p_0, \\ &\int_{z_0}^{z_1} g\frac{\partial \rho w}{\partial z}\, dz = g[(\rho w)_1 - (\rho w)_0] = g\left(\rho_1\frac{dz_1}{dt} - \rho_0\frac{dz_0}{dt}\right). \end{aligned}\right\} \qquad (7)$$

Combining equations (6) and (7), we get:

$$\frac{d}{dt}(p_0 - p_1) + (p_0 - p_1)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \qquad (8)$$

And finally, eliminating the horizontal divergence of velocity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, between equations (4) and (8) we obtain:

$$\frac{\zeta + 2\omega \sin\varphi}{p_0 - p_1} = \text{const.} \qquad (9)$$

The expression (9) remains constant, although the relative vorticity changes with latitude and mass of the layer. Consequently, if we reduce the air to a standard latitude φ and a standard pressure difference $p_0 - p_1$, the corresponding vorticity may be considered as a conservative characteristic or parameter of the air particle. This parameter is called the potential vorticity.

Studying on isentropic maps the motion of lines of equal specific humidity and considering this motion to be associated with lateral mixing, or which is almost the same, with horizontal exchange, we can evaluate the order of magnitude of the

horizontal exchange coefficient. Its value lies between $4 \cdot 10^5$ and $5 \cdot 10^7$ $g\,cm^{-1}\,sec^{-1}$.

Thus, the coefficient of horizontal exchange is at least 10^3 greater than the coefficient of vertical exchange. But vertical gradients of wind velocity and temperature are approximately 10^2 times greater than the corresponding horizontal gradients. Consequently, the horizontal exchange is not less important than the vertical exchange. We should note, however, that the order of magnitude of the coefficient of horizontal exchange must not be taken as finally established.

§ 8. Spectral Structure of Turbulence

We will consider in greater detail the internal structure of turbulent flow. The pulsative motion, superimposed on the laminar flow, can itself be regarded as a superposition of turbulent motions due to eddies (elements of turbulence) of different size.

The scale of motion is the order of magnitude of those distances over which the flow velocity varies appreciably. It was experimentally established that the appearance of eddies of different scales is connected with the Reynolds number, or specifically, as the Reynolds number increases the scale of the eddies decreases; at very high Reynolds numbers occur eddies of the most different scales—from very large to infinitesimally small.

Eddies of different scales play different roles in turbulent flow; the eddies of maximum scale, of the order of magnitude of the characteristic linear dimension L of the flow, have the greatest effect. These large scale eddy-pulsations have the largest "amplitudes" – of the order of the variation Δu of the mean velocity over a distance L. The "frequencies" of the large-scale pulsations are of the order u/L. The small-scale pulsations possess larger "frequencies" and smaller "amplitudes".

Thus we may speak of turbulence of different scales, and of the spectral structure of turbulence. In connection with this we mention the spectral distribution of the kinetic energy in a turbulent flow. The main part of the kinetic energy belongs to the large-scale turbulence, and only a small residual part of the kinetic energy—to the small-scale turbulence.

The spectral resolution of turbulence allows to clarify the peculiarities of pulsation velocity variation from point to point in the flow pattern, at a given instant of time. Changes of pulsation velocity over large distances are due to large-scale turbulence. Therefore, they are of an order comparable with Δu. Changes of pulsation velocity over short distances are due to small-scale turbulence. Therefore, they are small in comparison with Δu, though large relative to the changes of the mean velocity over the corresponding distances.

Similarly, in observations performed at different times at a fixed spatial point, considerable velocity variations of the order of Δu are recorded if the time interval is comparable with the characteristic time $T = L/u$ for the main flow, due to large-scale turbulence. On the other hand, the small changes of velocity at short time intervals $\tau < T$ are due to small-scale turbulence.

We have seen above that the Reynolds number characterizing the flow pattern depends greatly on the mixing length l. In order to determine the properties of the flow associated with the spectral structure of the turbulence, we heuristically introduce the Reynolds number Re_λ characteristic of the various scales of turbulence. Let λ denote the magnitude of the scale of turbulence and v_λ - the order of magnitude of the pulsation velocity; we define the corresponding number Re_λ by the formula:

$$Re_\lambda = \frac{v_\lambda \lambda}{\nu} \qquad (1)$$

The number Re is greater the larger the scale of motion. But at high Reynolds numbers, the Re_λ for large-scale turbulence is also high. High Reynolds numbers correspond to low viscosity. We therefore obtain the following results.

For large-scale turbulence, of essential importance in turbulent flow, viscosity does not play any significant role. It can therefore be ignored, and regarding the fluid as perfect, we may use Euler's equation to study its motion. It is obvious, that at a large-scale turbulence there is no noticeable dissipation of kinetic energy. On the other hand, the less the scale of turbulence, the smaller become Re_λ and consequently, the more pronounced the effect of viscosity.

These considerations lead to the following conclusion about the process of "transformation" and dissipation of kinetic energy in turbulent flow.

The kinetic energy of turbulence on a large-scale, comparable to the dimensions of the entire flow pattern, is due to the instability of the main laminar motion. As the large-scale eddies disintegrate, a part of their kinetic energy is gradually converted into kinetic energy of small-scale eddies. This conversion is effected with practically no dissipation of kinetic energy. Thus, in the turbulent stream energy is incessantly transferred downwards in the scale of turbulence. And only in the microscale elements of turbulence, of the order $\lambda_0 \ll \lambda$ is the kinetic energy dissipated, i. e., directly transformed into heat energy.

Therefore, all the quantities pertaining to turbulence of the order $\lambda \gg \lambda_0$ cannot depend on the viscosity ν. Strictly speaking, these quantities cannot vary with ν, if the other conditions determining the motion remain unchanged.

Consequently, the number of parameters characterizing the turbulent flow is reduced. This allows to use considerations of dimensional similitude and to obtain several important results.

As an example we shall evaluate the order of magnitude of energy dissipation in turbulent motion. We denote by ε the amount of energy dissipated per unit time in a unit volume of fluid:

$$[\varepsilon] = \frac{\text{erg}}{\text{cm}^3\text{sec}} = \frac{\text{g}}{\text{cm.sec}^3}$$

Although the final analysis shows that dissipation is caused by viscosity, the ε can be expressed only through the magnitudes $\rho, l, \Delta u$, characteristic of small scale turbulence. The only combination of these quantities having the dimensions of the dissipation ε, is $\frac{\rho(\Delta u)^3}{l}$. Therefore:

$$\varepsilon \sim \frac{\rho(\Delta u)^3}{l}. \qquad (2)$$

However, the expression (2) for the dissipation of kinetic energy is not convenient for comparison with the expression for energy dissipation derived earlier (§ 8 Chapter VIII).

As was indicated above, the semi-empirical theory of turbulence maintains that the motion of a turbulent fluid is governed by the equations of motion of a viscous fluid if the coefficient of kinematic viscosity ν in these equations is replaced by the coefficient of "kinematic eddy viscosity" ν_{turb}. We shall evaluate ν_{turb}, using the values of Δu, and l, characteristic for large-scale turbulence.

There is only one way to combine these values in order to get an expression having the dimensions of the kinematic viscosity: $\Delta u \cdot l$. Hence:

$$\nu_{turb} \sim \Delta u \cdot l. \tag{3}$$

The ratio between the coefficient of eddy viscosity and the coefficient of molecular viscosity gives the Reynolds number:

$$\nu_{turb} : \nu = \frac{\Delta u \cdot l}{\nu} = \mathrm{Re}. \tag{4}$$

Consequently, ν_{turb} increases together with Re. We shall now substitute ν_{turb} in expression (1), by the aid of (2); we obtain:

$$\varepsilon \sim \nu_{turb} \cdot \rho \left(\frac{\Delta u}{l}\right)^2. \tag{5}$$

This expression corresponds to the usual expression of energy dissipation (§§ 8 and 14 Chapter VIII), namely:

The coefficient of kinematic eddy viscosity ν determines the dissipation of energy as a function of the derivatives of the mean current velocity with respect to the coordinates.

§ 9. The "Two-Thirds" Law of Kolmogorov-Obukhov

The hypothesis of homogeneous and isotropic turbulence introduced in the statistical theory enables the elucidation of certain important characteristics of turbulent motion. This hypothesis is well substantiated by measurements conducted in wind tunnels, on turbulent flow artificially produced by passing the stream through a grid.

However, in most of the partically important cases, especially in the study of natural turbulent flow, this hypothesis gives merely a very rough approximation to reality. In fact, assuming a completely fortuitous nature of turbulent pulsations, it is reasonable to expect a tendency to isotropy in the turbulent flow. However, natural flow always occurs in the field of gravity. The field of gravity is not isotropic, therefore the presence of the force of gravity prevents the establishment of isotropy in a turbulent flow. Besides, isotropy is also disturbed in the proximity of boundaries.

But the role of the volume forces decreases with the diminution of the scale of the phenomenon, according to the general considerations of similitude. Therefore, for a sufficiently small scale the volume forces may be ignored.

Starting from the above considerations A.N. Kolmogorov assumes that in a sufficiently small region of an arbitrary flow pattern which is not located in the neighborhood of a boundary, the hypothesis of "local isotropy" is valid to a good enough approximation, for sufficiently large Reynolds numbers.

By a "small region" Kolmogorov means a region whose dimensions are small in comparison with the characteristic dimensions of the flow system L and whose characteristic time is small in comparison with the characteristic time $T = L/u$.

Thus, Kolmogorov considers a flow on a scale l, sufficiently small in comparison with the scale of the principal motion to be free of the influence of any external forces as might hinder the establishment of isotropy in the flow $l \ll L$. At the same time this scale is quite large in comparison with the dimensions of the smallest eddies which dissipate their energy into heat: $l \ll \lambda_0$

The turbulence on this scale possesses the following properties: 1. it is isotropic; 2. the kinetic energy gain of turbulence on the considered scale is surrendered by the large-scale kinetic energy; 3. the kinetic energy is spent by its transformation into turbulent energy of a smaller scale, and not by its dissipation into heat.

This simple scheme allowed Kolmogorov to establish a very important law of the structure of turbulent flow, the so-called "two-thirds law".

Let us first explain which parameters may characterize turbulent motion in the chosen region of scale 1, fulfilling:

$$\lambda_0 \ll l \ll L$$

The overall dimension of the flow L, Δu and the mean velocity of u at a given ε and ρ do not influence the local properties of turbulence, owing to the magnitude of the scale l chosen. For the same reason the viscosity ν is not a parameter of turbulence on the l scale.

The relevant parameters are ρ, l, and ε — the dissipation of energy per unit time, in a unit volume. Although energy dissipation is connected with microscale turbulence, the value of ε characterizes turbulent motion of a larger scale as well. This can be explained by the fact, that the value of ε, as has been shown above, represents an "energy flux" from the large-scale to the small-scale eddies.

Let us now consider the quantity:

$$R_{ll} = \overline{[v_l(M') - v_l(M'')]^2},$$

i. e., the mean square difference between the values of a velocity component along the straight connecting the points M^1 and M^2. In case of local isotropy the function R_{11} depends only on the distance r between the points M' and M'', and on the dissipation ε and density ρ.

Let us compare the dimensions of the quantities:

$$[R_{ll}] = L^2T^{-2}; \quad [\varepsilon] = ML^{-1}T^{-3}; \quad [\rho] = LT^{-3}; \quad [r] = L.$$

Their only possible combination yields the following relationship:

$$R_{ll} = C\left(\frac{\varepsilon}{\rho}\right)^{2/3} \cdot r^{2/3}. \qquad (1)$$

Formula (1) expresses the "two-thirds law" of Kolmogorov and Obukhov.

An analogous formula is obtained for:

$$R_{nn} = \overline{[v_n(M') - v_n(M'')]^2}, \qquad (2)$$

i.e., for the mean square difference between the values of velocity components normal to the straight segment connecting points M^1 and M^n.

This law means that the mean square difference of a certain velocity component v_f between two points is directly proportional to the 2/3 power of distance between these points.

The law of two-thirds established by Kolmogorov on the basis of dimensional considerations, was also derived by Obukhov on the basis of analysis of turbulent energy balance in turbulence flow.

Chapter XVI

WIND VARIATION WITH ALTITUDE IN A TURBULENT ATMOSPHERE

§ 1 Variations in the Wind Velocity and the Exchange Coefficient in the Earth's Surface Layer

The problem of wind variation with altitude, one of the most important in meteorology, is quite old. We shall not discuss the numerous empirical studies, and only note that the first theoretical studies devoted to this question date back to the eighties of the last century.

The altitude distribution of the wind is subject to the effect of turbulent exchange and friction with the earth's surface in the layer extending up to 500 - 1,000 m. However, the wind distribution in altitude over the entire thickness of the atmosphere is also affected by the Coriolis force and the altitude dependence of the pressure gradient. The effect of the latter on the variation of the geostrophic wind has already been discussed (cf. Chapter XI) and the role of the Coriolis force will be explained in § 3 of this chapter.

In the present section we restrict ourselves to the study of wind distribution in altitude, or briefly, to the study of the wind profile in the lowermost atmospheric layer, a few decameters thick, which is characterized by the constancy of the wind direction in altitude, indicating that the earth's rotation exerts only a negligible effect on the wind distribution in this layer. For the sake of brevity we shall refer to this as the "lowermost" layer.

In the lowermost layer the majority of the meteorological elements (wind velocity, temperature, humidity), are most susceptible to altitude variations; the effect of turbulent exchange is therefore especially pronounced in this layer.

Observations show that the wind velocity distribution in the lowermost layer sometimes resembles the velocity profile of a flow in rough-walled tubes. This resemblance, however, holds only for thermally uniform flow. Upon temperature stratification the wind profile in the atmosphere differs considerably from the velocity profile of the flow in tubes.

In the lowermost layer one may ignore the inertial and Coriolis forces and the horizontal pressure gradient, since these forces are small in comparison with the forces of friction. We shall regard the air motion in the lowermost layer as plane-parallel, and consider the tangential shear stress to be constant both in magnitude and direction. Then we have,

$$\tau = \rho \left(l \frac{du}{dz} \right)^2 = \text{const.}, \tag{1}$$

hence

$$A \frac{du}{dz} = \text{const.} \tag{2}$$

and

$$l \frac{du}{dz} = u^* = \text{const.}, \tag{3}$$

where

$$A = \rho l^2 \frac{du}{dz} \tag{4}$$

is the exchange coefficient and u* - a constant, the so-called friction velocity.

Let us further assume that the mixing length in the lowermost layer depends linearly on the altitude

$$l = \varkappa (z + z_0), \tag{5}$$

where $\varkappa = 0.4$ and z_0 is the roughness parameter. Eliminating the mixing length l between (3) and (5), we get:

$$\frac{du}{dz} = \frac{u^*}{\varkappa (z + z_0)}, \tag{6}$$

Integrating (6), we obtain:

$$u = \frac{u^*}{\varkappa} \ln \frac{z + z_0}{z_0} \tag{7}$$

(the boundary condition is taken in the form u = 0, at z = 0).

The logarithmic law of wind velocity distribution in the lowermost layer has been empirically confirmed more than once. The corresponding empirical formula has the form

$$u = a + b \ln (z + c). \tag{8}$$

However, only the introduction of the mixing length allows to grasp thoroughly the mechanism of the phenomenon and find the relationship between the coefficients of the empirical formula (8) and the surface roughness. Writing formula (8) in the form

$$u = a + b \ln c + b \ln \frac{z + c}{c}$$

and assuming:

$$c = z_0; \quad b = \frac{u^*}{\varkappa}; \quad a = -\frac{u^*}{\varkappa} \ln z_0,$$

we obtain formula (7).

We shall now determine the exchange coefficient (under the same conditions as before) for the lowermost layer. Taking into account (4), (5) and (6), we immediately find, that

$$A = \rho l^2 \frac{du}{dz} = \rho l u^* = \rho u^* \varkappa (z + z_0). \tag{9}$$

Thus, owing to the logarithmic wind distribution in the lowermost layer the exchange coefficient increases linearly with altitude.

Since the real mean wind velocity u is proportional to u*, the exchange coefficient is directly proportional to the wind velocity, and we may therefore write the following relations for A:

$$\frac{A}{\rho} = K = a_0 + a_1 z, \tag{10}$$

$$\frac{A}{\rho} = K = \alpha u. \tag{11}$$

The result obtained is of great importance for the proper understanding of the processes taking place in the lowermost layer.

We have repeatedly mentioned the analogy between irregular turbulent flow and chaotic molecular motion. But this analogy is limited. In fact, we have seen above that the coefficient of molecular viscosity is constant, depending neither on density, nor on flow velocity and determined solely by the nature of the gas. The exchange coefficient, however, which is the analogue of the coefficient of viscosity, depends, as we have just seen, on the following: 1) the density ρ, 2) the distance z from the underlying surface, 3) the mean velocity u, and 4) the roughness parameter z_0.

Therefore, the equations describing the processes connected with turbulent exchange are more complicated than those describing processes due to molecular exchange. Only in special cases can these equations be simplified by the assumption A = const. However, such simplification is ruled out for the lowermost layer.

The mean value of the exchange coefficient for an average wind and average roughness at the altitude of 2 m, is 1 - 2 g $cm^{-1}sec^{-1}$, while the mean value of the molecular viscosity of air is 1 - 2 $\times 10^{-4}$ g $cm^{-1}sec^{-1}$. Consequently, at this altitude the exchange coefficient is approximately 10,000 times greater than the coefficient of viscosity.

The exchange coefficient A and the roughness parameter z_0 can be calculated from the wind velocity at two given levels in the lowermost layer. Writing formula (7) for some definite level z_1:

$$u_1 = \frac{u^*}{\varkappa} \ln \frac{z_1 + z_0}{z_0}, \tag{12}$$

and eliminating the auxiliary quantity u* between (7) and (12), we obtain:

$$u = u_1 \frac{\ln \frac{z + z_0}{z_0}}{\ln \frac{z_1 + z_0}{z_0}}; \tag{13}$$

substituting for $\frac{du}{dz}$ in (9) the expression

$$\frac{du}{dz} = \frac{u_1}{z + z_0} \cdot \frac{1}{\ln \frac{z_1 + z_0}{z_0}},$$

we find the following expression for the exchange coefficient:

$$A = \rho \varkappa^2 (z + z_0) u_1 \frac{1}{\ln \frac{z_1 + z_0}{z_0}}. \tag{14}$$

These results remain valid for a uniform flow of an incompressible fluid; under the conditions prevalent in the lowermost atmospheric layer they are true only in the absence of temperature stratification. In that case, the nature of the flow is determined solely by viscosity and surface roughness. The only characteristic constant of the flow is the Reynolds number. In the lowermost layer, however, Re changes very little and it is of no importance in characterizing the atmospheric turbulence. Besides, temperature stratification occurs as a rule in the lowermost layer, and is a very important and often a decisive factor determining the nature of the flow. Therefore, the nature of the flow in the lowermost layer is determined not only by the Reynolds number Re, but also by the Richardson number:

$$\mathrm{Ri} = \frac{g \frac{d \ln \theta}{dz}}{\left(\frac{du}{dz}\right)^2}. \tag{15}$$

The Richardson number Ri in the lowermost layer may vary over quite a wide range, including positive as well as negative values. In case of adiabatic stratification of the atmosphere, Ri = 0 and the nature of the flow is determined solely by Re. In that case the previously derived formulas are correct. When the stratification is stable $Ri > 0$ and when unstable $Ri < 0$.

It is worthwhile to note that, owing to the small thickness of the lowermost layer, the adiabatic stratification is practically indistinguishable from the isothermal one. We therefore distinguish between the following types of temperature distribution in altitude: isothermality (adiabatic stratification), inversion (corresponding, of course, to a stable stratification) and temperature reduction, sometimes called convection, corresponding to superadiabatic lapse rates.

The fact has been repeatedly stressed that the logarithmic distribution of wind velocity in the lowermost layer is only a particular case of the more general power law (law of the 1/p power). It has also been noted many times that the logarithmic law is valid only for the adiabatic state of the atmosphere and that for inversions this law should be substituted by the power law in the form

$$u = u_1 \left(\frac{z}{z_1}\right)^{\frac{1}{p}}, \tag{16}$$

where p is some dimensionless parameter. Observations have shown that formula (16) holds only for strong inversions.

The Soviet researcher D. L. Laikhtman has proposed more general expressions for the wind velocity distribution and the exchange coefficient in the lowermost layer. Let us assume, following Laikhtman, that the mixing length is given by:

$$l = \varkappa \left(\frac{z}{z_0}\right)^n \cdot z_0. \tag{17}$$

where n is the yet undetermined exponent depending on the nature of the atmospheric stratification, i.e., on the Ri number. For n = 1 we have the particular case discussed above. Substituting (17) in (3), we obtain:

$$du = \frac{u^*}{\varkappa} \cdot \frac{1}{z_0} \left(\frac{z}{z_0}\right)^{-n} dz,$$

integrating, we find:

$$u = \frac{u^*}{\varkappa} \cdot \frac{z_0^{n-1}}{1-n} \left[z^{1-n} - z_0^{1-n}\right]; \tag{18}$$

since

$$u_1 = \frac{u^*}{\varkappa} \cdot \frac{z_0^{n-1}}{1-n} \left[z_1^{1-n} - z_0^{1-n}\right],$$

we finally obtain:

$$u = u_1 \frac{z^{1-n} - z_0^{1-n}}{z_1^{1-n} - z_0^{1-n}}. \tag{19}$$

Let us determine the value of the exponent n. In the lowermost layer $\frac{du}{dz} > 0$. But

$$\frac{du}{dz} = (1-n)\, u_1 \frac{z^{-n}}{z_1^{1-n} - z_0^{1-n}} > 0,$$

therefore, $1 - n > 0$, or $n < 1$.

In addition, it is known from observation that the vertical velocity gradient decreases with the distance from the rough surface, i.e.,

$$\frac{d^2u}{dz^2} = -n(1-n)\frac{u_1 z^{-n-1}}{z_1^{1-n} - z_0^{1-n}} < 0,$$

consequently, $n > 0$. Thus, n is a proper, positive fraction

$$0 < n < 1. \tag{20}$$

For the expression (17) to correspond to the 1/p power law, expressed by equation (16), we introduce a new parameter p instead of n, which is related to it by:

$$1 - n = \frac{1}{p}. \tag{21}$$

Obviously, p varies between the following limits:

$$1 \leqslant p \leqslant \infty. \tag{22}$$

Thus, the following formulas of Laikhtman are obtained :

$$l = \varkappa \left(\frac{z}{z_0}\right)^{1-\frac{1}{p}}, \tag{23}$$

$$u = u_1 \frac{z^{\frac{1}{p}} - z_0^{\frac{1}{p}}}{z_1^{\frac{1}{p}} - z_0^{\frac{1}{p}}}, \tag{24}$$

$$\frac{A}{\rho} = K = l^2 \frac{du}{dz} = \frac{\varkappa}{\rho} \frac{u_1 z^{1-\frac{1}{p}}}{z^{\frac{1}{p}} - z_0^{\frac{1}{p}}}. \tag{25}$$

Formula (24) reduces to the logarithmic law as $p \to \infty$, i.e., as $n \to 0$, and to the linear law as $p \to 1$, i.e., as $n \to 0$.

The correctness of Laikhtman's formulas was proved by the numerous observations on the layer extending up to 6 m altitude, performed at the Central Geophysical Observatory. The results agree well with Laikhtman's theory.

The ratio between the wind velocities at two different levels in the lowermost layer depends only on the parameter p, i.e., ultimately on the Richardson number.

Numerous observations have shown that the exchange coefficient in the lowermost layer rises rather rapidly with altitude; at a certain altitude its rise is usually almost checked, and sometimes it even starts decreasing. Such behavior of the exchange coefficient is qualitatively fully understandable. In the lowermost layer the proximity of the solid boundary interferes with the process of exchange; therefore, the exchange coefficient is very small near the earth's surface. With increasing distance from the solid boundary, the exchange effect becomes more pronounced, and the exchange coefficient increases with altitude. However, this increase cannot go on indefinitely. Since the vortexes "obstructing" the atmosphere originate mainly on the uneven surface of the earth, the exchange coefficient after having attained a maximum should decrease with further increase of distance from the earth's surface. However, this is often not the case, due to the unevenness of the earth's surface being not the only source of the mentioned "obstructing" vortexes. Vortexes are also generated far away from the earth's surface by the discontinuity surfaces present in the atmosphere. Therefore, the exchange coefficient, which is a characteristic statistical parameter of turbulence, does not fall to zero with altitude, but assumes a definite almost constant value.

Consequently, one can assert for the majority of cases that the exchange coefficient varies linearly with altitude up to a certain level h, and then becomes constant. This altitude h, which, generally speaking, does not coincide with the upper boundary of the lowermost layer, is not constant. It depends upon both the atmospheric stratification and the mean flow velocity. For an unstable stratification and at high flow velocities the level h is higher than for stable stratification and low velocities.

§ 2. Distribution of the Exchange Coefficient in the Boundary Layer

A.M. Obukhov has theoretically investigated the problem of the altitude variation of the exchange coefficient.

The boundary layer plays a dual role in the atmosphere. First, it transmits the friction shear stress into the atmosphere, a stress caused by the interaction between the underlying surface and the moving air mass. Second, through the boundary layer, heat is exchanged between the active layer of the soil and the free atmosphere above.

This dual role of the boundary layer is used by A.M. Obukhov for the theoretical determination of the altitude variation of the exchange coefficient in the boundary layer. In the present section we shall report on the investigations of Obukhov.

We shall denote the friction shear stress at the ground by τ_0 and the heat flux passing through a unit area of the horizontal surface per unit time by q_0. We shall assume that in the boundary layer a certain stationary state has been established. Let all the external forces be extremely small in comparison with the forces of turbulent friction. We also assume that internal heat sources (heat of condensation, radiation) are absent. Assuming all this, the momentum flux and the heat flux do not vary with altitude:

$$\tau(z) = \tau_0 = \text{const.}, \tag{1}$$
$$q(z) = q_0 = \text{const.} \tag{2}$$

Evidently, within the boundary layer, for small z values, these assumptions are well enough fulfilled.

Let us write the equations of turbulent momentum exchange and of heat exchange in the lowermost layer:

$$\tau = -\varphi(\mathrm{Ri})\,\rho l^2 \left|\frac{du}{dz}\right| \cdot \frac{du}{dz} \approx \tau_0 = \text{const.}, \tag{3}$$

$$q = -a\varphi(\mathrm{Ri})\,c_p \cdot \rho l^2 \left|\frac{du}{dz}\right| \cdot \frac{d\theta}{dz} \approx q_0 = \text{const.} \tag{4}$$

Instead of τ and q, Obukhov introduces the "friction velocity" u* and "heat transfer velocity" q*, defining them by the equations:

$$u^* = \sqrt{\frac{\tau}{\rho}}, \tag{5}$$

$$q^* = \frac{q}{c_p \rho T}. \tag{6}$$

Ignoring small changes of ρ and T within the lowermost layer, Obukhov assumes:

$$u^* = u_0^* = \text{const.}, \tag{7}$$

$$q^* = q_0^* = \text{const.} \tag{8}$$

We eliminate φ (Ri) between (3) and (4). Dividing (4) by (3), we get:

$$\frac{1}{T}\frac{d\theta}{dz} : \frac{du}{dz} = \frac{q^*}{\alpha u^{*2}} = \text{const.} \tag{9}$$

Integrating (9) with respect to z and ignoring the small changes in T, we find:

$$\theta(z) = \frac{q^* T}{\alpha u^{*2}} \cdot u(z) + \text{const.} \tag{10}$$

Since $\frac{q^* T}{\alpha u^{*2}} \approx$ const., the distribution of temperature in the lowermost layer is similar to the velocity distribution.

Let us turn to the derivation of the relationship determining the Richardson number. Introducing into formula (5) the values of τ from formula (3), determining the transfer of momentum, we get:

$$u^* = \sqrt{\varphi(\text{Ri})} \cdot l \cdot \frac{du}{dz}. \tag{11}$$

Substituting in formula (6) the expression (4) for the velocity of heat transfer, we obtain:

$$q^* = \frac{q}{c_p \rho T} = \frac{\alpha \varphi(\text{Ri})}{T} l^2 \left|\frac{du}{dz}\right| \cdot \frac{d\theta}{dz}.$$

Dividing all terms of this expression by the third power of the right and left sides of (11), we obtain:

$$\frac{q^*}{u^{*3}} = \frac{\alpha}{\sqrt{\varphi(\text{Ri})}} \cdot \frac{1}{lT} \frac{\frac{d\theta}{dz}}{\left(\frac{du}{dz}\right)^2}. \tag{12}$$

Multiplying both sides by g and recalling:

$$\text{Ri} = \frac{g}{T} \frac{\frac{d\theta}{dz}}{\left(\frac{du}{dz}\right)^2},$$

we obtain the equation for Ri:

$$\frac{\text{Ri}}{\sqrt{\varphi(\text{Ri})}} = \frac{1}{\alpha} \cdot g \cdot \frac{l q^*}{u^{*3}}, \tag{13}$$

where $l = \varkappa z$.

It was shown above (§ 5 Chapter XV), that φ (Ri) is a monotonically decreasing function. Therefore, the left-hand side of (13) is a monotonically increasing function of Ri. Hence, equation (13) has only one solution for Ri.

Obviously, at the earth's surface, i.e., for l = 0, we have Ri = 0. We shall evaluate Ri near the ground. For this purpose we will find the value of the derivative $\left(\frac{\partial \text{Ri}}{\partial z}\right)_{z=0}$; we rearrange equation (13) as follows:

$$\text{Ri} = \frac{\varkappa}{\alpha} \cdot g \cdot \frac{z q^*}{u^{*3}} \sqrt{\varphi(\text{Ri})}. \tag{14}$$

Differentiating equation (14) with respect to z, we obtain:

$$\frac{\partial \mathrm{Ri}}{\partial z} = \frac{\varkappa}{a} \cdot g \frac{q^*}{u^{*3}} \sqrt{\varphi(\mathrm{Ri})} + \frac{\varkappa}{a} \cdot g \frac{q^*}{u^{*3}} \cdot z \frac{\varphi'(\mathrm{Ri})}{2\sqrt{\varphi(\mathrm{Ri})}}, \tag{15}$$

hence

$$\left(\frac{\partial \mathrm{Ri}}{\partial z}\right)_{z=0} = \frac{\varkappa}{a} \cdot g \cdot \frac{q^*}{u^{*3}}, \tag{16}$$

since $\varphi(0) = 1$. Consequently, near the earth's surface $\mathrm{Ri} \approx 0$. On this basis Obukhov maintains that, for an arbitrary temperature distribution, a sublayer exists within the boundary layer, in which the effect of temperature stratification is exceedingly small, and the nature of turbulence is determined by dynamic factors only. This is called by Obukhov the sublayer of "dynamic turbulence".

Obukhov goes on to evaluate the height of the sublayer of "dynamic turbulence" or rather a certain characteristic dimension L of this sublayer, defined as follows:

$$L = \frac{1}{a\left(\frac{\partial \mathrm{Ri}}{\partial z}\right)_{z=0}} = \frac{1}{\varkappa} \frac{u^{*3}}{g \cdot q^*}, \tag{17}$$

or, going back to τ and q,

$$L = \frac{1}{\varkappa} \frac{c_p T \cdot \tau^{3/2}}{g\sqrt{\rho} \cdot q}. \tag{18}$$

This characteristic length does not depend on the form of the universal function φ (Ri), but is determined only by the momentum flux density τ and by the heat flux q.

Introducing instead of τ, the wind velocity u_0 at a certain altitude (≈ 30 m) depending on the surface roughness, through the empirical formula $\tau = 0.0025 \rho u_0^2$, Obukhov calculated the following values of the height of the dynamic-turbulent sublayer (Table 47):

Table 47

Height of the sublayer of dynamic turbulence (in meters)

q $\frac{\text{cal}}{\text{cm}^2\text{min}}$		0.01	0.02	0.05	0.1	0.15	0.2	0.3
$u_0 = 20u^*$ m/sec	u^* cm/sec							
0.5	2.5	0.182	0.091	0.036	0.018	0.120	0.008	0.006
1	5	1.45	0.73	0.29	0.14	0.10	0.07	0.05
2	10	11.63	5.81	2.34	1.16	0.78	0.58	0.39
3	15	39.2	19.6	7.9	3.9	2.6	1.9	1.3
4	20	93.0	46.5	18.7	9.2	6.2	4.6	3.1
5	25		90.8	36.5	18.2	12.1	9.1	6.0
6	30			63.1	31.4	21.0	15.7	10.5
10	50					97.0	72.7	48.4

As can be seen from the table, in the region of the likeliest values of u_0 and q the height of the sublayer of dynamic turbulence changes within the range of 8 to 50 m.

Obukhov has further studied the altitude distributions of the Richardson number and of the coefficient of turbulence. For convenience we introduce the following designations:

$$\frac{\mathrm{Ri}}{\mathrm{Ri}_{\mathrm{crit}}}=\eta;\quad \frac{z}{L_1}=\zeta;\quad \varphi(\mathrm{Ri})=\varphi_1(\eta);\quad \beta=\frac{1}{\alpha(\mathrm{Ri})_{\mathrm{crit}}}, \tag{19}$$

where ζ is a dimensionless altitude, β - a constant of the first order.

Since

$$\begin{aligned}&\varphi(0)=1,\\ &\varphi'(\mathrm{Ri})\leqslant 0,\\ &\varphi(\mathrm{Ri})=0 \quad (\text{for } \mathrm{Ri}\geqslant \mathrm{Ri}_{\mathrm{crit}}).\end{aligned}$$

the function $\varphi_1(\eta)$ satisfies the conditions:

$$\begin{aligned}&\varphi_1(0)=1,\\ &\varphi_1'(\eta)\leqslant 0,\\ &\varphi_1(\eta)=0 \quad (\text{for } \eta\geqslant 1).\end{aligned}$$

For definiteness we consider the case of stable temperature distribution, when $\mathrm{Ri}>0$ and the heat flux is directed from the atmosphere to the soil. In this case q and q* are regarded as positive:

$$\mathrm{Ri}=\frac{\varkappa}{\alpha}z\cdot\frac{gq^*}{u^{*3}}\sqrt{\varphi(\mathrm{Ri})}=\frac{z}{\alpha L}\sqrt{\varphi(\mathrm{Ri})}. \tag{20}$$

Then equation (13) is written in the following dimensionless form:

$$\frac{\eta}{\sqrt{\varphi_1(\eta)}}=\frac{z}{L}\frac{1}{\alpha(\mathrm{Ri})_{\mathrm{crit}}}=\beta\zeta=\xi. \tag{21}$$

Further, a new function is introduced:

$$\eta=\psi(\xi)$$

defined by the solution of equation (21) for η.

Returning to the former variables, Obukhov obtains the following expression, determining the altitude distribution of the Richardson number:

$$\mathrm{Ri}(z)=\mathrm{Ri}_{\mathrm{crit}}\cdot\psi\left(\beta\frac{z}{L}\right). \tag{22}$$

On the basis of the properties of the function φ_1 established in § 5 Chapter XV, Obukhov demonstrates that the functions ψ possess the following properties:

$$\left.\begin{aligned}&\psi(0)=0;\ \psi(\xi)\leqslant 1;\ \psi'(\xi)\geqslant 0,\\ &\varphi(\xi)\to 1 \quad \text{as} \quad \xi\to+\infty,\\ &\frac{\psi(\xi)}{\xi}\to 1 \quad \text{as} \quad \xi\to 0.\end{aligned}\right\} \tag{23}$$

Thus, Obukhov obtains theoretically a very interesting result, namely, that in the boundary layer, for $z>L$, the Ri number asymptotically approaches $\mathrm{Ri}_{\mathrm{crit}}$ in a stable atmosphere.

Obukhov also studied the altitude variation of the coefficient of turbulence K.

Since

$$\tau=K\rho\frac{du}{dz}, \tag{24}$$

we have

$$K = \frac{\tau}{\rho \frac{du}{dz}} = \frac{u^{*2}}{\frac{du}{dz}}. \tag{25}$$

Introducing the height parameter L of the sublayer of dynamic turbulence, Obukhov obtains:

$$K = u^* \sqrt{\varphi(\mathrm{Ri})}\, \varkappa z = L \varkappa u^* \sqrt{\varphi(\mathrm{Ri})}\, \frac{z}{L} \tag{26}$$

or in the variables η and ζ:

$$K = \varkappa L u^* \sqrt{\varphi(\eta)} \cdot \zeta. \tag{27}$$

But $\frac{\eta}{\sqrt{\varphi(\eta)}} = \beta\zeta$, consequently $\sqrt{\varphi(\eta)} \cdot \zeta = \frac{\eta}{\beta}$ and

$$K(z) = \frac{\varkappa \cdot u^* L}{\beta} \cdot \psi\left(\beta \frac{z}{L}\right). \tag{28}$$

The behavior of the function ψ is known in its broad outlines; it is therefore easy to study the changes of K(z) for relatively small (in comparison with L) and relatively great values of z.

Since $\frac{\psi(\xi)}{\xi} \longrightarrow 1$ as $\xi \longrightarrow 0$, for $z \ll L$ we have

$$\psi\left(\frac{\beta z}{L}\right) \approx \frac{\beta z}{L},$$

therefore:

$$K(z) = \varkappa u^* \cdot z. \tag{29}$$

Furthermore: $\psi(\xi) \longrightarrow 1$ as $\xi \longrightarrow +\infty$, hence for $z \gg L$

$$K(z) = \frac{\varkappa u^* L}{\beta}.$$

Thus Obukhov theoretically obtained the following result: for small z values the coefficient of turbulence K increases linearly with altitude, and for high z values it asymptotically approaches the limit:

$$K_\infty = \frac{\varkappa u^* L}{\beta}. \tag{30}$$

If the length L is eliminated between (30) and (17), K_∞ will be expressed as follows:

$$K_\infty = \frac{1}{\beta} \frac{u^{*4}}{gq^*}. \tag{31}$$

Obukhov has compiled a table of the limit values A_∞ g $\mathrm{cm}^{-1}\mathrm{sec}^{-1}$ (Table 48).

As can be seen from this table, the theoretically obtained values of the parameters of turbulence are of the same order as their empirical values.

Obukhov gives the following example, demonstrating the close correspondence between the theoretical and empirical values of the turbulence parameters.

When u_0 = 5 m/sec, u^* = 0.25 cm/sec and q = 0.1 cal/cm^2.

K_∞ = 1.82 m^2/sec, A_∞ = 22.4 g/$\mathrm{cm}^{-1}\mathrm{sec}^{-1}$, L = 18.2 m,

$\frac{\partial\theta}{\partial z} = 0.31\ ^\circ C/100\ m$, $\frac{\partial T}{\partial z} = -0.69\ ^\circ C/100 m$.

This example shows that the value of the lapse rate calculated from the theoretically derived values of K_∞ agrees well with the observed values.

Table 48

Limiting values of the exchange coefficient A_∞ g $cm^{-1} sec^{-1}$ for a stable atmosphere

q $\frac{cal}{cm^2 min}$		0.01	0.02	0.05	0.1	0.15	0.2	0.3
$u_0 = 20u^*$ m/sec	u^* cm/sec							
0.5	2.5	0.0224	0.0112	0.00443	0.00221	0.00158	0.00108	0.000738
1	5	0.3567	0.1722	0.0738	0.0369	0.0246	0.0123	0.0123
2	10	5.7195	2.8536	1.1439	0.5658	0.3813	0.2829	0.1845
3	15	28.905	14.514	5.781	2.829	1.968	1.476	0.98
4	20	91.512	45.756	18.327	9.102	6.150	4.551	3.08
5	25		111.684	44.895	22.386	14.883	11.193	7.38
6	30			93.111	46.371	30.873	23.124	15.38
10	50					238.743	178.719	119.19

We mention that for Ri_{crit} Obukhov adopted in his calculations the empirical value obtained by Sverdrup: $(Ri)_{crit} = \frac{1}{11}$.

§ 3. Wind Variation with Altitude and Turbulent Exchange in the Planetary Boundary Layer

One of the best known manifestations of the effect of turbulence in the atmosphere is undoubtedly the variation of wind velocity and direction with altitude in the lower layer of the atmosphere. In a homogeneous air mass the increase in wind velocity with altitude manifests itself quite clearly, as well as a as well as a veering of the wind in the Northern Hemisphere and a backing in the Southern Hemisphere.

Such wind variations start at an altitude of 10-20 m and extend at our latitudes up to about 1 km. Below 10-20 m the wind direction remains constant and the velocity changes either according to the logarithmic law, or according to the power law.

Altitude variation of the wind direction in a higher layer is probably caused by the joint action of the deflective force of the earth's rotation and of turbulence.

Higher up, the variation of the wind directions is probably caused by the joint action of the deflective force of the earth's rotation and of turbulence, and the layer in which this effect is significant is called the planetary boundary layer.

The most important parameter of this layer is the angle β, formed between the wind direction v and the pressure gradient (Figure 130). This angle is called the "angle of deviation". The angle θ between the wind direction and the isobar, which complements the angle of deviation to 90°, is called the "angle of inclination" of the wind. Within the planetary boundary layer

inclination angle differs to some extent from zero, and the deviation angle is always acute. The value of the deviation angle is a function of the altitude z.

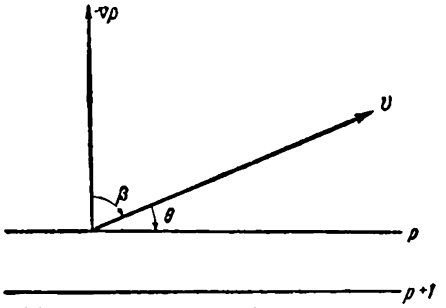

Figure 130. The angle of deviation β and angle of inclination θ of the wind

As shown by numerous investigations, the inclination angle of the wind depends not only on altitude, but varies also with time (possessing an annual and a diurnal cycle), with latitude and local orography. Thus, for example, at the earth's surface the inclination is usually smaller in the summer than in winter, smaller at high than at low latitudes, smaller above sea than above land.

Let us consider theoretically the problem of altitude variation of wind velocity and direction under the influence of turbulence and the deflective force of earth's rotation. Regarding the motion as horizontal and the velocity field as horizontally uniform, we base our calculation on the averaged equations of atmospheric motion, which can be written in the following form:

$$\left.\begin{aligned} \frac{du}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\omega_z v + \frac{\partial}{\partial z}\left(\frac{A}{\rho}\frac{\partial u}{\partial z}\right), \\ \frac{dv}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} - 2\omega_z u + \frac{\partial}{\partial z}\left(\frac{A}{\rho}\frac{\partial v}{\partial z}\right). \end{aligned}\right\} \qquad (1)$$

We also make the following simplifying assumptions:

1) The flow is uniform and rectilinear, whence

$$\frac{du}{dt} = \frac{dv}{dt} = 0.$$

2) The pressure gradient does not change with altitude. Therefore, directing the x - axis along the isobar and the y - axis along the gradient, we obtain:

$$\frac{\partial p}{\partial x} = 0; \quad \frac{\partial p}{\partial y} = \text{const}.$$

3) The density does not vary with altitude : ρ = const.

4) The exchange coefficient does not vary with altitude: A = const.

The equations of motion will take the form:

$$\left.\begin{aligned} 2\rho\omega_z \cdot v + A\frac{d^2u}{dz^2} &= 0, \\ -2\rho\omega_z \cdot u + A\frac{d^2v}{dz^2} &= \frac{\partial p}{\partial y}. \end{aligned}\right\} \qquad (2)$$

Let us define the boundary conditions:

1) The air adheres to the earth's surface, therefore:

$$u = v = 0 \text{ for } z = 0;$$

2) At high altitudes the wind velocity becomes geostrophic, equals u_g and consequently, as $z \to \infty$ $u \to u_g$, $v \to 0$.

The problem of integrating the system of equations (2) with two unknown functions u and v can be reduced to the integration of a single equation for a complex function $w = u + iv$. For this purpose we multiply the second of equations (2) by $i = \sqrt{-1}$ and add it to the first:

$$A\frac{d^2w}{dz^2} - 2\rho\omega_z \cdot i \cdot w = i\frac{\partial p}{\partial y}. \tag{3}$$

Thus, we obtain an ordinary linear differential equation of the second order with constant coefficients, which can be written down as follows:

$$-\frac{1}{a^2}\frac{d^2w}{dz^2} + w = u_g, \tag{4}$$

where

$$a^2 = \frac{2\rho\omega_z \cdot l}{A}; \quad u_g = -\frac{1}{2\rho\omega_z} \cdot \frac{\partial p}{\partial y}. \tag{5}$$

The general solution w of the inhomogeneous equation (4) is obtained as the sum of a particular solution w_1 of this equation and the general solution w_2 of the homogeneous equation:

$$\frac{d^2w_2}{dz^2} = a^2 w_2. \tag{6}$$

As a particular solution of equation (5) we can take:

$$w_1 = u_g = -\frac{1}{2\rho\omega_z}\frac{\partial p}{\partial y} = \text{const.}, \tag{7}$$

i.e., the geostrophic wind velocity. The general solution of equation (6) will then be:

$$w_2 = C_1 e^{az} + C_2 e^{-az}, \tag{8}$$

where C_1 and C_2 are arbitrary constants of integration.

Adding (7) and (8), we obtain the general solution of the inhomogeneous equation

$$w = C_1 e^{az} + C_2 e^{-az} + u_g. \tag{9}$$

The constants of integration are determined from the boundary conditions. Since with a limitless increase of z the wind velocity remains bounded, we have

$$C_1 = 0. \tag{10}$$

In order to fulfill the first boundary condition $u = v = w = 0$ for $z = 0$, we put:

$$C_2 = -u_g. \tag{11}$$

Consequently, the final solution of equation (4) will have the form:

$$w = u_g[1 - e^{-az}]. \tag{12}$$

We now separate the imaginary and real parts of the solution (12).

According to (5):

$$a = \sqrt{\frac{2\rho\omega_z i}{A}} = \sqrt{\frac{\rho\omega_z}{A}} \cdot \sqrt{2i} = \alpha(1+i), \tag{13}$$

where

$$\alpha = \sqrt{\frac{\rho\omega_z}{A}};$$

since

$$(1+i)^2 = 1 + 2i - 1 = 2i,$$

consequently,

$$w = u_g [1 - e^{-\alpha z} \cdot e^{-i\alpha z}]. \tag{14}$$

Employing Euler's formula:

$$e^{-i\alpha z} = \cos \alpha z - i \sin \alpha z,$$

we find

$$w = u + iv = u_g [1 - e^{-\alpha z} (\cos \alpha z - i \sin \alpha z)] \tag{15}$$

and finally

$$\left.\begin{aligned} u &= u_g [1 - e^{-\alpha z} \cos \alpha z], \\ v &= u_g \cdot e^{-\alpha z} \cdot \sin \alpha z. \end{aligned}\right\} \tag{16}$$

We have now the components of the wind velocity as functions of the altitude z. Let us find the absolute value of the velocity $c = \sqrt{u^2 + v^2}$ and the inclination angle θ:

$$c = u_g \sqrt{1 - 2e^{-\alpha z} \cos \alpha z + e^{-2\alpha z}}, \tag{17}$$

$$\operatorname{tg} \theta = \frac{v}{u} = \frac{e^{-\alpha z} \cdot \sin \alpha z}{1 - e^{-\alpha z} \cdot \cos \alpha z}. \tag{18}$$

Formulas (17) and (18) show, that with increasing altitude the velocity increases and the inclination angle decreases, so that the wind approaches the geostrophic, both in magnitude and direction. This process is faster the greater is $\alpha = \sqrt{\frac{\rho\omega \sin \varphi}{A}}$, i.e., the smaller the turbulent exchange and the higher the latitude of the locality.

It follows from (18) that at certain altitudes the wind assumes the direction of the geostrophic wind, so that $\theta = 0$ for $\alpha z = \pi, 2\pi, \ldots . n\pi$. The lowest level:

$$D = \frac{\pi}{\alpha} = \pi \sqrt{\frac{A}{\rho\omega \sin \varphi}}, \tag{19}$$

at which the wind becomes geostrophic in direction is called the friction level. Although with further increase of altitude the wind will somewhat diverge from the geostrophic, these deviations are so small as to be practically unnoticeable, as shown in Table 50. Therefore, the friction level determined by formula (19) may be regarded as the upper border of the terrestrial boundary layer. Table 49 gives the thickness of the boundary layer corresponding to different values of the exchange coefficient and latitude for $\rho = 1.25 \cdot 10^{-3} \text{g} \cdot \text{cm}^{-3}$.

Table 49

The friction level for a constant exchange coefficient

A \ φ	10°	30°	50°	60°	90°
10 g cm^{-1} sec^{-1}	790	470	380	350	330
50	1770	1040	840	790	740
100	2500	1470	1190	1120	1040
150	3060	1800	1460	1370	1280

As can be seen from formula (19) and from Table 49, high friction levels correspond to large exchange coefficients and low latitudes.

The velocity at the friction level, expressed by the formula:

$$c_D = u_g \sqrt{1 + 2e^{-2\pi} + e^{-2\pi}} = u_g (1 + e^{-\pi}) > u_g, \tag{20}$$

is higher than the geostrophic velocity, since $e^{-\pi}$ is a positive constant. Thus, the wind velocity reaches the absolute value of the geostrophic velocity somewhat below the friction level.

As $z \to \infty$, $e^{-z} \to 0$ and the wind becomes geostrophic.

For $z = 0$, formula (18) becomes indeterminate like $\frac{0}{0}$. Applying l'Hospital's rule we obtain:

$$\lim_{z \to 0} \operatorname{tg} \theta = \lim_{z \to 0} \frac{\frac{d}{dz}(e^{-\alpha z} \sin \alpha z)}{\frac{d}{dz}(1 - e^{-\alpha z} \cos \alpha z)} = 1.$$

Thus, at the earth's surface the deviation angle equals 45° under all conditions.

Using the values found for α, from (17) and (18) one can calculate the corresponding values of θ and $\frac{c}{u_g}$. Assuming that $\varphi = 45°$, $\rho = 10^{-3}$ g/cm^3, $A = 25.2$ g cm^{-1} sec^{-1} and consequently $\alpha = 0.004775$ m^{-1}, we obtain the following table of values of θ and $\frac{c}{u_g}$ for different altitudes (Table 50).

Table 50

Change of wind velocity and direction with altitude in the terrestrial boundary layer

$z =$	10	20	40	100	200	400	800	1200 m
$\frac{c}{u_g} =$	0.075	0.145	0.286	0.584	0.893	1.068	1.005	0.999
$\theta =$	43	42	39	31	20	5	—1	0

Drawing a diagram of the wind velocity as a function of the altitude z, according to Table 50, we obtain a curve known in geophysics as the Ekman spiral (in honor of Ekman, who first solved the problem of the deflection of sea currents under the joint influence of turbulence and Coriolis forces).

The Ekman spiral (Figure 131) is an equiangular logarithmic spiral with an angle of 45°. It passes through the origin of coordinates and asymptotically approaches the point $c = u_g$. Thus, even though we used simplifications, we have shown that the joint action of the Coriolis force and turbulence increases the wind velocity in the planetary boundary layer, causing at the same time its veering (in the Northern Hemisphere). At a certain altitude the wind equals the geostrophic, first in magnitude and then, continuing to increase, becomes at a certain altitude equal in direction too to the geostrophic wind. At the earth's surface the deviation angle is always 45°.

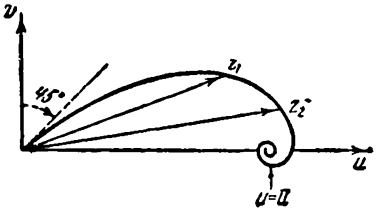

Figure 131. The Ekman spiral

It is evident from the vast empirical material extant that the actual mean wind distribution in altitude at a large enough distance from the earth agrees well with what may be expected on theoretical grounds, but in the lowest few tens of meters this correspondence breaks down. Thus, at the earth's surface the inclination angle is considerably less than 45° and, besides, does not remain constant, but varies over quite a wide range, diminishing over the sea in the summer and in daytime, and increasing over land in winter and during the night. The actual value of the wind velocity at weathervane level is considerably higher than the theoretical.

Further efforts of the investigators have been aimed at the perfection of the theory of wind variation with altitude, whose foundations have been laid in the work of Ekman and Åkerblom.

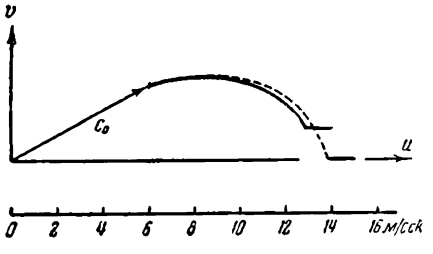

Figure 132. The Ekman spiral for the layer above anemometer level. Solid line - average empirical value, broken line - computed value

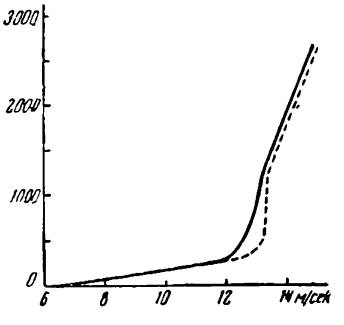

Figure 133. Absolute wind velocity as as function of height above anemometer level. Solid line - empirical, broken line - computed

Hesselberg and Sverdrup have taken into account the altitude variation of the pressure gradient, considering this variation to be linear, but assuming, as before, that ρ and A are constant. However, they completely excluded from their

consideration the layer below the weathervane level, correctly assuming that in this layer the conditions of turbulent exchange are rather different from those prevalent in the higher layers.

On these assumptions, Hesselberg and Sverdrup obtained a model of wind distribution in altitude, conforming quite well with observation. In Figure 132 is given an Ekman spiral corrected according to Hesselberg and Sverdrup, for the air layer above the anemometer level. In the same figure, for comparison, is drawn a spiral, based on 99 observations conducted in Lindesberg (Sweden). Figure 133 gives the theoretical altitude distribution of the magnitude of the velocity, versus the observed distribution. The agreement between the two curves is close indeed.

The studies of Hesselberg and Sverdrup improved the Ekman-Akerblom theory. However, their solution is far from being final. The question of change of the inclination angle with the season and orography still remains open.

Soviet research workers have made substantial contributions to the solution of the above-discussed problem, which will be dealt with in the next section.

§ 4. Wind Distribution in Altitude with a Variable Exchange Coefficient

The state of turbulence at any point is determined by one magnitude - the exchange coefficient. Regarding, as in the foregoing section, the exchange coefficient as constant for all altitudes, we by the same token considered the state of turbulence to be the same at any altitude.

Atmospheric turbulence, and consequently the exchange coefficient, are subject to considerable changes. The main factors determining the magnitude of the exchange coefficient are: geostrophic wind velocity, (or the pressure gradient, determining it), roughness of the underlying surface, stability of atmospheric stratification (lapse rate), and altitude above the earth's surface. Since the exchange coefficient is not directly measured, but is calculated from the measurements of other meteorological elements, a knowledge of its precise variation is essential in performing these calculations. The assumption of its constancy for different altitudes sometimes does not allow the correct evaluation from observations even of the order of magnitude of the exchange coefficient.

On the other hand, the assumption that the exchange coefficient increases linearly throughout the whole planetary boundary layer leads to incorrect results, namely, excessive values of the exchange coefficient.

In their recent works the Soviet scientists usually employ one of the two most widely used models of the altitude variation of the exchange coefficient.

Model A: the exchange coefficient increases with altitude, asymptotically approaching a constant value:

$$\nu = \nu_\infty (1 + \varepsilon - e^{-mz}), \tag{1}$$

where $\nu_\infty \cdot \varepsilon$ is the molecular viscosity. This model was suggested by B.I. Izvekov.

Model B: the exchange coefficient increases linearly with altitude up to a certain level h, and then remains constant, i.e.,

$$\left.\begin{aligned} \nu &= cz \quad \text{for} \quad z \leqslant h, \\ \nu &= ch \quad \text{for} \quad z > h. \end{aligned}\right\} \tag{2}$$

Model A suffers from the disadvantage that its use involves rather complicated calculation. Model B is more convenient, though it involves the solution of a two-layer problem.

In the present section we shall consider the problem of wind variation with altitude, using model B for the exchange coefficient. This model was first proposed by M.E. Shvetz and M.I. Yudin, who also gave a simple and quite accurate solution of this problem.

Yudin and Shvets, assuming a uniform motion , proceed from the following equations:

$$\left.\begin{aligned}\frac{\partial}{\partial z}\left(\nu\frac{\partial \rho u}{\partial z}\right)+2\omega_z\cdot\rho v=\frac{\partial p}{\partial x},\\ \frac{\partial}{\partial z}\left(\nu\frac{\partial \rho v}{\partial z}\right)-2\omega_z\rho u=\frac{\partial p}{\partial y}.\end{aligned}\right\} \tag{3}$$

Multiplying the second equation by i, subtracting from the first and introducing the complex velocity:

$$w=u-iv, \tag{4}$$

the two equations (3) are unified:

$$\frac{\partial}{\partial z}\left(\nu\frac{\partial \rho w}{\partial z}\right)+2i\omega_z\rho w=\frac{\partial p}{\partial x}-i\frac{\partial p}{\partial y}. \tag{5}$$

Introducing a new function - the complex momentum

$$M=M_x-iM_y=\rho w, \tag{6}$$

equation (5) can be written in the following form:

$$\frac{\partial}{\partial z}\left(\nu\frac{\partial M}{\partial z}\right)+2i\omega_z M=\frac{\partial p}{\partial x}-i\frac{\partial p}{\partial y}. \tag{7}$$

It is assumed that the pressure gradient does not change with altitude.

For a layer with linear change of the turbulence coefficient, located below the level h, equation (7) will assume the form:

$$c\cdot z\cdot\frac{\partial^2 M}{\partial z^2}+c\frac{\partial M}{\partial z}+2i\omega_z M=\frac{\partial p}{\partial x}-i\frac{\partial p}{\partial y}. \tag{8}$$

This is one of the forms of the well-known Bessel equation, which is integrated by the aid of the well-known cylindrical functions:

$$M=D\cdot I_0(2k\sqrt{i}\cdot z)+EN_0(2k\sqrt{i}\,z)+G, \tag{9}$$

where $D = D_1 + iD_2$ and $E = E_1 + iE_2$ are arbitrary complex constants of integration, to be determined by the boundary conditions; I_0 is the Bessel function of order zero, N_0 - Neumann's function of order zero, $k^2=\frac{2\omega_z}{c}$, $G=-\frac{1}{2\omega_z}\left(\frac{\partial p}{\partial y}-i\frac{\partial p}{\partial x}\right)$ is the geostrophic wind velocity.

For a layer above the h level we obtain the solution of Åkerblom, discussed before:

$$M=Ae^{-n\sqrt{-i}\,z}+B\cdot e^{n\sqrt{-i}\,z}+G, \tag{10}$$

where A and B are integration constants, determined by the boundary conditions, and $n^2 = \frac{2\omega_z}{hc}$.

Yudin and Shvets propose the following boundary conditions:

1) At the z_0 level the air adheres to the surface, therefore:

$$w = 0 \quad \text{at} \quad z = z_0. \tag{11}$$

2) At infinity the force of friction is bounded:

$$c \cdot h \left(\frac{\partial M}{\partial z}\right) < \infty \quad \text{at} \quad z = \infty. \tag{12}$$

3) For $z \to 0$ the Prandtl relation holds:

$$M = \frac{1}{\varkappa} \sqrt{\tau_0 \rho_0} \ln \frac{z}{z_0} \quad \text{as} \quad z \to 0 \tag{13}$$

and the Kibel' relation,too:

$$\frac{\partial M}{\partial z} \to \frac{\tau_0}{cz} \quad \text{as} \quad z \to 0. \tag{14}$$

Furthermore, at the level z = h the solution obtained for the lower layer should continuously pass into the solution obtained for the upper level. That means, that at the level z = h the function M and its derivative $\frac{\partial M}{\partial z}$ are continuous:

$$M_{h+0} = M_{h-0}; \quad \left(\frac{\partial M}{\partial z}\right)_{h+0} = \left(\frac{\partial M}{\partial z}\right)_{h-0}. \tag{15}$$

The boundary conditions formulated give 10 equations for the determination of 11 quantities: c, h, τ_0 and 8 arbitrary constants of integration. Taking one of these quantities as a parameter, we can express all the other by it.

Introducing yet another new variable:

$$x = 2k \sqrt{z},$$

we use for the functions I_0 and N_0 the following representations:

$$I_0 (x \sqrt{i}) = \operatorname{ber} x + i \operatorname{bei} x, \tag{16}*$$

$$N_0 (x \sqrt{i}) = \operatorname{ner} x + i \operatorname{nei} x. \tag{17}$$

The Bessel and Neumann functions have been thoroughly studied. Thus, when $x \to 0$:

$$\left.\begin{aligned} &\operatorname{ber} x \to 1, \\ &\operatorname{bei} x \to 0, \\ &\operatorname{ner} x \to -\frac{2}{\pi} \ln \frac{2}{\gamma x}, \\ &\operatorname{nei} x \to \frac{1}{2}, \end{aligned}\right\} \tag{18}$$

where γ is the Euler-Mascheroni constant, equal to 1.7811...

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* [Translator's note - the "ber" and "bei", "ner", and "nei" symbols obviously stand, respectively, for the real and imaginary parts of the Bessel and Neumann functions of complex argument].

Then we have:

$$|M| \to \frac{2}{\pi} \ln \sqrt{z}\, |E|. \tag{19}$$

Substitution of (19) in (13) gives:

$$\frac{\varkappa}{\pi} |E| = \sqrt{\tau_0 \rho_0}, \tag{20}$$

and from the Kibel' relation (14) follows:

$$|E| = \frac{\pi \tau_0}{c}. \tag{21}$$

Eliminating τ_0 between (20) and (21) we get:

$$|E| = \frac{\pi}{\varkappa^2} c \rho_0. \tag{22}$$

For further calculations Yudin and Shvetz assume:

$$\frac{\pi}{\varkappa^2} = 20 \quad (\varkappa = 0{,}395).$$

Then they introduce a new parameter α - the inclination angle of the wind, i.e., the angle the wind direction forms with the isobar:

$$\operatorname{tg} \alpha = \frac{v(0)}{u(0)} = -\frac{E_2}{E_1}. \tag{23}$$

Equation (22) can now be rearranged as follows:

$$|E| = \sqrt{E_1^2 + E_2^2} = \sqrt{E_1^2 (1 + \operatorname{tg}^2 \alpha)} = 20 c \rho_0,$$

hence:

$$\left.\begin{aligned} E_1 &= 20 c \rho_0 \cos \alpha, \\ E_2 &= 20 c \rho_0 \sin \alpha. \end{aligned}\right\} \tag{24}$$

According to the boundary conditions (13), we obtain:

$$DI_0(x_0 \sqrt{i}) + EN_0(x_0 \sqrt{i}) + G = 0. \tag{25}$$

Since x_0 is very small, the authors use approximate expressions for the cylindrical functions of small argument; with regard to (18), they obtain:

$$D + E\left(-\frac{2}{\pi} \ln \frac{2}{\gamma x_0} + \frac{1}{2} i\right) + G = 0. \tag{26}$$

In order to satisfy condition (12) it is necessary that B = 0. Finally, fulfilling condition (15) they find:

$$Ae^{-nh\sqrt{-1}} = DI_0(x_h \sqrt{i}) + EN_0(x_h \sqrt{i}), \tag{27}$$

$$-An\sqrt{-i}\, e^{-nh\sqrt{-i}} = \frac{k\sqrt{i}}{\sqrt{h}} [DI_0'(x_h \sqrt{i}) + EN_0'(x_h \sqrt{i})]. \tag{28}$$

Since

$$\frac{k}{\sqrt{h}} = n,$$

we get, combining (27) and (28):

$$D[I_0(x_h \sqrt{i}) + iI_0'(x_h \sqrt{i})] + E[N_0(x_h \sqrt{i}) + iN_0'(x_h \sqrt{i})] = 0. \tag{29}$$

The authors proceed in the following manner. Assuming for the sake of brevity, that:

$$I_0(x_h\sqrt{i})+iI_0'(x_h\sqrt{i})=j_1+ij_2,$$
$$N_0(x_h\sqrt{i})+iN_0'(x_h\sqrt{i})=n_1+in_2$$

and eliminating D by the use of (26), they get:

$$E(n_1+in_2)=\left[G+E\left(-\frac{2}{\pi}\ln\frac{2}{\gamma x_0}+\frac{1}{2}i\right)\right](j_1+ij_2)=0, \qquad (30)$$

and separating the real and imaginary parts:

$$\left.\begin{aligned} &E_1n_1-E_2n_2+j_1\left[E_1\frac{2}{\pi}\ln\frac{2}{\gamma x_0}+\frac{1}{2}E_0\right]-\\ &\quad-j_2\left[E_2\frac{2}{\pi}\ln\frac{2}{\gamma x_0}-\frac{1}{2}E_1\right]=Gj_1,\\ &E_1n_2+E_2\cdot n_1+j_1\left[E_2\frac{2}{\pi}\ln\frac{2}{\gamma x_0}-\frac{1}{2}E_0\right]+\\ &\quad+j_2\left(E_1\frac{2}{\pi}\ln\frac{2}{\gamma x_0}+\frac{1}{2}E_2\right)=Gj_2. \end{aligned}\right\} \qquad (31)$$

Dividing both sides of the equation by 20ρ and taking into account equations (24), the following are obtained:

$$\left.\begin{aligned} &c\left(n_1\cos\alpha+n_2\sin\alpha-\frac{1}{2}j_1\sin\alpha+\frac{1}{2}j_2\cos\alpha\right)+\\ &\quad+\frac{2}{\pi}c\ln\frac{2}{\gamma x_0}(j_1\cos\alpha+j_2\sin\alpha)=\frac{G}{20\rho_0}\cdot j_1,\\ &c\left(n_2\cos\alpha-n_1\sin\alpha-\frac{1}{2}j_1\cos\alpha-\frac{1}{2}j_2\sin\alpha\right)+\\ &\quad+\frac{2}{\pi}c\ln\frac{2}{\gamma x_0}(j_2\cos\alpha-j_1\sin\alpha)=\frac{G}{20\rho_0}\cdot j_2. \end{aligned}\right\} \qquad (32)$$

Thus, the following parameters enter in equations (32):

1) The velocity of the gradient wind at the earth's surface

$$\frac{G}{\rho_0}=U.$$

2) The inclination angle α for z = 0.

3) Quantities, determined by the law of variation of the coefficient of turbulence c

$$x_h=\sqrt{\frac{8\omega_z}{c}}\cdot h.$$

4) The value of $x_0=\sqrt{\frac{8\omega_z}{c}}z_0$, characterizing the roughness of the earth's surface.

Equations (32) allow to solve two very important practical problems.

A) Given: geostrophic wind U, inclination angle α and roughness parameter z_0. We wish to determine the altitude variation of the turbulence coefficient, i.e., the parameters c and h.

B) Given: geostrophic wind U, roughness z_0 and one of the parameters, determining the variation of the turbulence coefficient. We wish to determine the other parameter and the inclination angle α.

To illustrate the solution obtained and to perform the calculations, the authors compiled a number of auxiliary tables.

After having determined the law of altitude variation of the turbulence coefficient, one can determine the velocity components at any altitude. Thus for the layer $z \leqslant h$

$$M = -\left[E\left(-\frac{2}{\pi}\ln\frac{2}{\gamma x_0}+\frac{1}{2}i\right)+G\right]I_0(x\sqrt{i})+EN_0(x\sqrt{i})+G. \qquad (33)$$

Separating the real and imaginary parts, we find:

$$\begin{aligned}
u = &-20c\frac{\rho_0}{\rho}\left(-\frac{2}{\pi}\ln\frac{2}{\gamma x_0}\cos\alpha+\frac{1}{2}\sin\alpha\right)\operatorname{ber}x+\\
&+20c\frac{\rho_0}{\rho}\left(\frac{1}{2}\cos\alpha-\frac{2}{\pi}\ln\frac{2}{\gamma x_0}\sin\alpha\right)\operatorname{bei}x+\\
&+\frac{\rho_0}{\rho}U(1-\operatorname{ber}x)+20c\frac{\rho_0}{\rho}\cos x\operatorname{ner}x+20c\frac{\rho_0}{\rho}\sin\alpha\cdot\operatorname{nei}x,\\
v = &\,20c\frac{\rho_0}{\rho}\left(-\frac{2}{\pi}\ln\frac{2}{\gamma x_0}+\frac{1}{2}\sin\alpha\right)\operatorname{bei}x+\frac{\rho_0}{\rho}U\operatorname{bei}x-\\
&-20c\frac{\rho_0}{\rho}\cos\alpha\operatorname{nei}x+20c\frac{\rho_0}{\rho}\sin\alpha\cdot\operatorname{ner}x. \qquad (34)^*
\end{aligned}$$

For a layer where $z > h$:

$$M = Ae^{-n\sqrt{i}z}+G, \qquad (35)$$

whereby the arbitrary constant is determined by the condition (27).

Shvets and Yudin calculated a number of examples of wind distribution in altitude, adopting the following parameter values (see following table).

	I	II	III	IV	V	VI
U =	3.88	5.75	12.8	20	6.62	10.52 m/sec
z_0 =	0.1	0.1	0.1	0.1	0.1	190 m
ν =	0.077	3.44	3.82	42.3	20.9	1.77 m^2/sec

On the basis of the tables they have compiled, the authors were able to reach the following conclusions, which agree well with observations:

1) The wind inclination angle increases over a rough surface.

2) With the increase of the pressure gradient, eddy viscosity increases.

3) Wind velocity at weathervane level is about half of the geostrophic wind velocity.

4) With increasing altitude the wind approaches the geostrophic, both in magnitude and direction, attaining geostrophic speed sooner than geostrophic direction.

5) To a larger eddy viscosity correspond smaller inclination angles.

6) The altitude at which the wind reaches the geostrophic values, is higher for strong than for weak winds.

7) The ratio between wind velocity at weathervane level and the velocity of the geostrophic wind depends on the former velocity, on the roughness and on the inclination angle. When any of these quantities decreases the considered ratio increases.

* [Translator's note - see note to equations 16, 17, 18, this section].

8) At certain altitudes the wind speed exceeds the geostrophic wind speed at the earth's surface.

§ 5. The Diurnal Wind Cycle

At the earth's surface, if it is plane, the wind velocity displays a definite daily (24 hourly) cycle, unless disturbed by weather changes. The velocity usually attains its maximum in the early afternoon. At night a shallow and wide minimum is observed. The angle of wind deviation from the isobar attains a minimum during the day and a maximum at night. The diurnal cycle of the wind velocity depends on the season, the underlying surface, and the air mass. There exists almost no diurnal wind cycle above the sea.

Above a certain level the wind velocity undergoes an opposite diurnal cycle: maximum at night and minimum by day. The altitude of reversion of the diurnal cycle changes within a wide range, depending on the season and the wind force. In summer, for example, at a highly developed turbulence, the altitude of reversion rises sometimes up to 300 m, and in winter it drops to 30 – 50 m.

Calculation shows that the diurnal wind cycle cannot be explained by the pressure cycle, since the wind velocities due to the latter never exceed 10 cm/sec, while the actually observed daily amplitudes of wind variation are many tens of times greater.

On the other hand, the above enumerated characteristics of the diurnal wind cycle clearly show that it is connected with the diurnal cycle of turbulence.

The first attempt to explain the diurnal wind cycle on the basis of hydromechanical equations was made by B. I. Izvekov. He considered the coefficient of kinematic eddy viscosity to be constant at all altitudes, but a periodic function of time:

$$\nu = \nu_0 + \varepsilon \sin \omega t,$$

where ν_0 is the coefficient of kinematic eddy viscosity averaged over 24 hours, ε – the diurnal amplitude of this coefficient, and ω – the angular velocity of the earth.

However, assuming that the coefficient of eddy viscosity does not vary with altitude, Izvekov obtained unsatisfactory results. Thus, according to Izvekov the wind velocity does not tend with increasing altitude to the velocity of the gradient winds. The condition of air adhesion to the earth's surface is also not fulfilled. Finally, the reversal of the diurnal wind cycle at the higher levels of the boundary layer is not accounted for.

M. E. Shvets solved the problem of the diurnal wind cycle, assuming that ν increases linearly up to a certain altitude h and then remains constant. In addition, Shvets regards ν as a periodic function of time with a 24 hour period.

Writing the equations of motion for a turbulent atmosphere in the form:

$$\left.\begin{aligned} -\rho \frac{\partial u}{\partial t} + 2\omega_z \rho v + \frac{\partial}{\partial z} \nu \frac{\partial \rho u}{\partial z} &= \frac{\partial p}{\partial x}, \\ -\rho \frac{\partial v}{\partial t} - 2\omega_z \rho u + \frac{\partial}{\partial z} \nu \frac{\partial \rho v}{\partial z} &= \frac{\partial p}{\partial y} \end{aligned}\right\} \qquad (1)$$

and introducing the complex velocity $w = u - iv$, Shvets reduces the problem to the integration of a single partial differential equation:

$$-\rho \frac{\partial w}{\partial t} + \frac{\partial}{\partial z} \nu \frac{\partial \rho w}{\partial z} + 2i\omega_z \rho w = \frac{\partial p}{\partial x} - i \frac{\partial p}{\partial y}. \qquad (2)$$

Shvets further assumes, that in equation (2) one can disregard the term

$-\rho\frac{\partial w}{\partial t}$, since the amplitude of oscillation of the wind velocity does not usually exceed 10 – 20% of the mean velocity. Therefore, instead of equation (2) the following equation is used:

$$2i\omega_z\rho w+\frac{\partial}{\partial z}\nu\frac{\partial\rho w}{\partial z}=\frac{\partial p}{\partial x}-i\frac{\partial p}{\partial y}, \tag{3}$$

whereby ν, w, and p depend parametrically on the time t, so that:

$$\left.\begin{aligned}\nu&=c(t)\cdot z \quad \text{at} \quad z\leqslant h(t),\\ \nu&=c(t)\,h(t) \quad \text{at} \quad z>h(t).\end{aligned}\right\} \tag{4}$$

The boundary conditions were formulated by Shvets as follows:

1) At the level of the rough underlying surface the wind velocity vanishes:

$$w=0 \quad \text{at} \quad z=z_0. \tag{5}$$

2) The friction force is bounded at infinity:

$$\nu\frac{\partial\rho w}{\partial z}<\infty \quad \text{at} \quad z=\infty. \tag{6}$$

3) At the level z = h the condition of conjugation of the two solutions is fulfilled, i.e., the function ρw and $\frac{\partial\rho w}{\partial z}$ are continuous.

4) For $z\to 0$ Prandtl's boundary condition holds:

$$|\rho w|_{z\to 0}\to\frac{1}{\varkappa}\sqrt{\tau_0\rho_0}\ln\frac{z}{z_0};\quad \tau_0=\lim_{z\to 0}\nu\left|\frac{\partial\rho w}{\partial z}\right|. \tag{7}$$

A solution of equation (3) that satisfies the given boundary conditions was obtained above (cf. § 4):

$$\rho w=-\left[E\left(-\frac{2}{\pi}\ln\frac{2}{\gamma x_0}+\frac{1}{2}i\right)+G\right]\cdot I_0(x\sqrt{i})+EN_0(x\sqrt{i})+G \tag{8}$$

$$z\leqslant h,$$

where I_0 and N_0 are Bessel and Neumann functions respectively, and

$$x_0=\sqrt{\frac{8\omega_z\cdot z_0}{c}};\quad x=\sqrt{\frac{8\omega_z z}{c}};\quad G=-\frac{1}{2\omega_z}\left(\frac{\partial p}{\partial y}-i\frac{\partial p}{\partial x}\right); \tag{9}$$

E is the integration constant, γ = 1.17811 - Euler's constant, and

$$\rho w=Ae^{-\sqrt{\frac{-2\omega_z i}{ch}}}+G \quad \text{for} \quad z>h, \tag{10}$$

where A is a constant of integration.

It was also shown in the mentioned work that after determination of the integration constants, two equations remain, involving three functions: c(t), h(t) and α (t), where α is the angle between the wind direction and the direction of the isobar at the earth's surface.

These equations can be written in the form:

$$\frac{n_1j_2-n_2j_1}{j_1^2+j_2^2}+\frac{1}{2}=-\frac{G}{20c\rho_0}\sin\alpha;$$

$$\frac{n_1 j_1 + n_2 j_2}{j_1^2 + j_2^2} - \frac{2}{\pi} \ln \frac{2}{\gamma x_0} = -\frac{G}{20 c \rho_0} \cos \alpha, \Bigg\} \tag{11}$$

where:

$$\left.\begin{aligned} j_1 + i j_2 &= I_0\left(\sqrt{\frac{8\omega_z h}{c}}\, i\right) + i I_0'\left(\sqrt{\frac{8\omega_z h}{c}} \cdot i\right), \\ n_1 + i n_2 &= N_0\left(\sqrt{\frac{8\omega_z h}{c}}\, i\right) + i N_0'\left(\sqrt{\frac{8\omega_z h}{c}}\, i\right). \end{aligned}\right\} \tag{12}$$

Eliminating α from (11), we obtain a single equation, connecting c(t) and h(t)

$$\left(\frac{n_1 j_2 - n_2 j_1}{j_1^2 + j_2^2} + \frac{1}{2}\right)^2 + \left(\frac{n_1 j_1 + n_2 j_2}{j_1^2 + j_2^2} - \frac{2}{\pi} \ln \frac{2}{\gamma x_0}\right)^2 = \left(\frac{G}{20 c \rho_0}\right)^2. \tag{13}$$

Imposing on the coefficient of eddy viscosity ν above the level z = h the condition:

$$\nu = c(t)\, h(t) = \nu_0 (1 + \varepsilon \sin \omega t), \tag{14}$$

Shvets determines c(t) and h(t) from equations (13) and (14) and then finds the wind distribution in altitude from the tables he has constructed.

Notwithstanding the schematic nature of the time-dependence (14) of the coefficient of eddy viscosity, the theoretical deductions of Shvets conform well with observation.

As a result of absorption of radiation, internal and potential energies are accumulated, which can be further converted into kinetic energy of large-scale currents $(U + \Pi) \rightarrow E$, or eventually into eddy kinetic energy $(U + \Pi) \rightarrow E'$. The transformation of the potential and internal energy into kinetic energy is of utmost importance in modern dynamic meteorology. This is due to the fact that the general atmospheric circulation and the development of cyclones and anticyclones are closely connected with the mechanism of such transformations. Kinetic energy may be converted into potential or internal energy either through its direct transformation into potential energy $E \rightarrow (U + \Pi)$, or through dissipation of the mechanical energy in the form of heat. In the latter case kinetic energy is "degraded" through the disintegration of vortexes. The kinetic energy of large-scale currents is converted into the kinetic energy of eddies of progressively decreasing size, $E \rightarrow E'$, until it is dissipated into heat, $E' \rightarrow (U + \Pi)$, by the action of viscosity.

The process of conversion of kinetic energy of large-scale currents into eddy kinetic energy is irreversible.

Processes leading to rectification of streamlines and to an increase in the overall flow velocity at the expense of the velocity of local additional motions are not realizable in nature.

§ 2. Energy Balance of the Individual Air Parcel

We shall derive an equation of energy balance for a separate air parcel of unit mass. We assume that friction is absent and the air behaves like a perfect fluid. In this case the equations of motion can be written in the following manner:

$$\left.\begin{aligned} \frac{du}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\omega \sin\varphi v, \\ \frac{dv}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\omega \sin\varphi u, \\ \frac{dw}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \end{aligned}\right\} \tag{1}$$

Multiplying equations (1) by u, v, and w respectively and adding them, we obtain the equation of mechanical energy balance or living force [kinetic energy] equation

$$\frac{d}{dt}\left(\frac{c^2}{2} + gz\right) = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial s}\cdot c, \tag{2}$$

where c is the wind velocity, and $\frac{\partial p}{\partial s}$ – the component of the pressure ascendent in the direction of the wind vector.

We should note that the deflective force of the earth's rotation is not accounted for in this equation, as should be expected, since the work of the force of inertia identically vanishes, owing to the orthogonality of the velocity and this force.

If the motion is horizontal, $\frac{d}{dt}(gz) = 0$, and equation (2) is simplified:

$$\frac{d}{dt}\left(\frac{c^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial s}\cdot c. \tag{3}$$

It follows from equation (3) that if the pressure decreases in the direction of the flow $\left(\frac{\partial p}{\partial s} < 0\right)$ the kinetic energy increases and vice versa. Consequently, a gust of air contrary to the gradient results in a kinetic energy loss.

Combining equation (2) with the equation of heat balance for an ideal gas

$$\frac{dq}{dt} = c_v\frac{dT}{dt} + p\frac{d}{dt}\left(\frac{1}{\rho}\right), \tag{4}$$

we obtain the equation of energy balance for the frictionless flow of an ideal gas:

$$\frac{dq}{dt} = \frac{d}{dt}\left(\frac{c^2}{2} + gz + c_v T\right) + p\frac{d}{dt}\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\frac{\partial p}{\partial s}\cdot c. \tag{5}$$

Equation (5) gives the total energy absorbed from the surroundings by a unit of mass of an ideal gas in frictionless flow, as the sum of the respective kinetic, potential, and internal energy gains, of the work of expansion and of the work done against the pressure gradient.

Noting that

$$c\frac{\partial p}{\partial s} = \frac{dp}{dt} - \frac{\partial p}{\partial t},$$

we obtain the equation of energy balance in the following form:

$$\frac{dq}{dt} = \frac{d}{dt}\left(\frac{c^2}{2} + gz + c_v T + \frac{p}{\rho}\right) - \frac{1}{\rho}\frac{\partial p}{\partial t}. \tag{6}$$

If the flow is steady, $\frac{\partial p}{\partial t} = 0$, and if the state of the air varies adiabatically, $\frac{dq}{dt} = 0$, equation (6) can be integrated with respect to the time:

$$\frac{c^2}{2} + gz + c_v T + \frac{p}{\rho} = \text{const}. \tag{7}$$

But

$$c_v T + \frac{p}{\rho} = c_v T + RT = c_p T.$$

Consequently, the equation of energy balance assumes the following form:

$$\frac{c^2}{2} + gz + c_p T = \text{const}. \tag{8}$$

This equation connects the parameters of the particle at different times. Since we assume the flow to be steady, the streamlines coincide with the particle trajectories. Therefore, equation (8) connects also the parameters of state of various particles located on the same streamline. However, the constants for different streamlines generally differ.

Equation (7) represents the generalization of Bernoulli's law for an incompressible perfect fluid:

$$\frac{c^2}{2} + gz + \frac{p}{\rho} = \text{const}. \tag{9}$$

Bernoulli's equation and its generalization are widely applied, especially in instrumental meteorology. In high speed flow, for instance, it explains the distortion of the barometeric pressure field by the presence of obstacles. This equation furnishes a theoretical principle for the construction of flow velocity gauges (flowmeters) by means of the corresponding pressure variations. Bernoulli's equation allows to estimate error in temperature and pressure measurements by instruments installed on high speed aircraft.

§ 3. Energy Balance of a Given Air Mass

Proceeding from the law of conservation of energy we can derive an equation of energy balance for a given fluid mass, as follows:

$$\frac{dE}{dt} + \frac{dU}{dt} = Q + \int_{\tau} \rho(F, v)\, d\tau + \int_{\Sigma} (p_n, v)\, d\sigma \tag{1}$$

(see § 7 Chapter VIII) where $\frac{dE}{dt}$ is the kinetic energy variation of the given fluid mass per unit time; $\frac{dU}{dt}$ – the change per unit time in the internal energy of the same mass; Q – heat influx from external sources; $\int_{\tau} \rho(F, v)\, d\tau$ the total work of the volume forces per unit time; $\int_{\Sigma} (p_n, v)\, d\sigma$ the work of all surface forces per unit time.

Transforming the equation of energy balance into a form suitable for the study of the atmosphere, we are guided by the results obtained in § 13 Chapter VIII, allowing us to regard the variation of the total kinetic energy of the stream $\frac{dE}{dt}$ as a sum of the variation of the kinetic energy of the average motion $\frac{d\bar{E}}{dt}$ and of the kinetic energy of eddy pulsation $\frac{dE'}{dt}$,

$$\frac{dE}{dt} = \frac{d\bar{E}}{dt} + \frac{dE'}{dt}, \tag{2}$$

whereby the change of the eddy kinetic energy for the given air mass may be expressed as follows:

$$\frac{dE'}{dt} = \int_{\tau} T\,d\tau - \int_{\tau} D\,d\tau, \tag{3}$$

where T is the rate of transformation of average flow kinetic energy into eddy kinetic energy per unit volume, and D – the rate of dissipation of mechanical energy per unit mass.

(4)*

$$T = \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right\},$$

(5)

$$D = \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right\}.$$

Replacing $\frac{dE}{dt}$ in the equation of energy balance (1) by its expression obtained from equations (2) and (3):

$$\frac{dE}{dt} = \frac{d\bar{E}}{dt} + \int_{\tau} T\,d\tau - \int_{\tau} D\,d\tau, \tag{6}$$

we can write the equation of energy balance in the following manner:

$$\frac{d\bar{E}}{dt} + \frac{dU}{dt} = Q + \int_{\tau} \rho\,(F, v)\,d\tau + \int_{\Sigma} (p_n, v)\,d\sigma - \int_{\tau} T\,d\tau + \int D\,d\tau. \tag{7}$$

We shall now derive the expression for the work done by the mass forces. Let us recall that in dynamic meteorological studies of atmospheric motions we deal with only two mass forces: the force of gravity and the deflecting force of the earth's rotation. Since the latter is perpendicular to the velocity its work is equal to zero. Therefore, it does not enter in the equation of energy balance.

We now introduce into the equation of the energy balance the geopotential instead of the force of gravity Φ:

$$F = -\operatorname{grad}\Phi. \tag{8}$$

Since the geopotential Φ represents a potential energy referred to a unit of mass, the potential energy Π of the total fluid mass will be expressed by the formula

$$\Pi = \int_{\tau} \rho\Phi\,d\tau. \tag{9}$$

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* [Translator's note. The Russian text has μ instead of the coefficient of eddy viscosity μ_{turb}].

Taking into account that $\rho d\tau = \text{const}$. we obtain:

$$\frac{d\Pi}{dt} = \frac{d}{dt}\int_{\tau} \rho\Phi\, d\tau = \int_{\tau} \rho \frac{d\Phi}{dt}\, d\tau = \int_{\tau} \rho \frac{\partial\Phi}{\partial t}\, d\tau + \int_{\tau} \rho\,(\boldsymbol{v}, \operatorname{grad}\Phi)\, d\tau. \tag{10}$$

But the geopotential at any point is time-independent $\frac{\partial\Phi}{\partial t} = 0$, whence,

$$\frac{d\Pi}{dt} = \int_{\tau} \rho\,(\boldsymbol{v}, \operatorname{grad}\Phi)\, d\tau. \tag{11}$$

Transforming the integral expression for the work done by the mass forces, we obtain:

$$\int_{\tau} \rho\,(\boldsymbol{F}, \boldsymbol{v})\, d\tau = -\int_{\tau} \rho\,(\boldsymbol{v}, \operatorname{grad}\Phi)\, d\tau = -\frac{d\Pi}{dt}. \tag{12}$$

Consequently, the equation of the energy balance assumes the following form:

$$Q = \frac{d}{dt}(\bar{E} + E' + \Pi + U) - \int_{\Sigma} (p_n, v)\, d\sigma \tag{13}$$

or

$$Q = \frac{d}{dt}(\bar{E} + U + \Pi) - \int_{\Sigma} (p_n, v)\, d\sigma + \int_{\tau} T\, d\tau - \int_{\tau} D\, d\tau.$$

Equation (13) shows that the rate of heat influx from the surroundings makes up the sum of the variations of the kinetic energy (of average and eddy motions), of the potential and internal energies, and also of the work done by the pressure forces per unit time and by the viscosity forces acting on the surface of the fluid volume.

We now introduce the enthalpy $э$* into the equation of energy balance.

The enthalpy per mass unit is given by the formula $э = u + p \cdot \alpha$, where u is the internal energy of the mass unit and α - the specific volume; and the enthalpy of the total fluid mass - by the formula

$$э = \int_{\tau} \rho\,(u + p\alpha)\, d\tau = \int_{\tau} \rho u\, d\tau + \int_{\tau} \rho\,(p\alpha)\, d\tau. \tag{14}$$

Differentiating the equation with respect to the time and recalling that $\rho d\tau = \text{const.}$, we obtain:

$$\frac{dэ}{dt} = \frac{dU}{dt} + \int_{\tau} \rho \frac{d(p\alpha)}{dt}\, d\tau. \tag{15}$$

Eliminating the internal energy between equations (13) and (15), we obtain the equation of energy balance in the form:

$$Q = \frac{d}{dt}[\bar{E} + E' + \Pi + э] - \int_{\tau} \rho \frac{d(p\alpha)}{dt}\, d\tau - \int_{\Sigma} (p_n, v)\, d\sigma. \tag{16}$$

We transform the surface integral $\int_{\Sigma} (p_n, v)\, d\sigma$, which expresses the work per unit time done by the forces of pressure and viscosity acting on the boundary Σ of the volume τ:

$$\int_{\Sigma} (p_n, v)\, d\sigma = -\int_{\Sigma} p\,(n, v)\, d\sigma + \int_{\Sigma} (p_n + p \cdot n) \cdot v\, d\sigma,$$

where n is the unit normal to the surface element $d\sigma$; the first integral on the right-hand side gives the work done by the pressure and the second - the work done by the viscosity forces.

Applying Gauss' theorem and making use of the properties of Euler's

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* [Translator's note. The Russian character $э$ stands everywhere for the enthalpy].

operator, we get:

$$\int_{\Sigma} p(\boldsymbol{n}, \boldsymbol{v})\, d\sigma = \int_{\Sigma} pv_n\, d\sigma = \int_{\tau} \operatorname{div}(p\boldsymbol{v})\, d\tau = \int_{\tau} \left(\frac{\partial pu}{\partial x} + \frac{\partial pv}{\partial y} + \frac{\partial pw}{\partial z}\right) d\tau =$$
$$= \int_{\tau} \left(\frac{\partial p}{\partial x} u + \frac{\partial p}{\partial y} v + \frac{\partial p}{\partial z} w\right) d\tau + \int_{\tau} p \operatorname{div} \boldsymbol{v}\, d\tau = \tag{17}$$
$$= \int_{\tau} \left(\frac{dp}{dt} - \frac{\partial p}{\partial t}\right) d\tau + \int_{\tau} p \operatorname{div} \boldsymbol{v}\, d\tau.$$

In addition

$$\int_{\tau} \rho \frac{d(p\alpha)}{dt}\, d\tau = \int_{\tau} \rho\, p \frac{d\alpha}{dt}\, d\tau + \int_{\tau} \frac{dp}{dt}\, d\tau = \int_{\tau} p \operatorname{div} \boldsymbol{v}\, d\tau + \int_{\tau} \frac{dp}{dt}\, d\tau,$$

therefore

$$-\int_{\tau} \rho \frac{d(p\alpha)}{dt}\, d\tau - \int_{\Sigma} p(n, v)\, d\sigma = \int_{\tau} \frac{\partial p}{\partial t}\, d\tau.$$

Consequently, the equation of energy balance assumes the following form

$$Q = \frac{d}{dt}[\bar{E} + E' + \Pi + \mathfrak{z}] - \int_{\tau} \frac{\partial p}{\partial t}\, d\tau - \int_{\Sigma} \boldsymbol{v}(\boldsymbol{p}_n + p\boldsymbol{n})\, d\sigma. \tag{18}$$

This equation shows that the heat influx from external sources to the fluid volume is balanced by the variation of the sum of the kinetic energy (of the average and eddy motion), the potential energy and the enthalpy, of the work of local pressure variations and the work of the viscosity forces on the boundary of the volume.

§ 4. The Energy of a Vertical Air Column

We consider a vertical column of air of unit cross section, extending from the level z=0 up to the level z=h. The internal energy of this air column is determined by the formula

$$U = c_v \int_0^h T\rho dz. \tag{1}$$

Taking into account the fundamental hydrostatic equation

$$dp = -g\rho dz,$$

we obtain:

$$U = \frac{c_v}{g} \int_{p_h}^{p_0} T dp. \tag{2}$$

Formula (2) is sometimes more suitable than formula (1) for the calculation of the air column internal energy.

The potential energy of the given air column is equal to

$$\Pi = \int_0^h gz\rho dz, \tag{3}$$

hence, introducing the pressure through the fundamental hydrostatic equation, we obtain:

$$\Pi = \int_{p_h}^{p_0} z dp = -hp_h + \int_0^h p dz,$$

but

$$p = R\rho T;$$

whence

$$\Pi = -hp_h + R \int_0^h T\rho dz. \tag{4}$$

The enthalpy of this air column can be calculated by the formula

$$\vartheta = c_p \int_0^h T\rho dz = \frac{c_p}{g} \int_{p_h}^{p_0} T dp. \tag{5}$$

Combining equations (1) and (4) we obtain

$$\Pi = -h \cdot p_h + (\varkappa - 1) \cdot U, \tag{6}$$

where

$$\varkappa = \frac{c_p}{c_v} = 1{,}4,$$

consequently

$$U + \Pi = -hp_h + \varkappa U = -h \cdot p_h + \vartheta. \tag{7}$$

If the air column extends to infinity, then as $h \to \infty$ the product $h \cdot p_h \to 0$ since the pressure p drops exponentially with increasing altitude. Therefore, for the infinitely high air column the following relationships hold

$$U + \Pi = \vartheta, \tag{8}$$

$$\Pi : U = \varkappa - 1 = 0.4. \tag{9}$$

Consequently, any change in the potential energy of an infinitely high air column is accompanied by proportional changes in the internal energy of this air column, and vice versa. Therefore, in the study of atmospheric energy transformations, the sum of the potential and internal energies (or, alternatively, the enthalpy) should be considered. We stress again that such a relationship is valid only for infinitely high air columns; it does not hold for any finite volume, as may be seen from equation (7), derived for an air column of finite height. This equation shows that for a finite volume of air, Π by no means equals $(\varkappa - 1)U$, but always falls short of it.

§ 5. Energy Release in Vertical Redistribution of Air Masses

We shall discuss several examples of the conversion of potential and internal energy into kinetic energy. Let us assume that the atmosphere is in unstable equilibrium, due either to vertical stratification or to a horizontally unstable mass distribution. This unstable equilibrium is upset by a sufficiently strong external impulse. Once the equilibrium is disturbed the potential and internal energies are transformed into kinetic energy. As the sum of the potential and kinetic energies diminishes, tending to a minimum, the kinetic energy increases, approaching a maximum. We will call the state of maximum kinetic energy the "final" state. It is clear that this "final" state cannot be sustained, since once it is reached, the kinetic energy starts to be reconverted into potential and internal energy.

We consider an air column, bounded from below by the earth's surface, on the sides by solid vertical walls, and open at the top. Mass redistribution takes place only in the lower part of this air column, up to a certain height h, while its upper part moves up and down like a rigid piston of weight $p_h B$, where p_h is the pressure on the upper boundary of the redistributed masses, and B– the area of the column base.

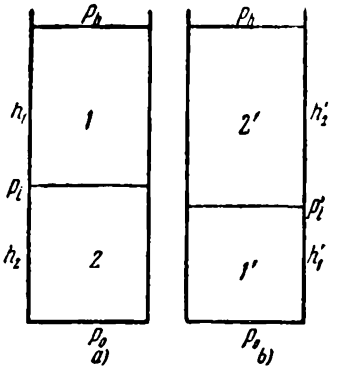


Figure 134. The calculation of the kinetic energy released upon disturbance of a vertically unstable mass equilibrium.

Example 1. In an air column (Figure 134), whose pressure near the ground is p_0, the two masses (designated by 1 and 2) are located one on top of the other. The pressure on the upper boundary of the first mass equals p_h. The second mass has a higher potential temperature than the first. As a result mass redistribution takes place, until in the final state the first mass lies under* the second (1' and 2'). During this process, the height of the two layers changes correspondingly from h_1 to h'_1 and from h_2 to h'_2. The pressure on the boundary changes from p_i to p'_i. The level of the p_h pressure changes too.

Upon mass redistribution the total potential and internal energies of the air column is converted partly into kinetic energy and partly spent as work of expansion of the air column, evidently equal to $p_h(h' - h)$. Consequently,

$$(\Pi + U) - (\Pi + U)' = E' + p_h(h' - h), \tag{1}$$

where the primes denote the values of the variables in the final state. But according to (7) § 4

$$\Pi + U = \vartheta - hp_h,$$

$$(\Pi + U)' = \vartheta' - h'p_h.$$

Consequently,

$$E' = \vartheta - \vartheta'. \tag{2}$$

Thus, it is sufficient to form the difference between the enthalpy values of an air column in order to obtain the value of the kinetic energy liberated in a vertical mass redistribution.

Let us assume, that both masses are initially in a state of indifferent equilibrium,, i. e., each of the masses has the same potential temperature at all points. Let us further assume that the redistribution takes place adiabatically, i. e., each of the masses maintains,in the course of the process,a constant potential temperature. Let the temperature have initially on the boundary i a discontinuous jump from T_{i2} to T_{i1}. Since p_0, T_{02}, h_2, h_1, T_{i1} are known values, we may calculate the enthalpy of the air column in the initial and final states. According to (5) § 4,

$$\vartheta = \frac{c_p}{g}\int_{p_h}^{p} T dp.$$

* [Translator's note. The Russian text has "over", an obvious misprint].

Since the equilibrium is indifferent, applying the adiabatic equation

$$T = T_0 \left(\frac{p}{p_0}\right)^{\frac{\varkappa-1}{\varkappa}},$$

we obtain:

$$\int_{p_h}^{p} T dp = T_0 p_0^{-\frac{\varkappa-1}{\varkappa}} \int_{p_h}^{p_0} p^{\frac{\varkappa-1}{\varkappa}} dp = \frac{\varkappa}{2\varkappa-1}(T_0 p_0 - Tp). \tag{3}$$

Therefore, the enthalpy of the air column in the initial state can be expressed in the following manner:

$$\vartheta = \frac{c_p}{g}\frac{\varkappa}{2\varkappa-1}(p_0 T_{02} - p_i T_{i2} + p_i T_{i1} - p_h T_{h1}),$$

where

$$T_{h1} = T_{i1} - \gamma_a h_1; \quad T_{i2} = T_{02} - \gamma_a h_2; \quad p_i = p_0 \left(\frac{T_{i2}}{T_{02}}\right)^{\frac{\varkappa}{\varkappa-1}}. \tag{4}$$

The enthalpy in the final state, i.e., after mass redistribution, will be:

$$\vartheta' = \frac{c_p}{g}\frac{\varkappa}{2\varkappa-1}(p_0 T'_{01} - p'_i T'_{i1} + p'_i T'_{i2} - p_h T'_{h2}), \tag{5}$$

whereas the pressures p_0 and p_h on the lower and the upper boundaries of the considered air column remain unchanged, and $p'_i = p_h + p_0 - p_i$. Since the mass redistribution proceeds adiabatically, the temperature distribution in each layer is determined by the dry-adiabatic equation, and we obtain

$$\left.\begin{aligned} T'_{01} &= T_{h1}\left(\frac{p_0}{p_h}\right)^{\frac{\varkappa-1}{\varkappa}}; \quad T'_{i1} = T_{h1}\left(\frac{p'_i}{p_h}\right)^{\frac{\varkappa-1}{\varkappa}}, \\ T'_{i2} &= T_{02}\left(\frac{p_i}{p_0}\right)^{\frac{\varkappa-1}{\varkappa}}; \quad T'_{h2} = T_{02}\left(\frac{p_h}{p_0}\right)^{\frac{\varkappa-1}{\varkappa}}. \end{aligned}\right\} \tag{6}$$

The thickness of the layers after mass redistribution is determined from the relationship

$$\left.\begin{aligned} T'_{i1} &= T'_{01} - \gamma_a h'_1, \\ T'_{h2} &= T'_{02} - \gamma_a h'_2. \end{aligned}\right\} \tag{7}$$

The mass of the air column is evidently equal to $\frac{p_0 - p_h}{g}$. The square of the mean velocity will be:

$$c^2 = \frac{2g}{p_0 - p_h}(\vartheta - \vartheta'). \tag{8}$$

Calculations performed according to the foregoing formulas show, that in vertical mass redistribution a large amount of kinetic energy is released. Therefore, as a result of such redistribution high velocities may develop. The kinetic energy of gales may be due to this kind of mass redistribution.

Example 2. Let two air masses rest side by side on the earth's surface (Figure 135). Their bases each equal B/2, they have the common height h and they are laterally bounded by vertical walls. Over these masses lies an air layer of a stable equilibrium, capable of moving up and down like a rigid piston. Let us assume that the first mass has a lower potential temperature than the second.

In the initial state the system consisting of the two air masses can be in equilibrium if they are divided by a rigid partition.

If the rigid partition is removed, the cold air will begin to flow underneath the warm air, as a result of the horizontal pressure gradient. Thus the cold air will contract, accumulating beneath the expanding warm air. The upper boundary will either lift or sink. If the upper boundary rises, the work is done by the pressure of air lying below it; in case it is lowered, the work is done by external pressure.

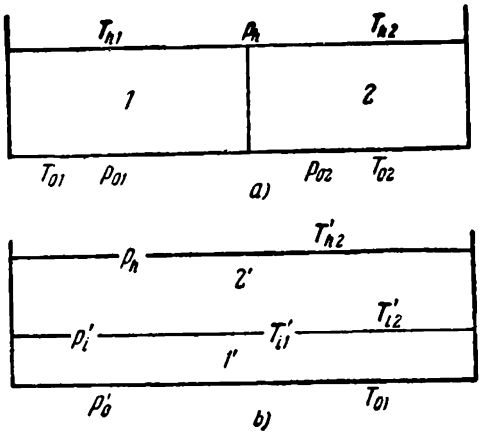


Figure 135. The calculation of the kinetic energy released upon disturbance of unstable equilibrium of adjacent masses

Let us assume that initially the two air masses are in indifferent equilibrium. After adiabatic mass redistribution they are again in indifferent equilibrium.

The enthalpy in the initial state is:

$$\mathfrak{э} = \frac{c_p}{g}\frac{\varkappa}{2\varkappa-1}\frac{B}{2}\left[p_{01}T_{01} - p_{h1}T_{h1} + p_{02}T_{02} - p_hT_{h2}\right]. \tag{9}$$

We assume that p_h, h, T_{h1}, T_{h2} are given. The remaining magnitudes are determined from the following relationships, which characterize the indifferent equilibrium and the adiabatic process:

$$T_{01} = T_{h1} + \gamma_a h; \qquad T_{02} = T_{h2} + \gamma_a h,$$

$$p_{01} = p_h\left(\frac{T_{01}}{T_{h1}}\right)^{\frac{\varkappa}{\varkappa-1}}; \quad p_{02} = p_h\left(\frac{T_{02}}{T_{h2}}\right)^{\frac{\varkappa}{\varkappa-1}}. \tag{10}$$

The enthalpy in the final state is:

$$\mathfrak{э}' = \frac{c_p}{g}\cdot\frac{\varkappa}{2\varkappa-1}B\left[p'_0T'_{01} - p'_iT'_{i1} + p'_iT'_{i2} - p_h'T'_{h2}\right]. \tag{11}$$

In the final state, the temperature at the interface between the two masses has a jump from the lower value T'_{i1} to the higher value T'_{i2}. The upper layer of the second mass remains at the same pressure p_h, and therefore $T'_{h2} = T_{h2}$. The interface is characterized by the corresponding pressure

$$p'_i = p_h + \frac{1}{2}(p_{02} - p_h),$$

as the second mass spreads over a twice larger area.

As a result of the adiabatic redistribution of masses and of the indifferent equilibrium in the initial and final states we have:

$$T'_{i2} = T_{h2}\left(\frac{p'_i}{p_h}\right)^{\frac{\varkappa-1}{\varkappa}}, \quad T'_{i1} = T_{h1}\left(\frac{p'_i}{p_h}\right)^{\frac{\varkappa-1}{\varkappa}}, \quad T'_{01} = T_{h1}\left(\frac{p'_0}{p_h}\right)^{\frac{\varkappa-1}{\varkappa}}, \tag{12}$$

where $p'_0 = \frac{p_{01}+p_{02}}{2}$.

The calculations show that at a 10° difference on each of the horizontal levels the following velocities, c, at various heights h are are developed:

h =	2000	3000	6000 m
c =	13	16	23 m/sec

The above given estimates of the velocity developing upon the transformation of the potential and internal energies into kinetic energy, are obtained by a rough scheme which is far from corresponding to actual atmospheric processes. However, the merit of these estimates consists in the fact that they demonstrate the essential role of the unstable mass distributions in the atmosphere as stores of immense potential energy which is transformed into kinetic energy upon mass redistribution.

Careful analysis shows that the main source of kinetic energy in the atmosphere is the unstable distribution of masses. The earlier prevailing view that the kinetic energy derives mainly from the pressure field has proved to be wrong.

Computations showed that the horizontal pressure distribution is insufficient to explain the kinetic energy which is developed in cyclones. The horizontal distribution of pressure can generate only about 5% of the total kinetic energy of the cyclone. The main source of the kinetic energy of atmospheric motions is the potential energy which increases with temperature, which in its turn rises as a result of heat influx. The gradients of pressure represent the difference in the potential energy of adjacent air columns. The pressure field is therefore merely a mechanism of transformation of potential into kinetic energy, since the horizontal mass transport is due to the gradients which develop in this field.

The above discussion of the transformation of potential into kinetic energy concerns dry air. However, similar equations are applicable to moist air even in the case of condensation, since this equation expresses only the principle of energy conservation. The sum of the potential and internal energies may, however, be substantially different for a column of dry and moist air at the same temperature distribution. If the stratification is relatively unstable, the column of dry air will be in stable equilibrium and no redistribution of masses will take place. In the case of moist air, redistribution may occur and result in vast release of kinetic energy. All this points to the important role of water vapor in the processes of energy transformation in the atmosphere.

§ 6. The Transformation of the Kinetic Energy of Mean Atmospheric Motion into Eddy Kinetic Energy

In § 13 Chapter VIII we have seen that the kinetic energy of the mean flow transformed into kinetic energy of eddy pulsation per unit volume is expressed by the following formula

$$T = A\left\{ 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \right.$$
$$\left. + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 \right\} \qquad (1)$$

(the symbol [bar] denoting average values is everywhere omitted, and the coefficient of eddy viscosity is replaced by the [equal] exchange coefficient A).

It is easy to estimate the transformation of the kinetic energy of the mean atmospheric flow for the simplest case only, when $u = u(z)$; $v = v(z)$; $w = 0$. In this case the formula for T is simplified and takes the following form:

$$T = A\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]. \qquad (2)$$

Let us assume, that the coefficient of turbulence, the density and the pressure do not change with altitude and that the velocity does not change with time. This assumption corresponds to the stationary condition of mean motion when the kinetic energy lost by transformation is continuously replenished. In § 3 Chapter XVI we have shown that the components of the wind velocity under these assumptions are

expressed as follows:

$$u = u_g(1 - e^{-\alpha z}\cos \alpha z),$$
$$v = u_g \cdot e^{-\alpha z}\sin \alpha z,$$

where

$$u_g = \frac{1}{2\rho\omega \sin \varphi}\frac{\partial p}{\partial y}$$

is the velocity of the geostrophic wind, and

$$\alpha = \sqrt{\frac{\rho\omega \sin \varphi}{A}},$$

consequently,

$$T = 2\rho\omega \sin \varphi u_g^2 e^{-2\alpha z}. \tag{3}$$

Since above the friction level $D = \pi/\alpha$ the altitude variation of the wind velocity due to turbulence is very small, the transformation of the kinetic energy of mean motion into eddy energy is also very slight above this level. For the estimation of the order of magnitude of the kinetic energy transformed in the atmosphere, it is sufficient to integrate the expression (3) from the earth's surface to the friction level D:

$$T^* = \int_0^D T dz = u_g^2(1 - e^{-2\pi})\sqrt{A\rho\omega \sin \varphi}. \tag{4}$$

We will compare the magnitude of the transformed kinetic energy with the total reserves of kinetic energy in a unit air column, extending from the earth's surface to the friction level D; since

$$\mathcal{E}^* = \frac{\rho}{2}\int_0^D (u^2 + v^2)\, dz = \frac{\rho}{2} D \cdot u_g^2 \left\{1 - \frac{(1 + e^{-\pi})^2}{2\pi}\right\}, \tag{5}$$

we get

$$\frac{T^*}{\mathcal{E}^*} = \frac{2\omega \sin \varphi}{\pi}\,\frac{1 - e^{-2\pi}}{1 - \frac{(1 + e^{-\pi})^2}{2\pi}}. \tag{6}$$

The calculations show that, for example, at a latitude $\simeq 45^\circ$, about 15% of the available kinetic energy is transformed in 1 hour. Consequently, 7 hours would have sufficed for the transformation of the entire kinetic energy into eddy energy, had it not been replenished at the expense of the solar energy converted into potential and internal energy.

More accurate estimates show that the transformation of the kinetic energy in an air column of cross section 1 m^2 and extending up to a height of 10 km (from the earth's surface) is equal to $5 \cdot 10^7$ erg m^2/sec, whereby the transformation in the lower part of this column (up to the altitude of 1 km) will be $3 \cdot 10^7$ erg m^2/sec. If the mean wind velocity in the entire air column is 10 m/sec, the total kinetic energy of the column will amount to $5 \cdot 10^{12}$ erg. Consequently, after 10^5 sec = 1.16 day, the entire kinetic energy of mean motion will be transformed into eddy energy. Elementary calculation shows that 2% of the available radiation is sufficient to compensate for the lost kinetic energy.

§ 7. Energy Balance of the Turbulent Atmosphere

As was shown earlier (§ 13, Chapter VIII), the variation per unit mass of the eddy energy of an incompressible fluid $\frac{dE'}{dt}$ is determined by the transformation of the kinetic energy T of the mean motion and by the dissipation D of the mechanical energy into heat:

$$\frac{dE'}{dt} = T - D \tag{1}$$

The transformation of the kinetic energy of the mean motion is not the only energy supply to the turbulent atmosphere. We have shown in § 5 that the change in the internal energy of an air column is closely linked with the changes of potential energy in this column. If the stratification of the atmosphere is stable, the air particle (individual vortex) upon entering a higher layer, becomes colder than the surrounding air; therefore it is forced to descend. Similarly, the individual vortex descending in a stable atmosphere will become warmer than the surrounding air and will tend to regain its former position. Since individual eddies continually move up and down in turbulent flow, the work of these motions is done at the expense of the eddy kinetic energy, which correspondingly decreases. Consequently, the variation of the atmospheric eddy kinetic energy is determined not only by transformation and dissipation of the energy but also by the work done in overcoming the stable vertical stratification of the atmosphere. It appears that the work performed substantially exceeds the amount of energy dissipated into heat.

A. M. Obukhov applies the equation of turbulent energy balance in the following form:

$$\frac{dF'}{dt} = T - \frac{d\Pi}{dt} - D, \tag{2}$$

where E' is the eddy kinetic energy, Π the potential energy of the flow, $\frac{d\Pi}{dt}$ the work done to overcome the stable stratification of the atmosphere.

For the analysis of this equation of turbulent energy balance A. M. Obukhov introduces two parameters of turbulence: v'—the eddy pulsation velocity and l—the scale of turbulency. This corresponds to the mixing length, and the equation of turbulent energy balance assumes the following form:

$$\frac{dE'}{dt} = K\left(\frac{\partial v}{\partial z}\right)^2 - K_T \frac{g}{T}\frac{\partial \theta}{\partial z} - D, \tag{3}$$

where

$$K = v'l; \quad K_T = \alpha v'l,$$
$$D = \beta \frac{v'^3}{l}; \quad E' = \frac{v'^2}{2}$$

(see § 5 and 8, Chapter XV).

The coefficients α and β are taken to be constants to a first approximation, whose value we will estimate in the following.

Obukhov examines the pattern of turbulence to which the following form of the equation of energy balance applies:

$$\frac{\partial E'}{\partial t} = v'l\left(\frac{\partial v}{\partial z}\right)^2 - \alpha v'l \frac{g}{T}\frac{\partial \theta}{\partial z} - \beta \frac{v'^3}{l} = 0$$

or

$$v'\left[l^2\left(\frac{\partial v}{\partial z}\right)^2 - \alpha l^2 \frac{g}{T}\frac{\partial \theta}{\partial z} - \beta v'^2\right] = 0. \tag{4}$$

Equation (4) is factored out into two equations

$$v' = 0, \tag{5}$$

$$\left(\frac{\partial v}{\partial z}\right)^2 - \alpha \frac{g}{T}\frac{\partial \theta}{\partial z} = \beta \frac{v'^2}{l}. \tag{6}$$

Equation (5) characterizes the steady laminar flow pattern, and equation (6) the steady turbulent flow pattern. Equation (6) may be solved only on condition that

$$\left(\frac{\partial v}{\partial z}\right)^2 - \alpha \frac{g}{T}\frac{\partial \theta}{\partial z} \geqslant 0. \tag{7}$$

Introducing the Richardson number

$$\mathrm{Ri} = \frac{g}{T}\frac{\frac{\partial \theta}{\partial z}}{\left(\frac{\partial v}{\partial z}\right)^2},$$

we find that for the steady turbulent regime it should satisfy

$$\mathrm{Ri} < (\mathrm{Ri})_{\mathrm{crit}} = \frac{1}{\alpha} = \frac{K}{K_T}. \tag{8}$$

For $\mathrm{Ri} > (\mathrm{Ri})_{\mathrm{crit}}$ the only solution of the equation of energy balance is $v' = 0$, which corresponds to steady laminar flow.

In order to estimate β we transform equation (6) to the form

$$\frac{\beta v'^2}{l^2\left(\frac{\partial v}{\partial z}\right)^2} = 1 - \frac{\mathrm{Ri}}{(\mathrm{Ri})_{\mathrm{crit}}},$$

whence

$$v' = \frac{1}{\beta}\sqrt{1 - \frac{\mathrm{Ri}}{(\mathrm{Ri})_{\mathrm{crit}}}}\left|\frac{\partial v}{\partial z}\right| \cdot l. \tag{9}$$

In order that formula (9) coincides with the basic relationship of the semiempirical theory $v' = l\frac{\partial v}{\partial z}$ for an adiabatic stratification of the atmosphere when $\mathrm{Ri} = 0$, it is necessary to assume $\beta = 1$.

The dissipation of energy may be neglected. Therefore, the equation of energy balance may be neglected. Therefore, the equation of energy balance may be put in the following form

$$\frac{dE'}{dt} = T\left(1 - \frac{\mathrm{Ri}}{(\mathrm{Ri})_{\mathrm{crit}}}\right). \tag{10}$$

In this case the onset of turbulence is possible only at definite values of $\mathrm{Ri} < (\mathrm{Ri})_{\mathrm{crit}}$. If $\mathrm{Ri} = (\mathrm{Ri})_{\mathrm{crit}}$, i.e., if the transformed energy exceeds the work done against the stable stratification, the eddy energy will increase. In the opposite case the eddy energy decreases. Considerable vertical gradients of the mean velocity are a sign, under otherwise similar conditions, of considerable energy transformation. Therefore, the eddy energy will increase even at an inversion, whereby the inversions would be destroyed, if, as it often happens in the atmosphere, they are accompanied by a sharp break in the velocity of wind.

The question of the numerical value of $(\mathrm{Ri})_{\mathrm{crit}}$ is at present unsolved.

Chapter XVIII

GENERAL CIRCULATION OF THE ATMOSPHERE

§1. General Considerations

The atmosphere is constantly in a state of complex motion, changing with time. Analyzing the synoptic maps, constructed for limited territories, we are struck by the variety of atmospheric flow patterns. All these diverse flows, whatever their cause, are included in the general circulation of the atmosphere. The flow of air in cyclones and anticyclones, orographic winds, breezes, and similar phenomena represent different instances of the general circulation of the atmosphere. The atmospheric motions are so manifold that the probability of recurrence of any flow pattern of the entire atmosphere may be taken to equal zero,

It is clearly impossible to describe the general circulation of the atmosphere in complete detail, and we must be content with the most characteristic and broad outlines of the general circulation which persevere from year to year. The theoretical study even of these most characteristic features of the general circulation also meets with insurmountable difficulties.

At the present state of our knowledge we are in possession of an unsatisfactory theoretical scheme of the general circulation.

In order to obtain such a scheme, we should first of all set up the mathematical laws that govern the general circulation of the atmosphere. The general physical and hydrodynamic laws are retained but they must be given an appropriate mathematical form. The derivation of the equation of heat balance is the most difficult. These difficulties are mostly due to the fact that the heat exchange in the atmosphere is accomplished through radiation, eddy heat conduction, and through water phase changes.

In the general circulation of the atmosphere all these forms of heat cxchange are of equal importance, and must be accounted for correspondingly. The radiation and eddy heat exchange are the most complex processes. In the derivation of the equations of motion a special difficulty arises when the eddy viscosity has to be introduced. Therefore, a rigorous mathematical formulation of the problem of the general atmospheric circulation is at present impossible. Nevertheless, the attempts to tackle the problem of the general atmospheric circulation have not been discontinued, and proceed in two directions: 1) some workers, proceeding from the analysis of the average climatological mean pressure charts, attempt to put forward empirical, often qualitative theories of the general circulation (Bjerknes, Bergeron, Prandtl); 2)other investigators offer semi-empirical theories of the general circulation, proceeding from the temperature distribution. The great achievement of the Soviet theoreticians (Kochin, Blinova) should be mentioned here.. However, the characterization of the real general atmospheric circulation is equally important for all investigators.

§2. A Survey of the General Circulation of the Atmosphere

In order to reveal the most typical features of the general circulation of the atmosphere, average weather maps, especially those compiled for January and July, are being used. The comparison of maps for January (Figure 136) and July

(Figure 137) of the mean pressure distribution, shows first of all that in summer low pressure prevails over the continents, while high pressure prevails over the oceans. In winter the picture is reversed. The pressure in the equatorial region is lowered. A sharply defined subtropical belt of high pressure centered around 30° southern latitude can be noticed in the Southern Hemisphere the year round, being more sharply defined over the oceans than over the continents. The maximum pressure in this region amounts on the average to 764-768 mm. In the Northern Hemisphere the high pressure belt can be observed only in winter, whereby it becomes strongly distorted by the presence of the Asian continent, where maximum pressure predominates (in winter). These subtropical regions of high pressure do not encircle the globe uninterruptedly, but fall into five closed regions, forming subtropical oceanic anticyclones, which are preserved without any special changes the year round. Two such oceanic subtropic anticyclones exist in the Northern Hemisphere: the Azores (North Atlantic) and the Honolulu (North Pacific) anticyclones. In the Southern Hemisphere three such anticyclones exist: the South Atlantic (island of St. Helena), the South Pacific, and the anticyclone of the island of Mauritius (in the Southern Indian Ocean).

On the equatorial periphery of these subtropical anticyclones, winds of moderate intensity — the trade winds, blow with exceptional constancy all the year round. In the Northern Hemisphere they have a northwesterly direction, and in the Southern-- southeasterly. Near the thermal equator the trade winds become weak easterlies, and this is the calm [doldrum] belt. The width of this belt is variable. Often it disappears completely. In such case the trade winds of the two hemispheres meet, forming the so-called tropical [intertropical] fronts. Since the air carried by the trade winds is very humid and encounters a much warmer ground surface at surface, the air masses of both trade winds converging at the equator possess huge energy reserves in an unstable state, which may lead to precipitation. Therefore, the calm equatorial [doldrum] belt is characterized by frequent downpours, squalls and storms.

The tropical [intertropical] front is characterized by abundant precipitations. Despite their constancy, the trade winds change somewhat according to the specific nature of the locality. For example, the trade winds of the Atlantic Ocean are more constant than those of the Pacific, which die out altogether in January and February in the vicinity of the Carolina Islands. More substantial variations are observed upon the change of seasons. For example, the entire trade wind system is displaced, following the sun either to the north or to the south. During this shift, the central region of the equatorial doldrum belt stays within the Northern Hemisphere. It occupies the northernmost position in September, reaching 10° Northern latitude It is characteristic that the central region of the high pressure belt is not displaced. The seasonal changes of the trade winds are conditioned not by the shift of the high pressure belt but by the change of its intensity.

As we have already said, the formation of a pressure high over a continent is impossible in the summer, owing to the strong heating of the land.

As a result of strong diurnal variations over the land, a considerable amount of energy is accrued in a state of moist instability which is discharged in downpours of short duration, extending above land more than above the ocean.

The most pronounced low pressure zone extends over the Southern part of the Asian continent, in July. Here occurs the most intense generation of cyclones, at the Asiatic depression which gives rise to the phenomenon, unique in the general circulation — the famous Asiatic or Indian southwesterly monsoon. The Indian monsoon is characterized by its exceptional constancy during the season and its extension over a vast region. In this respect the Indian monsoon may be compared with the trade winds.

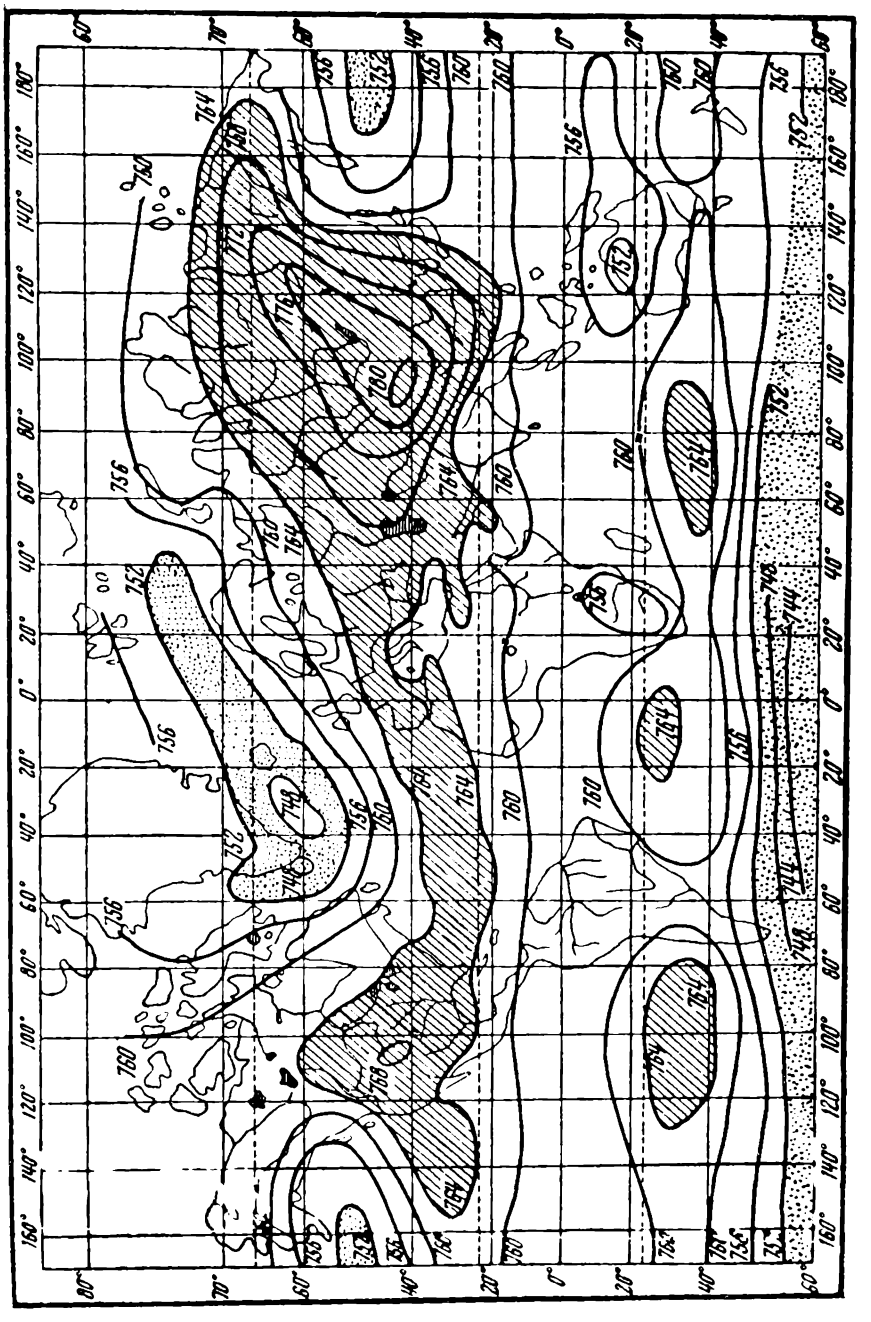


Figure 136. Average pressure distribution for January

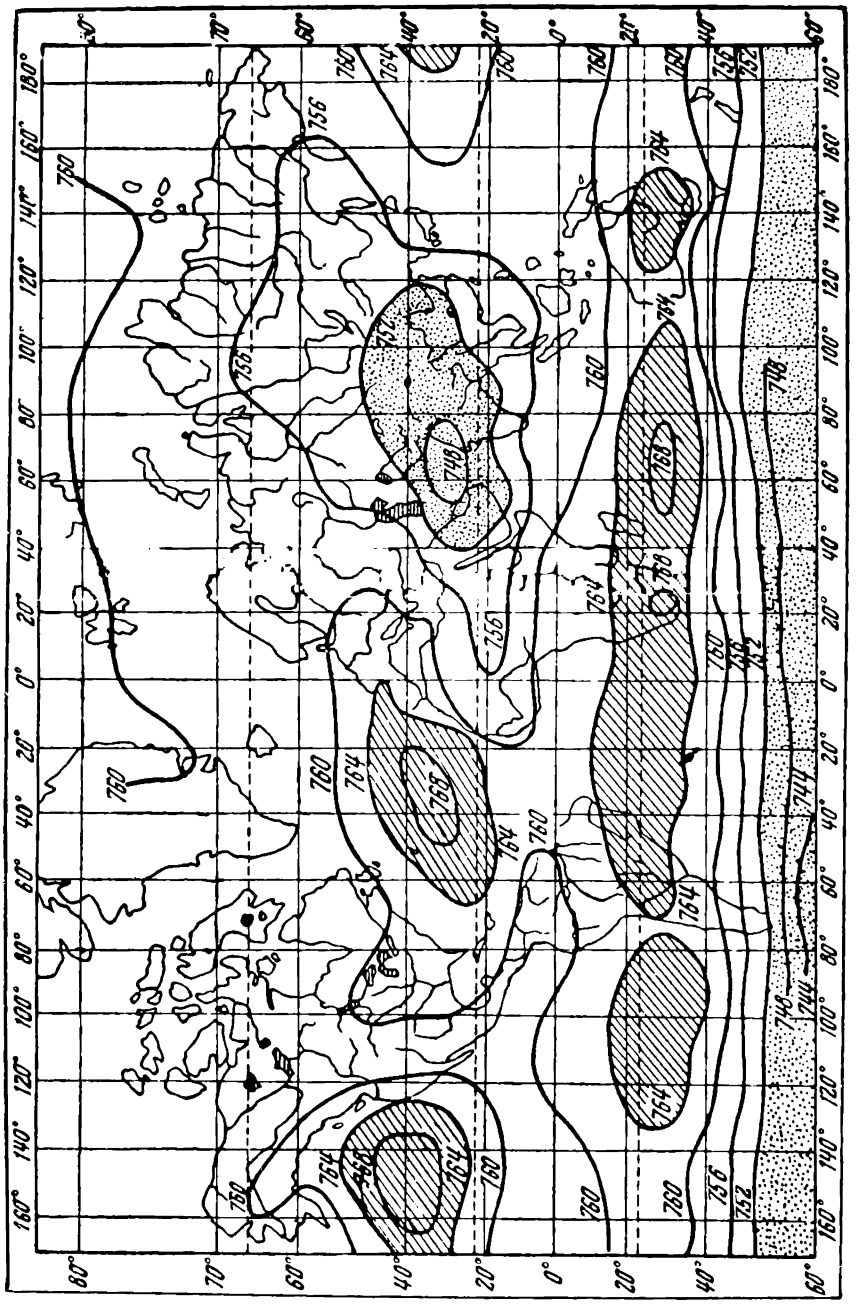


Figure 137. Average pressure distribution for July

The central belt of subtropical anticyclones corresponds to the zone of calm. These are the so-called "horse latitudes" of the subtropical seas.

Westerly winds blow on the polar periphery of the subtropical anticyclones. The zone of westerlies extends approximately to 65-70°.

We see that the extent of the belt of westerlies is not less than that of the trade winds. The westerlies, however, are not nearly as regular as the trade winds. Winds of different direction can often be observed in the belt of westerlies, due to the passage of cyclones and anticyclones.

In the Southern Hemisphere, the low pressure belt in the extratropical latitudes does not split into separate closed low pressure regions. This is explained by the uniform nature of the underlying surface.

It has been assumed till lately that near the poles high pressure continually predominates. Observations in the Arctic carried out by the Soviet investigators during the last years have shown, however, that shifting cyclones can penetrate into very high latitudes. Nonetheless, on average pressure maps regions of high pressure may be observed in the Arctic as well as in the Antarctic. On the peripheries of these high pressure regions blow easterlies which collide with westerlies of the middle latitudes. Arctic fronts are formed when this encounter takes place at high altitudes, and polar fronts when the easterlies penetrate into lower latitudes.

The meagre data presently at our disposal on the actual wind distribution in the free atmosphere cannot be completely integrated. However, it is possible to get a sufficiently good idea about the average wind field in the atmosphere from the altitude distribution of pressure, though even less data exists on the atmospheric pressure field (based on aerological ascents). However, the average pressure field for different levels in the free atmosphere can be computed from the known average pressure field at sea level and from the average lapse rates. The average altitude distribution of the temperature is known with sufficient reliability, for certain localities in the Northern Hemisphere, in winter and summer. Therefore, the estimation of the field of pressure at various levels in the free atmosphere of the Northern Hemisphere is quite free of major difficulties. For the Southern Hemisphere however, such estimation is yet impossible, owing to the absence of a sufficiently close-meshed network of stations for temperature sounding.

Owing to the presence of a horizontal temperature gradient, directed on the average from south to north in the troposphere of the Northern Hemisphere, the isobars on the higher levels approach the circles of latitude.

The pressure maximum at the North Pole disappears already on the level of 2 km (Figures 138-142). At all altitudes from 2 to 8 km the pole is a center of low pressure. The Azores and Honolulu maxima as well as the South Asiatic depression, though considerably weakened, still exist at the altitude of 2 km. A weak anticyclone appears over North Africa. At an altitude of 4 km the Honolulu maximum disappears completely. The Azores maximum and the South Asiatic depression are greatly reduced and the North African maximum is augmented. Besides, a weak maximum appears near the Himalayas.

At an altitude of 8 km the North African maximum is intensified, the Azores maximum is shifted westward and assumes a position over Mexico. The maximum which appears at an altitude of 1 km to the north of the Himalayas, now extends from the Persian Gulf to Eastern China.

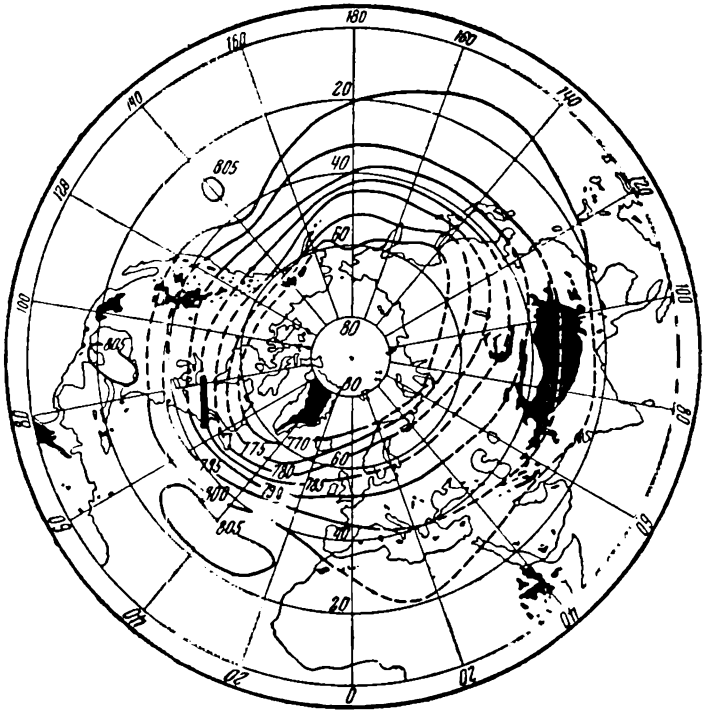


Figure 138. Average pressure distribution in the Northern Hemisphere at an altitude of 2 km, for January

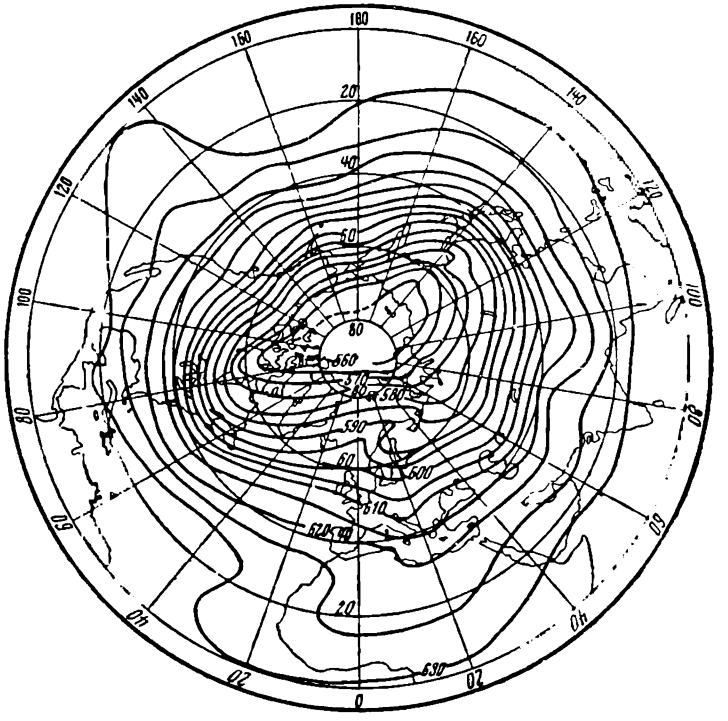


Figure 139. Average pressure distribution in the Northern Hemisphere at an altitude of 4 km, for January

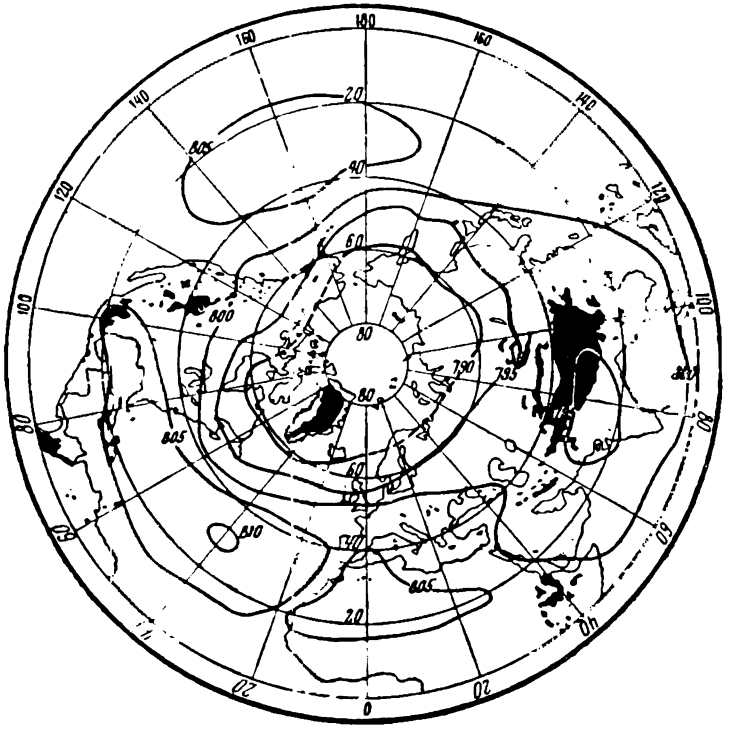


Figure 140. Average pressure distribution in the Northern Hemisphere at an altitude of 2 km, for July

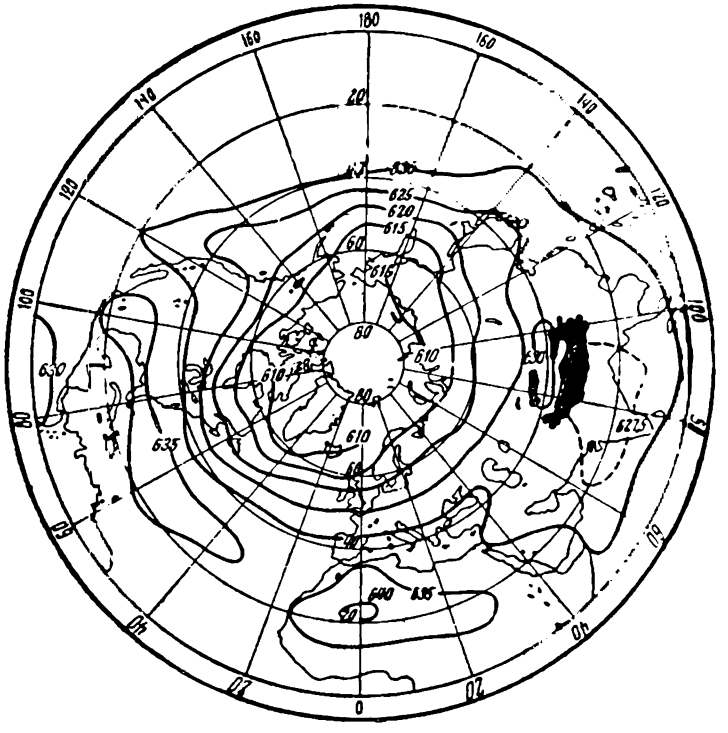


Figure 141. Average pressure distribution in the Northern Hemisphere at an altitude of 4 km, for July

The levelling of the pressure field with increasing altitude proceeds more rapidly in winter than in summer. The isobars lie almost on the circle of latitude already at the altitude of 4 km; nowhere are closely-spaced isobars to be found. Thus, the pressure field on this level represents a huge depression, spreading over the entire Northern Hemisphere and having its center near the North Pole.

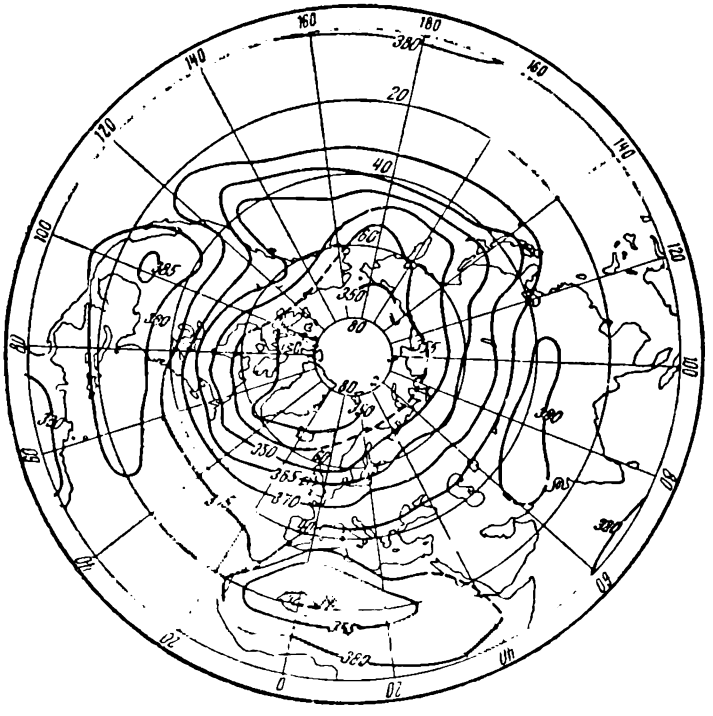


Figure 142. Average pressure distribution in the Northern hemisphere at an altitude of 8 km, for July

Since the wind in the free atmosphere approximates the geostrophic, the maps of the average pressure distribution at different levels characterize the field of wind at these levels.

It follows from the aforesaid that as a rule, westerlies must prevail at altitudes from 2 to 8 km in the Northern Hemisphere, up to 30° Northern latitude; however, at high altitudes the belt of westerlies extends almost to 10° Northern latitude. In winter the circulation of air in the subtropic anticyclones is completely arrested above the altitude of 4 km. On this level, the westerlies prevail at all altitudes. Such conditions predominate at all altitudes up to 12-16 km.

In the summer, at an altitude of 8 km, westerlies are prevalent over the tropic regions of Africa, India and North America.

From the given maps it is impossible to reach a final conclusion as to the winds prevailing over the Atlantic and Pacific Oceans, owing to the low values of the pressure gradients.

We may therefore draw the following conclusions about the distribution of wind in the upper layers of the troposphere. Near the equator, up to 10, 000 m, lies the zone of prevalent easterlies. As we move northward the wind at high altitudes becomes consecutively southeasterly, then southerly and southwesterly, and further to the north it becomes westerly. In the southern hemisphere an analogous distribution of wind over the latitudes should be expected; namely, the wind at the equator is westerly and as we progress to the south it becomes northeasterly, northern, northwesterly and westerly.

Thus, the wind in the upper layers of the atmosphere blows above the trade winds, in the opposite direction. Therefore, this wind is called an antitrade wind. Obviously, the antitrade wind does not represent a simple return of the air carried by the trade wind, and consequently, the trade wind and antitrade wind do not form together a closed circulation in the vertical plane. Observations showed that the vertical extent of the trade winds is small. It frequently amounts to several hundreds of meters, never exceeding 1.5 km. The trade wind inversion corresponds to the upper boundary of the trade winds. In between lies a transition layer where the wind is slow and vacillating in direction coinciding at times with the direction of the trade winds, but generally possessing a westerly compcnent.

Antitrade winds appear only at high altitudes, of the order of 2-7 km.

Observations in the North Atlantic showed that the transition from the trade wind to the antitrade wind is associated with a with a backing of the wind.

The upper boundary of the trade wind, as well as the lower boundary of the antitrade wind , ascend as we move from the pole to the equator. The altitude of these boundaries changes also with the longitude. In some places, above the anti-trade winds, upper trade winds blowing from the east can be observed.

Further to the north, predominantly western winds blow at all altitudes. These winds contain a component directed polewards in winter and towards the equator in summer.

Occasionally cyclones and anticyclones disturb this overall pattern, and in that case winds of any direction may be observed at all altitudes.

At high latitudes and at all altitudes, the westerlies generally predominate.

It is necessary to have a rough idea about the part played by the vertical motions of the air in the genral circulation. No methods of direct measurement of vertical velocities of the air are available. Therefore, no data on this subject exist and we are obliged to limit ourselves to indirect indications of only the outstanding characteristics of vertical velocities. Such, for example, is the analysis of the distribution of precipitations. In the subtropic zone of the pressure maxima, the precipitations are negligible. It is therefore natural to assume that in this zone the wind generally descends. The average vertical velocity of such subsiding currents seldom exceeds 0.5 cm/sec. In the northern part of the trade wind belt ($\sim$above 10°) descending motion is observed, and in the southern (below 10°), an ascending motion.

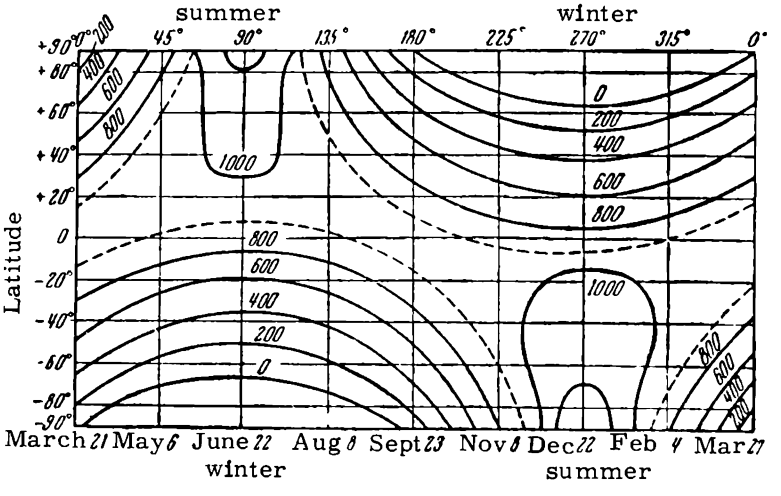
In conclusion to this short survey of the general circulation of the atmosphere we would like to point out that the presence of the tropopause must be of essential importance for the development of the atmospheric circulation. Schematically, the tropopause is a discontinuity surface of the first order in the temperature field, separating the tropospheric air from the stratospheric. The position of this interface undoubtedly affects the general circulation and the secondary circulations; processes of formation and dissolution of the tropopause are also connected with the general circulation. Thus, upon the development of cyclones the tropopause descends, and with it the stratospheric air. In this way the air is adiabatically heated and radiative equilibrium is disturbed. The descended stratospheric air starts to cool and as a result the tropopause forms at a high altitude and the descended stratospheric air is assimilated by the tropospheric air. Upon the development of anticyclones the tropopause is formed at a lower altitude and the tropospheric air ascends into the stratosphere.

§3. Geographical Distribution of the Solar and Outgoing Radiations

Chapter VII described an overall thermal balance for the entire globe. The real values of the insolational and outgoing radiations for different latitudes and at various times of the year significantly differ from the values given in Chapter VII.

Calculation of the value of the solar radiation incident on the boundary of the atmosphere is an astronomical rather than a meteorological problem. This problem was solved by Milankovich. Without giving the actual data we shall discuss now the diagram drawn from them (Figure 143), which characterizes the quantity of the energy incident on 1 cm^2 in the absence of any atmosphere, per 24 hours, for various latitudes at various times of the year. On the ordinates is plotted the geographical latitude and on the abscissa the time of the year, expressed by the right ascension of the sun (upper scale) and the corresponding months (lower scale). During the summer, the North Pole gets the maximum quantity of energy per 24 hours since there the sun shines continuously at this time of the year. A secondary maximum is observed at 40° latitude and manifests itself by weak inflection of the intensity isoline of 1000 cal/cm^2 per 24 hours. In the polar region of the Southern Hemisphere no solar radiation gets to the atmosphere in the summer, since the sun then lies under the horizon. In winter the opposite situation is observed. The insolation of the Southern Hemisphere during the southern summer is larger than that of the Northern Hemisphere in the northern summer. This is so because the earth is nearer to the sun during the northern winter (i.e., the southern summer) than during the northern summer.

The Northern Hemisphere



The Southern Hemisphere

Figure 143. Geographical distribution of insolation per 1 cm^2 of horizontal surface per 24 hrs in the absence of atmosphere

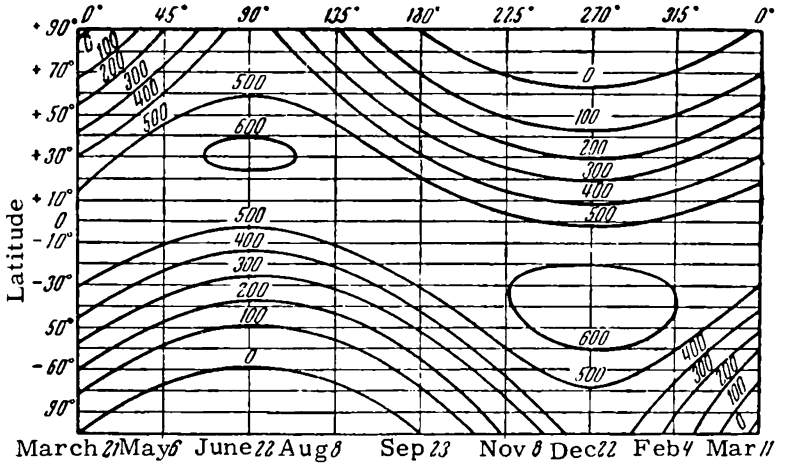


Figure 144. Geographical distribution of insolation incident on 1 cm^2 of horizontal surface per 24 hours, for an absorption [sic] coefficient of 0.7

Figure 144 shows the quantity of direct solar radiant energy incident on 1 cm^2 of the earth's surface per 24 hours, where the coefficient of transmittance is taken as 0.7. In this case the insolation of the earth's surface is considerably less than in the absence of atmosphere. The attenuation of the direct solar radiation is particularly high in the polar regions where the sun is low on the horizon. In low latitudes where the sun is nearer to the zenith, the solar radiation is correspondingly less attenuated. Therefore, the summer maximum is in this case not at the pole but where the secondary maximum would have been had the atmosphere been absent, i.e., at 40° latitude.

It is considerably more difficult to calculate the geographical distribution of the outgoing radiation. Simpson worked out a method for the calculation of the radiation leaving the earth and the atmosphere. This method is based on his simplified scheme of the absorption spectra of long-wave radiation in water vapor and carbon dioxide (see § 3, Chapter VII).

The results obtained by Simpson are given in Table 51.

The second column gives the insolational radiation from which the albedo is excluded. The third column contains data obtained by Simpson for the outgoing radiation. The average values of the incident and outgoing radiations coincide. According to Simpson, the outoing radiation changes but little with latitude, which he explains by the constant temperature he attributes to clouds at all latitudes. At a heavy overcast the outgoing radiation will increase towards the pole, in accordance with the higher stratospheric temperatures in the polar regions. Already at 35° latitude the outgoing radiation starts to exceed the insolational radiation. To compensate for the radiative energy losses in the higher latitudes, heat must be continuously transported from the lower to the higher latitudes by atmospheric currents. This circumstance is of essential importance for the maintenance of the general circulation of the atmosphere. Simpson also showed that the outgoing radiation is distributed quite uniformly not only over the surface of the earth but also in time, showing no significant annual variations.

Table 51

The insolational and outgoing radiations

Latitude	Insolational radiation (cal/cm^2 min)	Outgoing radiation (cal/cm^2 min)	
		According to Simpson	According to Albrecht
0	0.339	0.271	0.305
10	0.334	0.282	0.304
20	0.320	0.284	0.299
30	0.297	0.284	0.288
40	0.267	0.282	0.276
50	0.232	0.277	0.263
60	0.193	0.273	0.250
70	0.160	0.260	0.230
80	0.144	0.252	0.225
90	0.140	0.252	0.255
Mean	0.271	0.271	

However, Simpson's results stand in need of certain corrections. This is due to his overestimation of the magnitude of radiant energy absorption by atmospheric water, as shown by Gettner (see §3 Chapter VII). For the lower layers of the atmosphere this discrepancy is of no importance, but it leads to significantly different results if applied to the stratosphere. Besides, it appears that the stratosphere does not contain water vapor even in the amount equivalent to 0.3 mm of precipitation. Therefore, radiation in the region of total absorption by water vapor should occur not only in the stratosphere but also, partially, in the upper troposphere. Since water vapor concentration decreases exponentially with increasing altitude, the radiation will be maximal in the lowest part of the layer in question.

Albrecht examined this situation, and stressed the fact that the "emission layers", i.e., layers of maximum radiation, lie entirely in the upper troposphere and not in the stratosphere. The temperatures of these emission layers vary with latitude less than the corresponding stratospheric temperatures. Therefore, according to Albrecht, the outgoing radiation varies with latitude more than assumed by Simpson. According to the latter the cold surface at the pole radiates together with stratosphere, and on the equator the warm surface radiates together with cold stratosphere. As a result, some compensation takes place, which tends to equalize the latitude variations of the total radiation. According to Albrecht, such compensation does not exist, and the outgoing radiation decreases towards the pole together with the terrestrial surface temperature. Therefore, according to Albrecht the outgoing radiation varies with latitude more than Simpson assumed.

§4. Circulation Theorems

It was shown in the preceding section that the insolational radiation at low latitudes considerably exceeds the terrestrial and atmospheric radiation lost in outer space, and at high latitudes the outgoing radiation exceeds the insolation. Therefore, a continuous transport of heat by air currents must take place from low to high latitudes.

In this connection it is natural to raise the following question: is the heat transferred from low to high latitudes by closed currents which form a stationary circulation? Atmospheric motions are often schematized as closed circulations, though, strictly speaking, the existence of such circulations has never been demonstrated.

In this section we shall derive several general theorems concerning the circulation in the velocity field.

In the mechanics of a rigid body we consider the system of material points as a whole. Similarly, we may study in hydrodynamics the motion of an assembly of particles lying on a closed curve instead of considering each particle separately. The notion of circulation plays an important role in the study of fluid motion.

Let us consider a closed contour s, at a given instantaneous position in a moving fluid. We will calculate the tangential and the normal components of the velocity v_s and v_n, for every particle lying at the given moment on the curve s. Only the tangential component effects the transport of the fluid particles along the contour s. The normal component does not influence this transport at all. The particle is transported in the direction of the tangential component of its velocity, which will be positive in case it coincides with the given positive orientation of the contour. The orientation is defined so that the interior of the contour remains to the left when it is traveled in the positive sense. We wish to find the displacement of any fluid particle lying on the contour s from its original position. This displacement is determined not only by the direction of the tangential component of the velocity also by its magnitude. Thus, the value of this [infinitesimal] displacement of the individual particle is given by the scalar product (Figure 145)

$$(v, dr) = v\cos(v, dr)ds = udx + vdy + wdz \qquad (1)^*$$

where r is the radius vector of the particle, and $ds = |dr|$ is the line element of the curve at the position of the particle. Forming the sum of all the products (1) along the entire contour and passing to the limit as $dr \to 0$, we obtain a line integral around the contour

$$\Gamma = \oint_s (v, dr) = \oint v_s ds = \oint (udx + vdy + wdz) \qquad (2)$$
$$[\Gamma] = [L^2T^{-1}],$$

which expresses the total displacement, relative to the contour, of all particles lying on it at the given moment. The integral Γ is called the circulation (of the velocity v) around the closed contour s. The circulation is positive if the total displacement along the curve is effected in the positive sense defined on it, and vice versa. When the particles are not displaced along the curve the circulation is equal to zero.

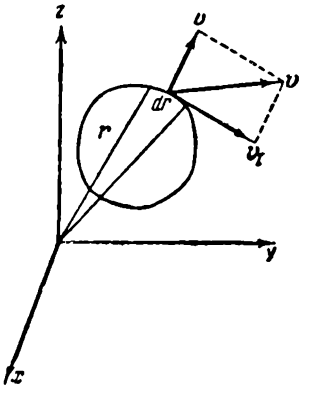


Figure 145. Circulation around a contour

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*[The Russian text uses the notation u,v,w for the components of the velocity v. The confusion which may arise between the two different quantities bearing the common designation v must be resolved according to the context. - Translator's note.]

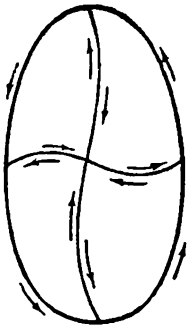

Figure 146. The circulation around a closed contour equals the sum of the elementary circulations

We will prove that the circulation around any contour s which encloses some given surface σ, is equal to the algebraic sum of the circulations along the contours of all domains that are obtained by a subdivision of this surface. Let us consider a contour enclosing a given surface σ (Figure 146). Dividing this surface into domains: $\Delta\sigma_1, \Delta\sigma_2, \ldots, \Delta\sigma_n$, we form the sum of the circulations along the boundaries of these domains: $\Delta\Gamma_1 \ldots \Delta\Gamma_n$. Obviously on summation the integrals $\int v_s\, ds$, taken along the paths lying within the contour cancel out in pairs, since each of them appears twice, with opposite signs, according to the sense in which these inner boundaries are passed, as indicated by the arrows.

The equation is connected with the vorticity by the equality

$$\oint_s (\boldsymbol{v}, d\boldsymbol{r}) = \int_\sigma (\operatorname{curl} \boldsymbol{v}, \boldsymbol{n})\, d\sigma, \tag{3}$$

known as Stokes' theorem and signifying that the circulation around a contour is equal to the vorticity flux across any surface bounded by the contour. It follows from (3) that the circulation characterizes vortex flows, since in the absence of vorticity the circulation vanishes.

We have discussed the circulation around a geometrical static contour lying in the moving fluid. Let us now consider a fluid contour composed continually of the same particles, and which may be displaced and deformed without breaking their association. A necessary and sufficient condition for the existence of such a fluid contour is the continuity of the functions

$$x = x(a, b, c, t); \quad y = y(a, b, c, t); \quad z = z(a, b, c, t),$$

expressing the laws of motion for the fluid in Lagrangean variables. The circulation around the fluid contour s is naturally determined by the same formula as the circulation around the geometrical contour, the only difference being that s is now regarded as composed at every moment of the same liquid particles. Evidently the circulation around the fluid contour generally changes with time. Let us consider the temporal variation of the circulation. For this purpose we differentiate the two sides of (2) with respect to the time:

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \oint (\boldsymbol{v}, d\boldsymbol{r}) = \oint \left(\frac{d\boldsymbol{v}}{dt}, d\boldsymbol{r}\right) + \oint \left(\boldsymbol{v}, \frac{d}{dt}(d\boldsymbol{r})\right).$$

Since the circulation in question is taken around a fluid contour consisting all the time of the same particles, individual derivatives are required. The first integral on the right side represents the circulation of the acceleration around the fluid contour. The second integral is equal to zero. Indeed,

$$\frac{d}{dt}(dr) = d\left(\frac{dr}{dt}\right) = dv;$$

therefore

$$\oint\left(v, \frac{d}{dt}(dr)\right) = \oint (v, dr)^{*} = \oint d\left(\frac{v^2}{2}\right) = 0.$$

Thus we have proved Thomson's theorem, stating that the time variation of the circulation around a closed contour is equal to the circulation of the acceleration around the same contour:

$$\frac{d\Gamma}{dt} = \oint\left(\frac{dv}{dt}, dr\right) = \oint\left(\frac{dv}{dt}dx + \frac{dv}{dt}dy + \frac{dv}{dt}dz\right),^{**} \qquad (4)$$

$$\left[\frac{d\Gamma}{dt}\right] = \left[L^2T^{-2}\right].$$

Replacing the components of the acceleration in equation (4) by their equivalent expressions from the equations of motion of the ideal atmosphere, we find:

$$\left.\begin{aligned} \frac{du}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\omega \sin\varphi v, \\ \frac{dv}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\omega \sin\varphi u, \\ \frac{dw}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\omega\cos\varphi u. \end{aligned}\right\} \qquad (5)$$

We thus obtain

$$\frac{d\Gamma}{dt} = -\oint\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) - \oint g dz - \\ - 2\omega\oint \sin\varphi\,(udy - vdx) + \cos\varphi u\, dz.$$

The first integral in equation (5) can evidently be written in the form: $\oint\frac{dp}{\rho}$, where the pressure variation dp is taken along the contour s.

The second integral vanishes, since the line integral of a total differential (the constant g) around a closed contour equals zero.

Let us assume that the contour around which the integrals are taken is not too large, so that the variation of latitude φ in its interior can be neglected. Then sin φ may be put in front of the integral sign. We will explain the geometrical significance of the integral $\oint(udy - vdx + \operatorname{ctg}\varphi\ udz)$. Let us project the contour s onto a horizontal plane. The area of the domain bounded by the horizontal projection of s will be denoted by F. Evidently the temporal variation of this area is connected with the motion of the elements of the contour s. Thus, when a line element ds moves in the x-direction, the area F varies by udy per time unit, and when the element moves in the y-direction, the area varies by (−vdx). Consequently,

$$\oint(udy - vdx + \operatorname{ctg}\varphi\, udz) = \frac{dF}{dt},$$

and the change of circulation per unit time is expressed by the following formula:

$$\frac{d\Gamma}{dt} = -\oint\frac{dp}{\rho} - 2\omega\sin\varphi\frac{dF}{dt}. \qquad (6)$$

Formula (6) is the main circulation theorem, due to V. Bjerknes.

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* [Translator's note: Misprint in Russian text. Should be dv.]

** [Translator's note: Misprint in Russian text. Should be $\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz$].

Formula (6) shows that when the fluid is barotropic, the change in circulation depends only on the rotation of the earth. In a resting reference frame the circulation of a barotropic fluid remains constant with time. The circulation theorem was proved in this particular form by Helmholtz for an incompressible fluid.

In a baroclinic atmosphere the variation of the circulation and the variation of the vorticity associated with it take place at the expense of the atmospheric baroclinicity.

The circulation theorem is expressed by a prognostic equation, permitting the calculation of the variation of the circulation up to some future moment from the present given distribution of density and pressure. As soon as the fluid contour is in motion, the circulation starts to vary under the additional influence of the rotation of the earth, as indicated by the second term on the right side of equation (6).

The change in circulation on a resting earth under the influence of atmospheric baroclinicity may be represented in various forms obtained by elementary transformations.

Thus, since $\frac{1}{\rho} = \alpha$, we have

$$\frac{d\Gamma}{dt} = -\oint \alpha dp. \tag{7}$$

Recalling that $\alpha p = RT$, we obtain

$$\frac{d\Gamma}{dt} = \oint p d\alpha, \tag{8}$$

because $\oint dT = 0$.

Eliminating the specific volume by the equation of state we obtain:

$$\frac{d\Gamma}{dt} = -R \oint T d \ln p. \tag{9}$$

Introducing the potential temperature, according to (8) §16 Chapter II, we obtain:

$$\frac{d\Gamma}{dt} = -c_p \oint T d \ln \theta. \tag{10}$$

And finally, by introducing the entropy according to the equation $d\varphi = c_p d \ln \theta$, we obtain:

$$\frac{d\Gamma}{dt} = -\oint T d\varphi. \tag{11}$$

The calculation of $\frac{d\Gamma}{dt}$ is simplified when the fluid contour of integration is specially chosen. Let us draw a system of isobaric surfaces at intervals of one pressure unit, and a system of isosteric surfaces for α, at intervals of one specific volume unit. The entire surface is thus decomposed into isobaric-isosteric solenoids. Let us consider a fluid quadrilateral contour, formed by two isobaric and two isosteric surfaces (Figure 147). The number of solenoids encircled by this contour is equal to mn. We will prove that in this case

$$\frac{d\Gamma}{dt} = m \cdot n. \tag{12}$$

Indeed

$$\frac{d\Gamma}{dt} = \oint_{12\,341} p\,d\alpha = \int_3^4 p\,d\alpha + \int_1^2 p\,d\alpha =$$
$$= (p - m)[\alpha - (\alpha + n)] + p(\alpha + n - \alpha) = m \cdot n.$$

The relationship (12) holds for any fluid contour. Thus, the variation of circulation for any given fluid contour equals the number of separate isobaric-isosteric solenoids encircled by this contour.

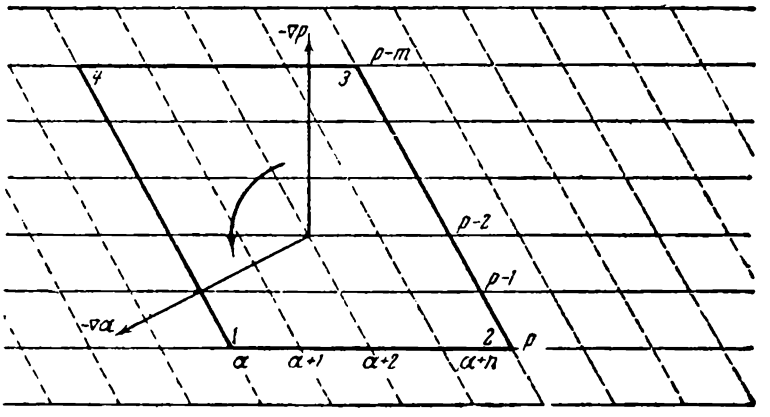

Figure 147. The change in the circulation around a fluid contour is expressed by the number of isobaric-isosteric solenoids encircled by the contour

The circulation increment is directed from the pressure gradient to the gradient of the specific volume. When the fluid contour is sufficiently large it might contain positive as well as negative solenoids, around which the direction of circulation is opposite. In this case the change in the circulation around the fluid contour is equal to the difference between the number of positive and the negative isobaric-isosteric solenoids.

By such reasoning it can be shown that the variation of circulation around a contou is equal to the difference between the number of positive and the number of negative isothermal-isentropic solenoids. The circulation increment is directed from the entropy gradient to the temperature gradient.

It is convenient to use another contour for the calculation of the change in circulation. This contour consists of two vertical lines and two isobars (Figure 148). In this case

$$\frac{d\Gamma}{dt} = -\oint_{12\,341} \frac{dp}{\rho} = -\int_2^3 \frac{dp}{\rho} - \int_4^1 \frac{dp}{\rho}. \qquad (13)$$

Obviously the hydrostatic equation $dp = -g\rho\,dz$ is fulfilled along the vertical; consequently,

$$\frac{d\Gamma}{dt} = g(h_2 - h_1) \qquad (14)$$

or

$$\frac{d\Gamma}{dt} = \Phi_2 - \Phi_1, \qquad (15)$$

where Φ is the geopotential of the isobaric surface p relative to the surface p_0.

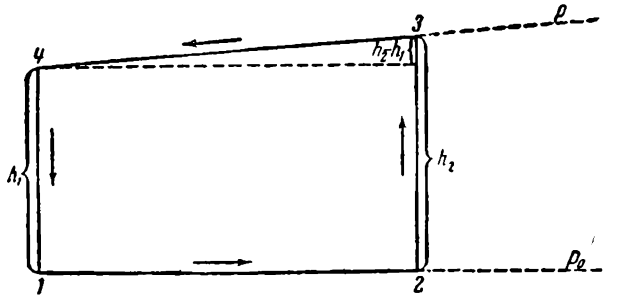

Figure 148. Variation of the circulation around the fluid contour, as expressed by the difference of the corresponding geopotentials of the isobaric surfaces

Let us calculate the change in the circulation for the same contour consisting of two isobars and two vertical lines using formula (9):

$$\frac{d\Gamma}{dt} = -R \oint_{12\,341} T d\ln p = -R \int_2^3 T d\ln p - R \int_4^1 T d\ln p.$$

Denoting the mean temperatures on the vertical lines by T_2 and T_1 respectively, we obtain:

$$\frac{d\Gamma}{dt} = R \ln \frac{p_0}{p} (T_2 - T_1). \qquad (16)$$

The circulation theorems can be used to explain such local wind circulations as breezes, mountain-valley winds [slope winds], etc. In these cases compensatory currents aloft are less pronounced than the lower flows. Nonetheless such circulations may be regarded as closed.

A breeze is formed where the air lying over the land is vigorously heated during the day and cooled at night, while the water surface maintains an almost constant temperature, due to mixing with deep layers. Therefore, the slope of the isobaric surface undergoes a 24 hour variation, and as a result the sense of this closed circulation is reversed (Figure 149).

In the same way, the temperature over a mountain slope rises by day and drops by night more than in the adjacent valley. Therefore the wind blows up the slope by day and down the slope by night.

§ 5. Application of the Circulation Theorems to the General Circulation of the Atmosphere

We had already mentioned that the present state of our knowledge does not permit the formulation of rigorous equations for the general circulation of the atmosphere. Therefore, we must be content with certain rough schemes, far from describing the real atmospheric circulation, in the hope that these schemes will be of some help in explaining, at least in part, the most important features of the general circulation of the atmosphere.

Let us assume that the earth is immobile and of uniform surface, and that the air is heated at the equator and cooled at the poles; then the general circulation of the atmosphere would consist of one meridional "ring" of circulation in each hemisphere; the warm air ascends at the equator and flows, in the upper layers, to the pole. At the pole it descends. The place it has initially occupied is filled with cold air flowing from the pole to the equator.

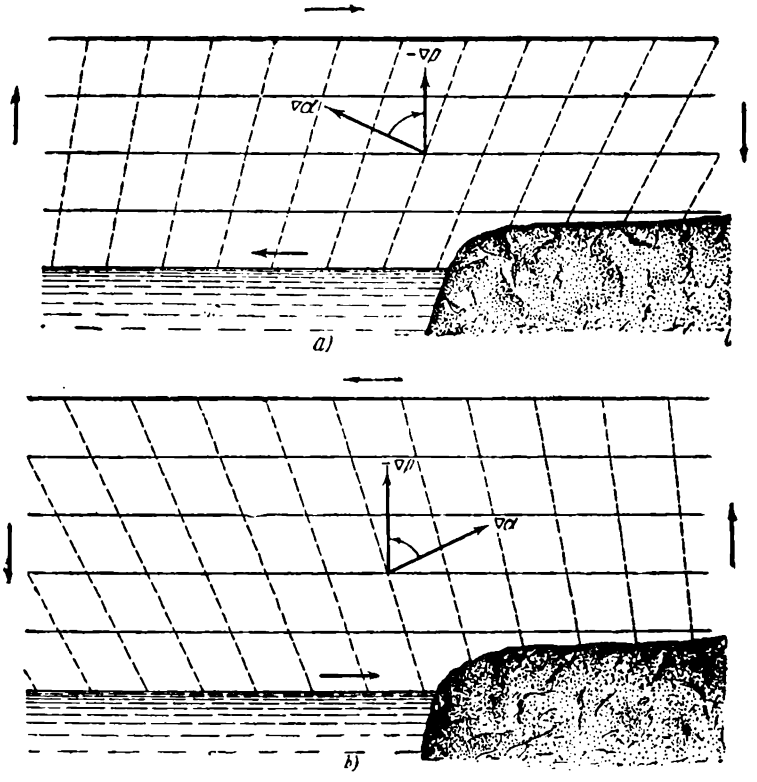

Figure 149. a) breeze by night, b) breeze by day

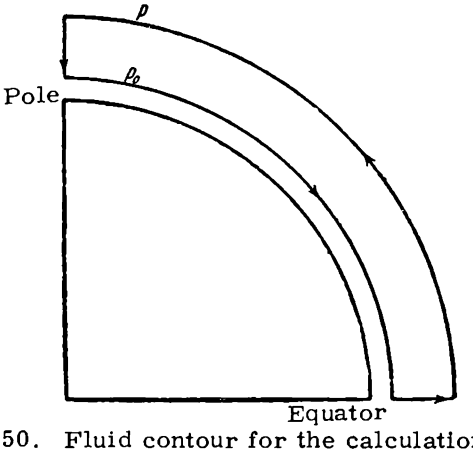

Figure 150. Fluid contour for the calculation of circulation changes in a frictionless atmosphere, for an immobile earth

Had the atmosphere been a perfect fluid, such circulation would have strengthened progressively. Using the contour (Figure 150) consisting of two isobars p_0 and p, and two verticals, one at the equator and the other at the pole, we can calculate the change of the circulation around it by the formula

$$\frac{d\Gamma}{dt} = R\left(T_e - T_p\right) \ln \frac{p_0}{p},$$

where T_e and T_p are the respective average temperatures at the equator and at the pole. Taking $T_e - T_p = 40^\circ$; $p_0 = 1{,}000$ mb and $p = 300$ mb (which corresponds to the level of cirrus clouds ~ 9,000 m), we obtain

$$\frac{d\Gamma}{dt} = 138 \cdot 10^6 \text{ cm}^2/\text{sec}^2$$

Recalling that the variation of the circulation around the contour is equal to the circulation of the acceleration, and putting the circumference of the contour equal to half the earth's diameter, i.e., $2 \cdot 10^9$ cm, we obtain for the average meridional acceleration.

$$\frac{dv}{dt} = 6.9 \cdot 10^{-2} \text{ cm/sec}^2$$

Regarding the average acceleration as constant, it is easy to calculate that already after six hours the circulation will attain a considerable velocity and after 24 hours—the velocity of a hurricane (see Table 52).

Table 52

Theoretical velocity of the general circulation in the absence of friction

Time	1 sec	3 hours	6 hours	24 hours
Average meridional velocity in m/sec	$6.9\ 10^{-4}$	7.4	14.9	59.6
Average meridional displacement (in kilometers)	$0.345\ 10^{-6}$	40.4	160.5	2.580
Average zonal velocity at 30° latitude in m/sec	$2.52\ 10^{-8}$	2.94	11.7	188
Average acceleration at 30° latitude in cm/sec^2	$1.84\ 10^{-10}$	$2.14\ 10^{-2}$	$8.51\ 10^{-2}$	1.37

Calculating the distance covered by an air parcel in this process, we find that after no more than 67 hours it would return to its starting point, having described a close path coinciding with the contour of integration.

However, these calculations stand in need of an essential correction. The temperatures at the equator and at the pole would rapidly fall as a result of such velocities, and the change in circulation will be considerably smaller, though in the absence of other factors the circulation would still continue to increase. Since there are no perfect fluids in nature, we must make allowance for the viscosity in those calculations. Viscosity effects hinder the radiant energy from being converted into kinetic energy in its entirety. Part of the kinetic energy will be dissipated by the action of viscosity and be accordingly transformed into potential and internal energy. In this manner a stationary closed circulation would eventually be established on an immobile earth.

However, such a circulation on the rotating earth is impossible. Owing to the action of the Coriolis forces the air currents do not flow in the direction of the pressure gradient, and in uniform flow are perpendicular to it. All currents will tend to assume the latitudinal direction, deviating in the Northern Hemisphere to the right of the meridian, and in the Southern Hemisphere to the left. Thus, under the influence of the rotation of the earth the air currents over a uniform underlying surface tend to become purely zonal.

§ 6. Heat Transfer from the Equator to the Pole

Helmholtz and other investigators, proceeding from different considerations, have shown that purely zonal atmospheric circulation is impossible, even if the earth's surface were completely uniform, i.e., in the absence of continents and oceans.

Therefore, heat must be transported from the equator to the pole by the general circulation. Otherwise, the temperature difference between the pole and the equator would have increased indefinitely. In reality it changes little even with the seasons.

Heat transfer in a meridional direction can be accomplished in two ways: by advection of masses of different temperatures or though the large-scale turbulent mass exchange.

When two opposite air currents possess meridional components of velocity and different mean temperatures, heat will flow advectively along the meridian, the magnitude of its flux being easily computed.

Heat may be transferred from the equator to the pole also by means of the so-called macroturbulent mass exchange.

In the latter case the general circulation of the atmosphere may be regarded as a large-scale eddy exchange. From this point of view, cyclones and anticyclones moving in various directions may be regarded as the elements of turbulency of the main latitudinal current.

This analogy between the general circulation and the eddy exchange of masses is of course very rough. The main condition for the existence of a state of turbulence is the large number of turbulence elements. But in our case only few distinct vortexes occur. Nevertheless this analogy is useful.

Defant has studied heat transfer for the extratropical latitudes where the meridional velocities are small, and consequently the heat flux is accomplished mainly by turbulent exchange.

The formula for the meridional heat flux Q, due to large-scale turbulency may be constructed by analogy with formula (3) §4 Chapter XV for the vertical eddy flux of heat, namely:

$$Q = -c_p \cdot A \cdot \frac{d\theta}{Rd\varphi}, \qquad (1)$$

where c_p is the specific heat at constant pressure, A—the coefficient of macroturbulent exchange, θ—the temperature, R—the terrestrial radius, φ—the latitude.

The magnitude of A, according to Defant, is of the order $4 \cdot 10^{17}$ g/cm sec. This value is 10^5-10^6 times larger than the coefficient of small-scale turbulent exchange and 10^2 times greater than the coefficient of lateral mixing or, in other words, the coefficient of horizontal exchange. The magnitude A can be determined by the formula:

$$A = \Sigma \frac{ml}{Ft}, \qquad (2)$$

derived for vertical turbulent exchange. In this formula, m is the mass of an individual vortex (cyclone or anticyclone), F—the area of the circle of latitude; l—the distance covered by the vortex up to its incidence on this circle, t—the period of averaging.

A may be evaluated indirectly, without having recourse to formula (2), by using formula (1), when the heat flux and the horizontal temperature variation along the meridian are known.

Ångstrom studied the heat transfer at all latitudes, including the tropical. His results agreed well with the direct determinations of the temperature at different latitudes. However, in the tropical latitudes, trade winds and antitrade winds prevail. There are no anticyclones in these latitudes, and tropical cyclones are extremely rare. Therefore, heat flux at the tropical latitudes is associated with advection, while at the extratropical latitudes this flux is associated with turbulency.

V.V. Shuleikin has shown that Ångstrom's data on the distribution of temperature agree with observations along the entire meridian, though he did not differentiate between tropical and extratropical latitudes. This procedure may be justified by the fact that both turbulent exchange and advection cause a meridional heat flux, which is proportional to the component of the temperature gradient in

this direction. Consequently both processes may be described by the same equations in which the meridional component of the temperature gradient is multiplied by some relative coefficient of heat conductivity. It turned out that this coefficient of heat conductivity is of the same order for thermal advection and for turbulent exchange, although the origin and physical significance of heat advection and turbulent exchange are of course quite different.

§7. Heat Transfer from Ocean to Continent

The atmosphere can be regarded as a thermodynamic engine, operating between a heat source and a heat sink. The energy source of the general circulation is the sun. In tropical latitudes the flux of the insolational radiation considerably exceeds the outgoing radiation. Therefore, the tropics may be considered as the heat reservoir of the atmospheric heat engine. On the other hand, in both polar regions the terrestrial surface radiational output exceed the received insolation. Therefore, the polar regions represent the heat sinks of this heat engine.

The tropical heater and the two polar coolers work the year around without changing their roles. Additional heat sources and sinks occur in the atmosphere, active over a period of several months in the year and changing their sign with the change of season. Thus, for example, the continents serve as heaters in summer and as coolers in wintertime. The vast regions of warm oceanic currents are powerful heaters during the winter. In the summer, the surface of the ocean in most cases functions as a cooler.

Academician Shuleikin paid special attention to the important role played by the seasonal heat sources and sinks in the general circulation of the atmosphere.

Permanent coolers and heaters determine the zonal and interzonal circulation, while the seasonal heaters and coolers cause the monsoon circulation.

Shuleikin was the first (in 1935) to calculate the heat influx from the ocean to the continents. No serious attempt at a quantitative evaluation of this flux has ever been made prior to his work. Defant, Ångström and other workers have averaged the air temperature along the entire circle of latitude, whereby the effect of heat flux from the ocean to the continents was, of course, entirely eliminated. However, the effect of this flux is immense.

Shuleikin points out that in climate formation of many regions this heat flux might have a role considerably greater than the meridional heat flux. Thus, for example, the heat flux from the Atlantic Ocean leads to the development of extreme temperature gradients on the Eurasian continent. We recall that the average temperature for January in Murmansk is $-10°C$ and in Verkhoyansk (lying even a little to the south of Murmansk) this mean temperature equals $-49°$. Thus the difference between the average temperatures of two points lying at the same latitude is 39°C. It is characteristic that the average temperature in Yalta (for January) is $+4°C$, i. e, it exceeds the average temperature for January in Murmansk by 14°, though Murmansk lies 25° to the north of Yalta.

We shall describe the main features of Shuleikin's method of calculation of the heat influx from the ocean to the continents, without going into details.

Shuleikin starts from the analysis of isothermal charts. The average horizontal temperature gradient determines the total heat flux which consists of the meridional flux and that due to the influence of the ocean. To find the heat flux caused by the influence of the ocean it is sufficient to subtract the meridional flux vector from the total flux. It is thus necessary to subtract the field which would have existed

over a uniform earth surface, continents and oceans being absent, from the average temperature field.

Having formed in this way the difference between the temperature fields, Shuleikin obtained isanomalous charts, i.e., charts on which are plotted the lines (isanomals) of equal difference (anomaly) between the actual temperature and that which would have existed over a uniform underlying surface. These charts provide a visual image of the effect of heat exchange between ocean and continent on the temperature distribution.

Thus, for example, in January, the isanomal of $+11^{\circ}$ passes through Murmansk, the isanomal of -25° through Verkhoyansk and the isanomal of 0° through Yalta. In July the isanomal of Verkhoyansk is $+7^{\circ}$.

Shuleikin proceeds to determine the coefficient of relative heat conductivity regarding it as a product of the form AH, where H is the altitude of the active layer, i.e., the atmospheric layer participating in the heat exchange between the ocean and the continent, and A—a constant numerical coefficient of the dimension of heat conductivity. Shuleikin finds that

$$AH = 1.28 \cdot 10^{20} \frac{\text{calorie}}{\text{year degree}}.$$

The coefficient of relative heat conductivity shows how many calories pass during a year through a strip 1 cm wide lying at the level H, when the temperature gradient is 1° cm.

On the basis of the isanomalous charts and the value of the coefficient of relative conductivity, Shuleikin found that through each centimeter of the coastline $4.1 \cdot 10^{12}$ cal pass on the average per year into the USSR. Since this heat is lost by the sea, it seems possible to estimate the quantity of heat entering the atmosphere from 1 cm^2 of sea surface. This quantity of heat amounts to 41,000 cal a year per 1 cm^2.

In another paper Shuleikin studied the heat budget of the Kara Sea, and found that this sea gives up, through heat exchange with the atmosphere and by radiation, a total of 48,000 cal/ cm^2 per year. Subtracting the radiation losses from this value, 41,000 cal remain. The different evaluations of this quantity obtained by Shuleikin using different methods are all of the same order of magnitude.

Shuleikin's investigations of heat flux from ocean to continent offer explanations for many important meteorological phenomena. Thus, for example, they disclose the causes of the formation of the cold center near Verkhoyansk. Proceeding from the analysis of perennial averages to the analysis of individual years, Shuleikin and his collaborators succeeded in finding a close connection between the abnormally warm winters in Eurasia and the abnormally high temperature of the Gulfstream. Employing the same method they succeeded in showing that no continent beside Eurasia is affected by the oceans to such an extent.

The works of Shuleikin reveal the essential part played by the monsoon currents, generated in the ocean-continent system. They show that no circulation theory can be complete if it does not allow for this all-important mechanism.

§8. Kochin's Mathematical Theory of the General Circulation

The construction of a theory of the general circulation of the atmosphere is one of the most basic problems of dynamic meteorology. The few attempts to construct a mathematical theory of the general circulation are far

from complete. The greatest advance in this direction was achieved by N. E. Kochin and E. N. Blinova. In the following we shall present the main outlines of their work.

We have repeatedly stressed the difficulties faced in the solution of that fundamental practical problem of dynamic meteorology, forecasting the weather. The construction of a rigorous theory of the general circulation meets with the same difficulties.

Kochin simplifies this problem, disregarding all effects connected with heat influx, and assuming the temperature distribution in the concerned part of the atmosphere to be known. The problem of the generation of this distribution is not touched upon. The given temperature distribution, is used to find the distribution of pressure, and from it the atmospheric flow pattern is determined.

Kochin has studied in detail the effects of eddy viscosity on the general circulation of the atmosphere, introducing the methods of the boundary layer into dynamic meteorology.

In §9 Chapter IX we discussed in detail the simplifications which can be introduced, according to Kochin, into the equations of motion of the atmosphere, drawing upon the theory of the boundary layer.

We start with a discussion of the stationary zonal circulation of the atmosphere. The equations of atmospheric flow for this case are written, following Kochin, in spherical coordinates r, $\theta = \frac{\pi}{2} - \varphi$, ψ in the following manner:

$$\mu \frac{\partial^2 v_\theta}{\partial r^2} + 2\omega\rho \cos\theta v_\psi = \frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{1}$$

$$\mu \frac{\partial^2 v_\psi}{\partial r^2} - 2\omega\rho \cos\theta \cdot v_\theta = 0, \tag{2}$$

$$\frac{\partial p}{\partial r} + g\rho = 0. \tag{3}$$

$$v_r = -\frac{1}{\rho r^2 \sin\theta}\int_a^r \frac{\partial(\rho v_\theta r \sin\theta)}{\partial\theta} dr, \tag{4}$$

where v_θ, v_ψ, v_r are the components of the wind velocity, a the terrestrial radius, and μ the coefficient of eddy viscosity.

From these equations one can determine all the elements of the stationary zonal circulation when the following data are given:

1) the temperature distribution over the entire atmosphere, $T(r, \theta)$;

2) the pressure distribution over the earth's surface

$$p_0(\theta) = p(a, \theta);$$

3) the viscosity distribution over the entire atmosphere.

In the first place, the pressure $p(r, \theta)$ at any point of the atmosphere can be determined according to the barometric formula

$$p(r,\theta) = p_0(\theta) \cdot e^{-\int \frac{g \cdot ds}{RT(r,\theta)}}. \tag{5}*$$

* [Translator's note: Obviously dz = dr, since z = r-a.]

Knowing the pressure p and the temperature T, the density can be determined from the equation of state:

$$\rho(r,\vartheta)=\frac{p(r,\vartheta)}{RT(r,\vartheta)}. \tag{6}$$

In the absence of viscosity the determined pressure distribution would give rise to gradient wind along the latitudinal parallels, of a velocity determined by the formula

$$v_\psi=\frac{1}{2\omega\rho\cos\theta}\cdot\frac{1}{a}\frac{\partial p}{\partial\theta}. \tag{7}$$

However, the presence of viscosity adds a meridional component v_θ to the zonal component v_ψ. The two components can be determined, for the entire height of the atmosphere, by the integration of the system of equations (1) and (2). These equations can be replaced by one equation, by introducing the complex velocity:

$$v_\theta+iv_\psi=u.$$

Multiplying equation (2) by i and adding to equation (1) we obtain:

$$\mu\frac{\partial^2 u}{\partial r^2}-2\omega\rho\cos\theta iu=\frac{1}{r}\frac{\partial p}{\partial\theta}. \tag{8}$$

Integrating equation (8), we obtain two arbitrary complex constants. They are determined from the boundary conditions

$$v_\theta=v_\psi=0 \quad \text{at} \quad r=a, \tag{9}$$

$$v_\theta,\ v_\psi \ \text{are bounded for}\ r\to\infty. \tag{10}$$

The vertical component of the velocity v_r can be determined with sufficient accuracy by equation (4).

The foregoing argument concerning the zonal circulation can be extended to the general case of an arbitrary atmospheric circulation. If the distribution of temperature T (r,θ,ϕ) is given throughout the atmosphere and the pressure distribution at the earth's surface p(a,θ,ϕ) is given too, and if the coefficient of turbulence μ is constant, the distribution of pressure and velocity can be determined over the entire atmosphere.

This is carried out in three stages, as in the case of stationary zonal circulation.

1) the pressure distribution is found from the barometric formula

$$p(r,\theta,t)=p(a,\theta,t)\,e^{-\frac{g}{R}\int_a^r\frac{ds}{T(r,\theta,\psi)}}. \tag{11}$$

2) the horizontal wind velocity field is found from the two equations

$$\mu\frac{\partial^2 v_\theta}{\partial r^2}+2\omega\rho\cos\theta\cdot v_\psi=\frac{1}{r}\frac{\partial p}{\partial\theta}, \tag{12}$$

$$\mu\frac{\partial^2 v_\psi}{\partial r^2}-2\omega\rho\cos\theta\cdot v_\theta=\frac{1}{r}\frac{\partial p}{\partial\psi}\cdot\frac{1}{\sin\theta} \tag{13}$$

3) The vertical component of the velocity is then determined from the equation of continuity

$$v_r = -\frac{1}{\rho r^2 \sin\theta}\int_a^r \left[\frac{\partial}{\partial\theta}(\rho v_\theta r \sin\theta) + \frac{\partial \rho v_\psi r}{\partial\psi}\right] dr. \qquad (14)^*$$

Finally, in the general case of a nonzonal, nonstationary circulation with varying coefficient of turbulence, μ Kochin reduces the task to the determination of

1) the pressure, by the barometric formula

$$p(r, \theta, \psi, t) = p(a, \theta, \psi, t)\cdot e^{-\frac{g}{R}\int_a^r \frac{dz}{T(r,\theta,\psi,t)}} \qquad (15)$$

the velocity $u = v_\theta + iv_\psi$ by the equation

$$\rho\frac{\partial u}{\partial t} = -2i\omega\rho\cos\theta u + \frac{\partial}{\partial r}\mu\frac{\partial u}{\partial r} - \frac{i}{r\sin\theta}\frac{\partial p}{\partial\psi} - \frac{1}{r}\frac{\partial p}{\partial\theta} \qquad (16)^{**}$$

and 2) the vertical velocity v_r, by the equation

$$v_r = -\frac{1}{\rho r^2 \sin\theta}\int_a^r \left[\frac{\partial \rho r v_\theta \sin\theta}{\partial\theta} + \frac{\partial \rho r v_\varphi}{\partial\psi} + \frac{\partial \rho r^2 \sin\theta}{\partial t}\right] dz. \qquad (17)^{***}$$

If the pressure gradient would not have changed with altitude, i.e., if the atmosphere could be regarded an incompressible fluid, the velocity field could be described by the equations of Åkerblom. Within the boundary layer the complex velocity would have satisfied these equations and above it gradient wind would have prevailed. The atmosphere, however, is a baroclinic fluid, and thus the pressure gradient does vary with altitude, the eddy viscosity consequently having some effect in the atmosphere wherever the pressure gradient changes appreciably with altitude.

It may happen, therefore, that the thickness δ of the boundary layer will be many times greater than $\sqrt{\frac{\mu}{\omega\rho}}$ (see § 9 Chapter IX), owing to the fact that the influence of the eddy viscosity may extend up to well over several kilometers, i.e., almost over the entire troposphere. This remarkable result of Kochin contradicts the views held until now. Thus, for instance, Jeffreys in a well-known paper finds that the influence of viscosity is sensible in a layer of the order of 300 m. From this premise Jeffreys drew several erroneous conclusions, stating for example that stationary zonal circulation is impossible.

In 1936 Kochin constructed a model of a stationary zonal circulation, adopting the constant value of μ = 2 ton/m sec. Analogous calculations were performed by Helbke for nonconstant μ. Dorodnitsyn proved that the foreknowledge of the pressure distribution is not an indispensable prerequisite for Kochin's method. It can be replaced by another condition. Dorodnitsyn supposed that the atmosphere is involved in the flow to a certain sufficiently high altitude H and also took into consideration the general lag of the atmosphere behind the rotating earth.

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* [The Russian text has 2 instead of r, obviously a misprint. Translator's note.]

** [The Russian text has $\frac{1}{2}$ instead of $\frac{1}{r}$, obviously a misprint. Translator's note.]

*** [The Russian text has z instead of r, obviously a misprint. Translator's note.]

In 1938, Dorodnitsyn, Izvekov and Shvets reconstructed a model of the zonal circulation according to Kochin's method, using an improved system of equations and new data on the distribution of temperature and pressure. They succeeded in showing that in the case of zonal circulation, there exist in the atmosphere three main "rings" of well-defined circulation in the vertical plane. Tropical and polar rings are of thermal origin. They "work" like simple thermodynamic heat engines. The heat sources are located at the lower latitudes where the pressure is high and the heat sinks at higher latitudes and correspondingly low pressure. Between them, at middle latitudes, lies the third ring. This ring is of dynamic origin. The meridional currents diverge at the belt of high pressure (horse latitudes) and the mass loss is compensated by subsiding masses. In the low pressure belt (subarctic) the convergent flow is compensated by ascending currents.

§ 9. Blinova's Hydrodynamic Theory of Pressure Waves and Action Centers in the Atmosphere

E. N. Blinova studied the problem of long-range forecast of pressure, temperature, and velocity, proceeding from the equations of hydromechanics. The results she obtained represent a valuable contribution towards the construction of a mathematical theory of the general circulation of the atmosphere.

Blinova considered a spherical earth of radius a. Gravity and Coriolis forces act on the atmosphere. The force of friction is disregarded. Fridman's equation (16) § 1 Chapter IX is taken as the starting point. This equation determines the variation of the vorticity vector in a fluid particle. To obtain this equation we write Euler's equation in vectorial form

$$\frac{dv}{dt} = -\frac{1}{\rho}\operatorname{grad} p + \operatorname{grad}\Phi + 2[\omega, v]. \tag{1}$$

Forming the curl of both sides of the equation, and putting $\Omega = \operatorname{curl} v$, we obtain the following equation (Fridman's equation):

$$\frac{d\Omega}{dt} - (\Omega, \operatorname{grad})\, v + \Omega \operatorname{div} v = -2\operatorname{curl}[\omega, v] - \frac{1}{\rho T}[\operatorname{grad} T \operatorname{grad} P]. \tag{2}$$

Passing to explicit coordinate notation, Blinova introduces spherical coordinates: r – the distance from the center of the earth, θ the complementary angle of the latitude ($\theta = 0$ at the North Pole and $\theta = \pi$ at the South Pole), and λ the longitude of the locality.

The atmospheric flow is regarded as consisting of the main flow – the zonal circulation, and of the perturbations of this flow; therefore, the θ and λ components of the velocity, and the local pressure can be written in the form

$$v_\theta = v_\theta(\theta, \lambda, t); \quad v_\lambda = \bar{v}_\lambda(\theta) + v_\lambda(\theta, \lambda, t); \quad P = \bar{p}(\theta) + p(\theta, \lambda, t). \tag{3}$$

We emphasize that here and henceforth all magnitudes are averaged over the altitude. The perturbation terms v_θ, v_λ, p are assumed to be relatively small.

We assume the validity of

$$\bar{v}_\lambda = \alpha \cdot a \cdot \sin\theta, \tag{4}$$

where α is the angular velocity of the flowing air relative to the earth, being independent of θ. The ratio α/ω is called the circulation index.

Regarding the flow as gradient wind, we obtain for the pressure the following formula:

$$\bar{p} = p_0 + \rho a^2 \alpha \omega \sin^2 \theta. \tag{5}$$

For the temperature the following formula is used:

$$T = \bar{T}(\theta) + T'(\theta, \lambda) + T''(\theta, \lambda, t), \tag{6}$$

where $\bar{T} + T'$ is the mean (monthly or seasonal) temperature distribution, depending on radiation, turbulence and position of land masses and oceans. This distribution is assumed to be known, whereby the zonal temperature is determined by the following formula, analogous to formula (5) for the zonal pressure:

$$\bar{T}(\theta) = T_0 + M \sin^2 \theta, \tag{7}$$

where T_0 and M are constants, and T'' is the deviation from the stationary temperature distribution. The quantities T' and T'', expressing the perturbation of the zonal temperature, are considered to be small.

Assuming that vertical velocities are absent and neglecting the variations of the density, Blinova introduces the stream function ψ, according to the equation of continuity we obtain

$$v_\theta = -\frac{\partial \psi}{a \sin \theta \partial \lambda}; \quad v_\lambda = -\frac{\partial \psi}{a \partial \theta}. \tag{8}$$

It is known that the stream function ψ for small perturbations may be represented as a sum of two stream functions

$$\psi(\theta, \lambda, t) = \psi'(\theta, \lambda) + \psi''(\theta, \lambda, t). \tag{9}$$

Here $\psi'(\theta, \lambda)$ is a time-independent stream function, standing for the stationary part of the perturbation of the temperature distribution in the zonal circulation, representing the stationary perturbation of the zonal temperature distribution, expressed in equation (6) by the term $T'(\theta, \lambda)$ $\psi''(\theta, \lambda, t)$* is a stream function representing the nonstationary perturbation of the zonal circulation.

The pressure perturbation $p(\theta, \lambda, t)$ is treated in exactly the same way, namely, as a sum of two terms: $p'(\theta, \lambda)$ — the stationary perturbation of the zonal pressure distribution and $p''(\theta, \lambda, t)$ — the nonstationary perturbation:

$$p(\theta, \lambda, t) = p'(\theta, \lambda) + p''(\theta, \lambda, t). \tag{10}$$

Projecting the terms of Fridman's equation on the r-axis, replacing the velocity components by the corresponding derivatives of the stream function and omitting the comparatively small terms, Blinova obtains the following fundamental equation of the problem:

$$\left(\frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial \lambda}\right)\left[\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \psi}{\partial \theta} + \frac{1}{\sin \theta}\frac{\partial^2 \psi}{\partial \lambda^2}\right)\right] + 2(\alpha + \omega)\sin \theta \frac{\partial \psi}{\partial \lambda} = \\ = \frac{1}{\rho \bar{T}}\left[\frac{\partial \bar{p}}{\partial \theta} \cdot \frac{\partial (T' + T'')}{\partial \lambda} - \frac{\partial \bar{T}}{\partial \theta} \cdot \frac{\partial p}{\partial \lambda}\right]. \tag{11}$$

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* [The Russian text has $\psi''(\theta, \lambda)$, an obvious misprint. Translator's note.]

Blinova simplifies this equation by omitting the terms containing T'' (which is small) in comparison with T', and the term $\frac{\partial \bar{T}}{\partial \theta} \cdot \frac{\partial p}{\partial \lambda}$ (which is small) in comparison with $\frac{\partial \bar{p}}{\partial \theta} \cdot \frac{\partial T'}{\partial \lambda}$, justifying this by subsequent calculations. Equation (11) thus assumes the following form:

$$\left(\frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial \lambda}\right) \cdot \left[\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial \lambda^2}\right] + 2(\alpha + \omega)\sin\theta \frac{\partial \psi}{\partial \lambda} = \\ = a^2 \alpha\omega \frac{\sin 2\theta}{T}\frac{\partial T'}{\partial \lambda}. \quad (12)$$

The equation of the λ -component of the flow has the form

$$\frac{\partial p}{\rho \sin\theta \partial \lambda} = -\left(\frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial \lambda}\right)\frac{\partial \psi}{\partial \theta} + 2(\alpha + \omega)\operatorname{ctg}\theta \frac{\partial \psi}{\partial \lambda}. \quad (13)$$

Finally, Blinova writes the equation of heat balance in the form

$$\frac{\partial T''}{\partial t} + \alpha \frac{\partial T'}{\partial \lambda} = \frac{1}{a^2 \sin\theta} \cdot \frac{\partial \bar{T}}{\partial \theta} \cdot \frac{\partial \psi''}{\partial \lambda}. \quad (14)$$

Thus, the system of equations (12-14) contains three unknown functions: ψ, p, T'.

We seek the stationary part ψ' of the solution of equation (12) in the form of an infinite expansion in spherical harmonics:

$$\psi'(\theta, \lambda) = \sum_{n=1}^{\infty}\sum_{m=1}^{n}(C_n^m \cos m\lambda + C'^m_n \sin m\lambda) \cdot P_n^m(\cos\theta), \quad (15)$$

the coefficients C_n^m and C'^m_n being determined by the formulas

$$\left.\begin{aligned} C_n^m &= \frac{2a^2\omega}{\bar{T}} \frac{\frac{n-m}{2n-1}\cdot\delta^m_{n-1} + \frac{m+n+1}{2n+3}\cdot\delta^m_{n+1}}{2\frac{\alpha+\omega}{\alpha} - n(n+1)}, \\ C'^m_n &= \frac{2a^2\omega}{\bar{T}} \frac{\frac{n-m}{2n-1}\cdot\delta'^m_{n-1} + \frac{m+n+1}{2n+\delta}\cdot\delta'^m_{n+1}}{2\frac{\alpha+\omega}{\alpha} - n(n+1)}, \end{aligned}\right\} \quad (16)$$

where δ^m_n and δ'^m_n are the coefficients of the spherical harmonic expansion of T', and T is the mean value of the temperature over the entire atmosphere.

The stationary perturbation of the zonal pressure distribution is found from equation (13):

$$p'(\theta, \lambda) = 2\rho(\alpha + \omega)\sum_{n=1}^{\infty}\sum_{m=1}^{n}\left\{\left[C^m_{n-1}\frac{(n-m)(n-1)}{(2n-1)\cdot n} + \right.\right. \\ \left. + C^m_{n+1}\frac{(n+2)(n+m+1)}{(n+1)(2n+3)}\right]\cos m\lambda + \left[C'^m_{n-1}\frac{(n-m)(n-1)}{2n-1} + \right. \\ \left.\left. + C'^m_{n+1}\frac{(n+2)(n+m+1)}{(n+1)(2n+3)}\right]\sin m\lambda\right\} P_n^m(\cos\theta); \quad (17)$$

the nonstationary part of the solution of the equation (12) is sought in the form

$$\psi''(\theta, \lambda, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{n} [D_n^m \cos(m\lambda + \sigma t) + D'^m_n \sin(m\lambda + \sigma t)] P_n^m(\cos\theta), \tag{18}$$

where

$$\sigma = \frac{2(\alpha + \omega) m}{n(n+1)} - \alpha m, \tag{19}$$

and D_n^m, D'^m_n are unknown coefficients which must be determined. For the nonstationary pressure perturbation $p''(\theta, \lambda, t)$ equation (13) yields an expression, analogous to (17) but with the $m\lambda$ replaced by $(m\lambda + \sigma t)$ and the constant C_n^m and C'^m_n replaced by D_n^m and D'^m_n. Such representation of the nonstationary part of the pressure perturbation is employed by Blinova for the determination of the coefficients D_n^m and D'^m_n.

In this way we obtain

$$\begin{aligned} A_n^m - 2\rho(\alpha + \omega)\left[D_{n-1}^m \frac{(n-m)(n-1)}{(2n-1)\cdot n} + D_{n+1}^m \frac{(n+2)(n+m+1)}{(n+3)(2n+3)}\right] = 0, \\ A'^m_n - 2\rho(\alpha+\omega)\left[D'^m_{n-1} \frac{(n-m)(n-1)}{(2n-1)\cdot n} + D'^m_{n+1} \frac{(n+2)(n+m+1)}{(n+1)(2n+3)}\right] = 0, \end{aligned} \tag{20}$$

where A_n^m and A'^m_n are the expansion coefficients of $\psi''(\theta, \lambda, t)$ in terms of spherical harmonics at $t = 0$.

Blinova looks for a solution symmetrical with respect to the equator. To this end, in the expressions (17) and (20) all n should be taken with the same parity as m. Then all coefficients D_n^m and D'^m_n are determined successively by equation (20).

The expression (17) determines the location and the intensity of the centers of action of the atmosphere. An analogous expression for p'' permits the calculation of the pressure distribution when the initial distribution is known.

Equation (14) determines the nonstationary temperature perturbation:

$$\begin{aligned} T''(\theta, \lambda, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \{ & [B_n^m \cos m(\lambda - \alpha t) + B'^m_n \sin m(\lambda - \alpha t)] P_n^m - \\ & - 2N_n \left[D_n^m \sin\left(m\lambda + \frac{\sigma - m\alpha}{2} t\right) - \right. \\ & \left. - D'^m_n \cos\left(m\lambda + \frac{\sigma - m\alpha}{2} t\right) \right] \left[(n+m) P_{n-1}^m + \right. \\ & \left. + (n - m + 1) P_{n+1}^m \right] \cdot \sin \frac{\sigma + m\alpha}{2} \cdot t \}, \end{aligned} \tag{21}$$

where

$$N_n = \frac{M}{a^2} \frac{n(n+1)}{(2n+1)(\omega + \alpha)}, \tag{22}$$

and B_n^m and B'^m_n are the expansion coefficients of $T''(\theta, \lambda, t)$ in spherical harmonics, at $t = 0$.

Blinova's computations have permitted to obtain theoretically the minima of Iceland and the Aleutians, and the Azores, Pacific, Siberian, and Canadian **maxima.**

Chapter XIX

THE THEORY OF PERTURBATION OF ATMOSPHERIC FLOWS

§ 1. The Application of the Theory of Small Oscillations to Atmospheric Flows

In the foregoing chapters we have discussed mainly the stationary flows of the atmosphere. However, real atmospheric flows are not stationary. The study of stationary flows enables us to obtain some idea of the background, on which other flows have to be superimposed to give the diversity of real atmospheric flows.

The mathematical study of the majority of problems of dynamic meteorology connected with nonstationary atmospheric motions encounters unsurmountable difficulties. Let us assume for the sake of argument that the problem is simplified by making certain broad approximations, i. e., that the atmosphere is incompressible, that there is no influx of heat, that turbulence and viscosity are absent. Then the problem will be reduced to the solution of a system of hydrodynamical equations for a perfect incompressible fluid. However, complete integration is impossible even for this simplified system of equations. This is due to the nonlinear terms entering in the equations of hydrodynamics

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z},\quad u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z},\quad u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z},$$

no method being available at present for complete integration of such equations.

We can, however, indicate one fruitful method, permitting to bring the equations of atmospheric motion into linear form. This is the so-called method of small oscillations or method of small perturbations. The principle of this method consists in regarding the examined motion as a sum of two motions – a stationary motion and a small perturbation superimposed on it. One of the most important problems of dynamic meteorology, namely the problem of the genesis and evolution of extratropical cyclones, is solved by the method of small perturbations. According to modern ideas, extratropical cyclones develop as a wave disturbance on the partition surface (interface) between two air masses having different temperatures and velocities. If the perturbation grows with time, the wave gradually transforms into a vortex – the cyclone. Thus, the problem of the extratropical cyclogenesis is reduced to the problem of the stability of the interface.

In this case, the main atmospheric motion may be taken as the geostrophic motion of the two air masses, separated by a stationary interface, whose slope is determined by Margules' formula.

Small perturbations are imposed on this stationary motion, and we search for the conditions under which these perturbations lead to small velocity fluctuations about the stationary geostrophic flow and to small oscillations of the interface about its equilibrium position, given by Margules' formula. When these conditions are not fulfilled, the perturbation grows with time and the method of small oscillations is no longer applicable.

The final criterion of applicability of the obtained solutions is their accord with actual atmospheric conditions.

§ 2. The Differential Equations of Small Perturbations

We now proceed to derive the differential equations of small perturbations for a given fundamental motion. Owing to their generality, these equations possess a symmetry which is usually lost in particular cases.

We shall derive the equations of small perturbations for a rectangular Cartesian Cartesian coordinate system.

We denote the functions referring to the fundamental state by capital letters, and the functions referring to the perturbation with the corresponding small letters.

Using Euler's method, the fundamental state of the atmospheric system is determined by the following functions:

$$U,\ V,\ W,\ P,\ Q,\ T,\ F. \tag{1}$$

Here, U, V, W are the velocity components of the fundamental motion, P – the pressure, Q – the density, T – the local air temperature, F – a function of point coordinates and time, which when set equal to zero gives the equation of the external or internal interface.

The functions U, V, W, P, Q, T, F are considered as known and are bound by the following conditions: they should satisfy the three equations of motion:

$$\left.\begin{aligned}
\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}&=-\frac{1}{Q}\frac{\partial P}{\partial x}+X-2\omega_y W+2\omega_z V,\\
\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}&=-\frac{1}{Q}\frac{\partial P}{\partial y}+Y-2\omega_z U+2\omega_x W,\\
\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}&=-\frac{1}{Q}\frac{\partial P}{\partial z}+Z-2\omega_x V+2\omega_y U,
\end{aligned}\right\} \tag{2}$$

where X, Y, Z are the components of the mass force;
the equation of continuity

$$\frac{\partial Q}{\partial t}+U\frac{\partial Q}{\partial x}+V\frac{\partial Q}{\partial y}+W\frac{\partial Q}{\partial z}+Q\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)=0, \tag{3}$$

the equation of state

$$P=QRT, \tag{4}$$

the equation of heat balance which will be taken in the form of the polytropic equation

$$\frac{1}{P}\left(\frac{\partial P}{\partial t}+U\frac{\partial P}{\partial x}+V\frac{\partial P}{\partial y}+W\frac{\partial P}{\partial z}\right)-\\
-\frac{K}{Q}\left(\frac{\partial Q}{\partial t}+U\frac{\partial Q}{\partial y}+V\frac{\partial Q}{\partial y}+W\frac{\partial Q}{\partial z}\right)=0, \tag{5}$$

and also the interface conditions: the kinematic condition

$$F(x, y, z, t) = 0 \tag{6}$$

or, when the boundary surface is external, in the differential form:

$$\frac{\partial F}{\partial t} + U\frac{\partial F}{\partial x} + V\frac{\partial F}{\partial y} + W\frac{\partial F}{\partial z} = 0. \tag{7}$$

When the boundary is internal, the interface condition will assume the form

$$\left.\begin{aligned} &\frac{\partial F}{\partial t} + U\frac{\partial F}{\partial x} + V\frac{\partial F}{\partial y} + W\frac{\partial F}{\partial z} = 0, \\ &\frac{\partial F}{\partial t} + U'\frac{\partial F}{\partial x} + V'\frac{\partial F}{\partial y} + W'\frac{\partial F}{\partial z} = 0. \end{aligned}\right\} \tag{7'}$$

This form of the interface condition stems from the fact that the interface should continually consist of the same fluid particles. Finally, the functions should satisfy the dynamic condition which for a free boundary will be written as follows:

$$\frac{\partial P}{\partial t} + U\frac{\partial P}{\partial x} + V\frac{\partial P}{\partial y} + W\frac{\partial P}{\partial z} = 0, \tag{8}$$

and on an inner boundary, an interface between two fluid masses will have the form:

$$P - P' = 0 \quad \text{for} \quad F(x, y, z, t) = 0, \tag{9}$$

or in the differential form:

$$\left.\begin{aligned} &\frac{\partial (P-P')}{\partial t} + U\frac{\partial (P-P')}{\partial x} + V\frac{\partial (P-P')}{\partial y} + W\frac{\partial (P-P')}{\partial z} = 0, \\ &\frac{\partial (P-P')}{\partial t} + U'\frac{\partial (P-P')}{\partial x} + V'\frac{\partial (P-P')}{\partial y} + W'\frac{\partial (P-P')}{\partial z} = 0. \end{aligned}\right\} \tag{10}$$

We consider now a motion slightly departing from the fundamental, characterized by the functions

$$\left.\begin{aligned} &U(x, y, z, t) + u(x, y, z, t), \\ &V(x, y, z, t) + v(x, y, z, t), \\ &W(x, y, z, t) + w(x, y, z, t), \\ &P(x, y, z, t) + p(x, y, z, t), \\ &Q(x, y, z, t) + q(x, y, z, t), \\ &T(x, y, z, t) + \tau(x, y, z, t), \\ &F(x, y, z, t) + f(x, y, z, t), \end{aligned}\right\} \tag{11}$$

which must also satisfy all the preceding equations.

Since the functions U, V, W, P, Q, T, F are known, the small perturbations represented by the additional terms

$$\boldsymbol{v}(x, y, z, t)\left\{\begin{array}{lll} u(x, y, z, t); & q(x, y, z, t); & p(x, y, z, t); \\ v(x, y, z, t); & \tau(x, y, z, t); & f(x, y, z, t), \\ w(x, y, z, t); & & \end{array}\right. \tag{12}$$

are fully determined.

For the derivation of the equations of small perturbations we introduce the functions (11) into equations (2)-(10) and stipulate the following:

1) the equations (2)-(10) characterize the fundamental motion as well as the motion obtained by the superposition of small perturbations.

2) the superimposed perturbations are sufficiently small, so that the terms (12) and their derivatives may be taken as first order infinitesimals, and consequently their nonlinear combinations may be disregarded.

This approach may, however, be replaced by another, which is more convenient.

Let us form the variations of all equations of the fundamental state with respect to the functions (1), regarding the variation δ as independent of the differentiations occurring in these equations. Let us then replace the variations of the functions (1) by the corresponding functions (12), in this way:

$$\delta U = u; \quad \delta V = v \quad \text{etc.}$$

We thus obtain the following system of equations of small perturbations, including the boundary conditions:

The equations of motion

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+V\frac{\partial u}{\partial y}+W\frac{\partial u}{\partial z}+u\frac{\partial U}{\partial x}+v\frac{\partial U}{\partial y}+w\frac{\partial U}{\partial z}= \\ &\qquad =-\frac{1}{Q}\frac{\partial p}{\partial x}+\frac{1}{Q^2}\frac{\partial P}{\partial x}q-2\omega_y w+2\omega_z v, \\ &\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}+V\frac{\partial v}{\partial y}+W\frac{\partial v}{\partial z}+u\frac{\partial V}{\partial x}+v\frac{\partial V}{\partial y}+w\frac{\partial V}{\partial z}= \\ &\qquad =-\frac{1}{Q}\frac{\partial p}{\partial y}+\frac{1}{Q^2}\frac{\partial P}{\partial y}q-2\omega_z u+2\omega_x w \\ &\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}+V\frac{\partial w}{\partial y}+W\frac{\partial w}{\partial z}+u\frac{\partial W}{\partial x}+v\frac{\partial W}{\partial y}+w\frac{\partial W}{\partial z}= \\ &\qquad =-\frac{1}{Q}\frac{\partial p}{\partial z}+\frac{1}{Q^2}\frac{\partial P}{\partial z}q-2\omega_x v+2\omega_y u. \end{aligned}\right\} \tag{13}$$

The equation of continuity

$$\frac{\partial q}{\partial t}+\frac{\partial(qU)}{\partial x}+\frac{\partial(qV)}{\partial y}+\frac{\partial(qW)}{\partial z}+\frac{\partial(uQ)}{\partial x}+\frac{\partial(vQ)}{\partial y}+\frac{\partial(wQ)}{\partial z}=0. \tag{14}$$

The equation of state

$$p = RqT + RQ\tau. \tag{15}$$

The polytropic equation:

$$\begin{aligned}&\frac{1}{P}\left(\frac{\partial p}{\partial t}+U\frac{\partial p}{\partial x}+V\frac{\partial p}{\partial y}+W\frac{\partial p}{\partial z}+u\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial y}+w\frac{\partial P}{\partial z}\right)-\\&-\frac{1}{P^2}\left(\frac{\partial P}{\partial t}+U\frac{\partial P}{\partial x}+V\frac{\partial P}{\partial y}+W\frac{\partial P}{\partial z}\right)p=\frac{k}{Q}\left(\frac{\partial Q}{\partial t}+U\frac{\partial q}{\partial x}+V\frac{\partial q}{\partial y}+\right.\\&\left.+W\frac{\partial q}{\partial z}+u\frac{\partial Q}{\partial x}+v\frac{\partial Q}{\partial y}+w\frac{\partial Q}{\partial z}\right)-\frac{k}{Q^2}\left(\frac{\partial Q}{\partial t}+U\frac{\partial Q}{\partial x}+V\frac{\partial Q}{\partial y}+W\frac{\partial Q}{\partial z}\right)q.\end{aligned} \tag{16}$$

The kinematic interface condition

$$F(x, y, z, t)+f(x, y, z, t)=0 \tag{17}$$

or in the differential form:

$$\left.\begin{aligned}&\frac{\partial f}{\partial t}+U\frac{\partial f}{\partial x}+V\frac{\partial f}{\partial y}+W\frac{\partial f}{\partial z}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0,\\&\frac{\partial f}{\partial t}+U'\frac{\partial f}{\partial x}+V'\frac{\partial f}{\partial y}+W'\frac{\partial f}{\partial z}+u'\frac{\partial F}{\partial x}+v'\frac{\partial F}{\partial y}+w'\frac{\partial F}{\partial z}=0.\end{aligned}\right\} \tag{18}$$

The dynamic free boundary conditions:

$$\frac{\partial p}{\partial t}+U'\frac{\partial p}{\partial x}+V\frac{\partial p}{\partial y}+W\frac{\partial p}{\partial z}+u\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial y}+w\frac{\partial P}{\partial z}=0 \quad \text{for } F+f=0. \tag{19}$$

The dynamic condition on an interface:

$$P+p-P'-p'=0 \quad \text{where } F+f=0 \tag{20}$$

or in the differential form:

$$\left.\begin{aligned}&\frac{\partial(p-p')}{\partial t}+U\frac{\partial(p-p')}{\partial x}+V\frac{\partial(p-p')}{\partial y}+W\frac{\partial(p-p')}{\partial z}+\\&+u\frac{\partial(P-P')}{\partial x}+v\frac{\partial(P-P')}{\partial y}+w\frac{\partial(P-P')}{\partial z}=0,\\&\frac{\partial(p-p')}{\partial t}+U'\frac{\partial(p-p')}{\partial x}+V'\frac{\partial(p-p')}{\partial y}+W'\frac{\partial(p-p')}{\partial z}+\\&+u'\frac{\partial(P-P')}{\partial x}+v'\frac{\partial(P-P')}{\partial y}+w'\frac{\partial(P-P')}{\partial z}=0.\end{aligned}\right\} \tag{21}$$

The equations of small perturbations (13)-(21) might seem even bulkier than the previous equations (2)-(11); the essential difference between them is that the functions U, V, W, P, Q, T, F in the original differential equations are unknown, i.e, the equations are nonlinear. In the equations of small perturbations the functions U, V, W, P, Q, T, F are regarded as known; therefore the equations (13) - (21) are linear with respect to the unknown functions u, v, w, p, q, τ, f. Thus equations (13 - (21) are simpler than equations (2) - (11).

§ 3. Wave Motion in the Atmosphere

Wave motion is characterized by the small oscillations performed by the particles about their equilibrium position or about their fundamental motion. Simultaneously, some elements of the wave, for instance, the wave front, the free boundary etc,undergo large displacements.

The classic example of wave motion are water surface waves. Suppose that by the instantaneous action of some forces the horizontal form of the free water surface is disturbed. After the influence of the momentarily applied forces ceases, owing to the cohesion between the water particles, this disturbance is propagated to farther lying water particles. Thus, the water surface becomes quite rapidly undulated.

The most intense oscillations will be performed by the particles on the free surface; such waves are called surface waves. The force of gravity, viscosity, and the surface tension tend to damp the disturbances. If the wave motion draws on the potential energy of position, the waves are called gravitational. Gravitational waves may arise not only on the free surface of the fluid but also on a surface of discontinuity within the fluid. Such internal gravitational waves are of special importance in meteorology.

Elastic waves may propagate in air, whereby the potential energy of the compressed or rarefied air transforms into the kinetic energy. These include, for example, sound and blast waves. Elastic waves are of no importance in meteorology; therefore they will not be discussed. On the other hand, inertial waves propagate by inertia only and do not involve the transformation of kinetic energy into other forms of energy. These waves are of great importance in a number of basic problems of dynamic meteorology.

The classic example of atmospheric wave motion are the waves formed on the interface between two air layers moving with different velocities and of different densities. Under favorable conditions these waves become visible. Namely, when particles in the crest of the wave reach the level of condensation and the particles descending in the trough move away from the saturation level, the well known undulated clouds are formed. The fact that such clouds are formed over immense territories indicates the probability of undulatory movements existing also when no clouds are formed. The upper boundary of a continuous overcast has usually the form of a choppy sea.

The aforementioned wave types are not the only forms of atmospheric wave motion.

The terrestrial atmosphere, as well as the ocean, undergoes tides, caused mainly by the gravitation of the moon and to a lesser degree by the gravitation of the sun.

The waves most important in meteorology are those formed on the frontal surfaces separating two air masses. This type of waves is especially important

since, according to modern ideas, the formation and development of extratropical cyclones is closely associated with them. However, the great difficulties encountered in the studies of wave disturbances on inclined frontal surfaces have baffled all past attempts at a solution. At the present time only the problem of stability of frontal surface wave motion has been solved.

§ 4. Gravitational Waves on the Free Boundary of an Incompressible Fluid

We will first discuss the simplest case of wave motion on the free boundary of a layer of incompressible fluid of thickness h, extending horizontally in all directions without limit. The effect of the rotation of the earth is not taken into account.

Let us assume that the fundamental motion of the fluid is uniformly rectilinear, due to one external force – the force of gravity.

We place the origin of coordinates on the free boundary of this layer, direct the x-axis parallel to the fundamental motion of the fluid and the z-axis vertically upwards.

The fundamental motion will be characterized by:

$$U=\text{const.}, \quad V=W=0, \quad Q=\text{const.}$$

Since the only external force is the force of gravity, we have

$$X=Y=0; \quad Z=-g.$$

The equations of the fundamental motion will be written in this manner

$$\frac{\partial P}{\partial x}=\frac{\partial P}{\partial y}=0; \quad \frac{\partial P}{\partial z}=-g\rho; \quad \frac{\partial U}{\partial x}=0. \tag{1}$$

Integrating the second equation (the hydrostatic equation), we obtain

$$P=-g\rho z+\varphi(t). \tag{2}$$

Concerning the small perturbations, it is natural to assume that they are periodic functions of the time and their amplitudes are small first order quantities. Besides, we assume that the motion is plane-parallel so as to be characterized by the following functions:

$$U+u, \quad v=0; \quad w, \quad P+p, \quad Q.$$

Then the equations of small perturbations will assume the form

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}; \quad \frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial z}; \quad \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0. \tag{3}$$

Let us determine the relevant boundary conditions. At the solid bottom we have

$$w = 0 \quad \text{at} \quad z = -h. \tag{4}$$

On the free boundary the pressure should change continuously; denoting the atmospheric pressure (on the unperturbed surface $z(x) = 0$) by p_0, we obtain:

$$P = p_0; \tag{5}$$

on the free surface under the action of a small perturbation given by

$$z(x, t) = \zeta(x, t), \tag{6}$$

we obtain:

$$P + p = p_0. \tag{7}$$

Applying (2) to the unperturbed surface, we obtain:

$$\varphi(t) = p_0,$$

and for the perturbed surface

$$-g\rho\zeta + \varphi(t) + p = p_0.$$

Eliminating p_0, we obtain the following condition:

$$p = g\rho\zeta \quad \text{at} \quad z = 0. \tag{8}$$

The quantity ζ is a new undetermined function expressing the profile of the perturbed surface.

Differentiating the equation of the surface with respect to t, we obtain the relation between ζ and the vertical velocity of the particle. Thus,

$$\frac{dz}{dt} = \frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + \frac{\partial\zeta}{\partial x}\frac{dx}{dt}.$$

But

$$\frac{dz}{dt} = w; \quad \frac{dx}{dt} = U + u.$$

Therefore, omitting the second order term $u \cdot \frac{\partial\zeta}{\partial x}$ we obtain:

$$w = \frac{\partial\zeta}{\partial t} + U\frac{\partial\zeta}{\partial x}. \tag{9}$$

Usually the integration of differential equations necessitates, beside the boundary conditions, that the initial conditions should be specified. In our case, however, this requirement is superfluous, since we consider oscillations. In this case we look for such a solution u, w, p, into which the time enters only through a periodic factor which is a trigonometric function of the form

$$\cos(\sigma t), \quad \sin(\sigma t),$$

Where $\sigma = 2\pi/\tau$ circular frequency, and τ the period of oscillation.

Besides, the profile is expressed in the simplest case by a function in which the co-ordinate x of the free surface enters in the form of a periodic trigonometric function $\cos kx$ or $\sin kx$, where $k = 2\pi/\lambda$ is the wave number, and λ the wave length. Thus, it is natural to seek a solution in which the time t and the coordinate x enter through the argument of a trigonometric function:

$$\cos(\sigma t - kx), \quad \sin(\sigma t - kx).$$

However, this problem may be solved in a simpler and more general way, by making use of complex functions and recalling the formula [De Moivre's formula] connecting the exponential and the trigonometric functions:

$$e^{i(\sigma t - kx)} = \cos(\sigma t - kx) + i \sin(\sigma t - kx).$$

Since all equations of small perturbations were shown to be linear we shall first look for complex solutions for u, w, p, ζ.

$$u = u_1 + iu_2; \quad w = w_1 + iw_2; \quad p = p_1 + ip_2; \quad \zeta = \zeta_1 + i\zeta_2;$$

since both the real and imaginary parts of the complex functions are solutions of the same problem.

The solution may thus be put in the following form:

$$\left.\begin{aligned} u(x, z, t) &= \bar{u}(z) \cdot e^{i(\sigma t - kx)}, \\ w(x, z, t) &= \bar{w}(z) \cdot e^{i(\sigma t - kx)}, \\ p(x, z, t) &= \bar{p}(z) \cdot e^{i(\sigma t - kx)}, \\ \zeta(x, t) &= \bar{\zeta} \cdot e^{i(\sigma t - kx)}. \end{aligned}\right\} \tag{10}$$

Substituting (10) in (3) we obtain the equations of the problem in the following form:

$$\left.\begin{aligned} i(\sigma - kU) \cdot u &= \frac{ik}{\rho} \cdot p, \\ i(\sigma - kU) \cdot w &= -\frac{1}{\rho} \frac{\partial p}{\partial z}, \\ iku &= \frac{\partial w}{\partial z}. \end{aligned}\right\} \tag{11}$$

The free boundary condition will be written in this form:

$$\left.\begin{aligned} p(x, 0, t) &= g\rho\zeta(x, t), \\ w(x, 0, t) &= i(\sigma - kU)\zeta(x, t). \end{aligned}\right\} \tag{12}$$

Strictly speaking, we should have put $p(x, \zeta, t)$ and $w(x, \zeta, t)$, in these equations, but since ζ is a very small quantity we may substitute zero in its place.

Differentiating the first equation (11) with respect to x, and the second with respect to z, and taking the third equation into account, we obtain:

$$k^2 p = \frac{\partial^2 p}{\partial z^2},$$

or, dividing by the common factor, we obtain the following linear homogeneous equation with constant coefficients for the determination of $p(z)$:

$$\frac{d^2\bar{p}}{dz^2} - k^2\bar{p} = 0. \tag{12'}$$

The general solution of this equation is

$$\bar{p}(z) = C_1 e^{kz} + C_2 e^{-kz}. \tag{13}$$

Consequently

$$p = (C_1 e^{kz} + C_2 e^{-kz})\, e^{i(\sigma t - kx)}. \tag{14}$$

Further, from (11) we have

$$\left.\begin{aligned} u &= \frac{k}{\rho(\sigma - kU)} (C_1 e^{kz} + C_2 e^{-kz})\, e^{i(\sigma t - kx)}, \\ w &= i \frac{k}{\rho(\sigma - kU)} (C_1 e^{kz} - C_2 e^{-kz})\, e^{i(\sigma t - kx)}. \end{aligned}\right\} \tag{15}$$

We will apply now the boundary condition.

Since $z = -h$ $w = 0$, for $C_1 e^{-kh} - C_2 e^{+kh} = 0$,

we have $$C_1 e^{-kh} = C_2 e^{+kh} = \frac{C}{2},$$

where C is an arbitrary constant. Thus we have

$$\begin{aligned} C_1 e^{kz} + C_2 e^{-kz} &= \frac{C}{2} [e^{k(z+h)} + e^{-k(z+h)}], \\ C_1 e^{kz} - C_2 e^{-kz} &= \frac{C}{2} [e^{k(z+h)} - e^{-k(z+h)}]. \end{aligned}$$

But the expressions in the square brackets are respectively the hyperbolic cosine chk(z+h) and hyperbolic sine shk(z+h). *

Consequently the final solution of this problem will be:

$$\left.\begin{aligned} u &= \frac{Ck}{\rho(\sigma - kU)} \operatorname{ch} k(z+h)\, e^{i(\sigma t - kx)}, \\ w &= i \frac{Ck}{\rho(\sigma - kU)} \operatorname{sh} k(z+h)\, e^{i(\sigma t - kx)}, \\ p &= C \cdot \operatorname{ch} k(z+h)\, e^{i(\sigma t - kx)}. \end{aligned}\right\} \tag{16}$$

For the profile of the wave we obtain the following two expressions

$$\left.\begin{aligned} \zeta(x, t) &= \frac{1}{g\rho} \cdot p(x, 0, t) = \frac{C}{g\rho} \operatorname{ch} kh e^{i(\sigma t - kx)}, \\ \zeta(x, t) &= \frac{w(x, 0, t)}{i(\sigma - kU)} = \frac{Ck}{\rho(\sigma - kU)^2} \cdot \operatorname{sh} kh \cdot e^{i(\sigma t - kx)}. \end{aligned}\right\} \tag{17}$$

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* [Translator's note - in the Russian text the k is missing, an obvious misprint.]

Evidently these two expressions should be identical. Hence, denoting the amplitude of the wave by A:

$$A = \frac{C}{g\rho} \operatorname{ch} kh = \frac{Ck}{\rho(\sigma - kU)^2} \operatorname{sh} kh, \tag{18}$$

and equating the right sides of equations (17), we obtain:

$$(\sigma - kU)^2 = gk \operatorname{th} kh, \tag{19}$$

where th denotes the hyperbolic tangent

$$\operatorname{th} x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Retaining only the real part we obtain the solution of our problem in the following form:

$$\left.\begin{aligned} u &= A(\sigma - kU)\frac{\operatorname{ch} k(h+z)}{\operatorname{sh} kh} \cos(\sigma t - kx), \\ w &= -A(\sigma - kU)\frac{\operatorname{sh} k(h+z)}{\operatorname{sh} kh} \cdot \sin(\sigma t - kx), \\ p &= A g\rho \cdot \frac{\operatorname{ch} k(h+z)}{\operatorname{ch} kh} \cdot \cos(\sigma t - kx), \\ \zeta &= A \cdot \cos(\sigma t - kx) = A\cos\sigma\left(t - \frac{k}{\sigma}x\right). \end{aligned}\right\} \tag{20}$$

Thus, the wave profile represents a cosinusoid, progressing with the velocity

$$c = \frac{\sigma}{k} \tag{21}$$

in the positive x-direction.

Expressing the velocity of propagation of the wave by the equation (19) and setting $k = \frac{2\pi}{\lambda}$, we obtain

$$c = \frac{\sigma}{k} = U \pm \sqrt{\frac{g\lambda}{2\pi} \operatorname{th} \frac{2\pi h}{\lambda}}. \tag{22}$$

Thus, the velocity of propagation of the wave is composed of two terms. The first term U indicates that the wave follows the current with the same velocity U. This is the "convective term". When $U = 0$, the velocity of propagation of the wave is determined by the second term. The second term depends on the gravity g, upon the wave length λ and on the ratio h/λ between the depth of the layer h and the wave length. This term is called the "dynamic term":

$$c = \pm \sqrt{\frac{g\lambda}{2\pi} \operatorname{th} \frac{2\pi h}{\lambda}}. \tag{23}$$

The alternative sign in front of the root indicates that the wave may propagate in either the positive or negative direction.

Let us consider two extreme cases, namely, when the waves move on a very deep or a very shallow fluid layer.

For very large values of x

$$\text{th}\, x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \approx 1,$$

and for very small values of x

$$\text{th}\, x = \frac{1 + x - (1 - x)}{(1 + x) + (1 - x)} = x.$$

Consequently, for a very deep layer

$$c = \sqrt{\frac{g\lambda}{2\pi}}, \tag{24}$$

and for a very shallow layer

$$c = \sqrt{gh}. \tag{25}$$

Thus, we have obtained Airy's formula (24) for the wave velocity in very deep water and Lagrange's formula (25) for the velocity of "long waves" (long in comparison with the depth) in shallow water.

Table 53 gives the propagation velocities for different wave lengths at different depths of the fluid layer, calculated according to formula (23).

Table 53

Velocities of propagation of gravitational waves on the surface of a resting fluid

h \ λ	1	10	100	500	1000	5000	10 000	100 000	100 000	∞
1	1	3	3	3	3	3	3	3	3	3
10	1	4	9	10	10	10	10	10	10	10
100	1	4	12	26	29	31	31	31	31	31
1000	1	4	12	28	39	81	93	99	99	99
10 000	1	4	12	28	39	88	125	295	313	313
∞	1	4	12	28	39	88	129	394	1250	

Let us discuss in detail the applicability of the formulas (24) and (25).

For $h/\lambda = 0.42$ the value of $\text{th}(2\pi h/\lambda) = 0.99$, i.e., only 1 % less than $2\pi h/\lambda$, and the velocity c, obtained from the formula (24) would be by 0.5 % greater than the wave velocity, as determined by formula (23). Consequently, formula (24) is sufficiently accurate for cases in which the depth is not less than 0.4 of the wave length: $h > 0{,}4\,\lambda$. It is clear from the foregoing what is the meaning of a very deep fluid. The concept of a "very deep" fluid may be misleading without this estimate. When $h/\lambda = 0.027$, the value of $\text{th}(2\pi h/\lambda)$ will differ from $2\pi h/\lambda$ only by 1 %. Therefore the speed c calculated by the approximate formula (25) will differ from the exact value as determined by formula (23) by less than 0.5 %.

For the approximate determination of the velocity of waves propagating in the atmosphere we often employ formula (25), since the ratio $h/\lambda \ll 1$ as a result of the vast horizontal extension of the atmosphere as compared with the thickness of atmospheric layers.

As an example we calculate the velocity of a wave propagating on the surface of a homogeneous (equally dense) atmosphere. The height of a homogeneous atmosphere is $H = RT_0/g$. Therefore for $T_0 = 293$ $c = \sqrt{RT_0}$,

$$c = \sqrt{RT_0} = \sqrt{293 \cdot 287} = 290 \text{ m/sec}$$

Thus, the speed of propagation of long waves on the free surface of a homogeneous atmosphere is equal to the speed of sound.

§ 5. Plane Waves on the Interface between Two Currents

We shall now consider the wave motion developing on a surface separating two fluids of different densities ρ_1 and ρ_2 and different fundamental velocities U_1 and U_2.

In the atmosphere this corresponds to waves on a discontinuity surface.

A necessary condition for the development of such wave motion is the presence of a cool air mass below a warm mass. Designating, as in the preceding section, the values referring to the warm air by index 1 and those referring to the cold air by the index 2, we assume that the first mass lies above the second. At first we neglect the rotation of the earth. In this case the unperturbed surface of discontinuity will be horizontal.

We identify the surface of discontinuity with the (x, y) plane, and mark the origin of coordinates on this plane; the z-axis will be directed vertically upwards and the x-axis parallel to the (common) direction of motion of the two fluids.

We assume that the upper fluid is limited from above by a rigid plane boundary whose equation is $z = h_1$, and the lower fluid limited by the rigid plane $z = -h_2$.

Assuming the motion to be plane parallel, the components of the velocity of the upper fluid will be $U_1 + u_1$, w_1 and of the lower fluid $U_2 + u_2$, w_2; the pressure in the upper fluid $P_1 + p_1$ and in the lower $P_2 + p_2$. Each of the two fluids is regarded as incompressible so that $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$, whereby $\rho_1 > \rho_2$. The equation of the perturbed interface will be $z = \zeta(x, t)$.

The equations of small perturbations will then have the form

$$\left.\begin{aligned} &\frac{\partial u_1}{\partial t} + U_1 \frac{\partial u_1}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}; \quad \frac{\partial u_2}{\partial t} + U_2 \frac{\partial u_2}{\partial x} = -\frac{1}{\rho_2}\frac{\partial p_2}{\partial x}, \\ &\frac{\partial w_1}{\partial t} + U_1 \frac{\partial w_1}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial z}; \quad \frac{\partial w_2}{\partial t} + U_2 \frac{\partial w_2}{\partial x} = -\frac{1}{\rho_2}\frac{\partial p_2}{\partial z}, \\ &\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0, \quad \frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial z} = 0. \end{aligned}\right\} \qquad (1)$$

The boundary conditions will be

$$\left.\begin{aligned} w_1 &= 0 \quad \text{at} \quad z = h_1, \\ w_2 &= 0 \quad \text{at} \quad z = -h_2, \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} p_1 - p_2 &= g(\rho_1 - \rho_2)\zeta, \\ w_1 &= \frac{\partial \zeta}{\partial t} + U_1 \frac{\partial \zeta}{\partial x}, \\ w_2 &= \frac{\partial \zeta}{\partial t} + U_2 \frac{\partial \zeta}{\partial x} \end{aligned}\right\} \quad \text{at} \quad z = 0. \tag{3}$$

As in the case of a layer consisting of a single fluid layer we seek a solution in the form

$$\left.\begin{aligned} u_1(x, z, t) &= \bar{u}_1(z)\, e^{i(\sigma t - kx)}; \quad u_2(x, z, t) = \bar{u}_2(z)\, e^{i(\sigma t - kx)}; \\ w_1(x, z, t) &= \bar{w}_1(z)\, e^{i(\sigma t - kx)}; \quad w_2(x, z, t) = \bar{w}_2(z)\, e^{i(\sigma t - kx)}; \\ p_1(x, z, t) &= \bar{p}_1(z)\, e^{i(\sigma t - kx)}; \quad p_2(x, z, t) = \bar{p}_2(z)\, e^{i(\sigma t - kx)}; \end{aligned}\right\} \tag{4}$$

$$\zeta(x, t) = \zeta_0\, e^{i(\sigma t - kx)}.$$

Repeating the discussion of the preceding section and adjusting the conditions (2) on the rigid boundaries we obtain

$$\left.\begin{aligned} u_1 &= C_1 \operatorname{ch} k(z + h_1) \cos(\sigma t - kx); \\ u_2 &= C_2 \operatorname{ch} k(z - h_2) \cos(\sigma t - kx); \\ w_1 &= -\frac{C_1 k}{\rho_1(\sigma - kU_1)} \operatorname{sh} k(z + h_1) \sin(\sigma t - kx); \\ w_2 &= -\frac{C_2 k}{\rho_2(\sigma - kU_2)} \operatorname{sh} k(z - h_2) \sin(\sigma t - kx); \\ p_1 &= C_1 \operatorname{ch} k(z + h_1) \cos(\sigma t - kx); \\ p_2 &= C_2 \operatorname{ch} k(z - h_2) \cos(\sigma t - kx); \\ \zeta &= \zeta_0 \cos(\sigma t - kx). \end{aligned}\right\} \tag{5}$$

Imposing the boundary condition (3) we obtain

$$\left.\begin{aligned} C_1 \operatorname{ch} kh_1 - C_2 \operatorname{ch} kh_2 &= g(\rho_1 - \rho_2)\zeta_0, \\ -\frac{C_1 k}{\rho_1(\sigma - kU_1)} \operatorname{sh} kh_1 &= \zeta_0(\sigma - kU_1), \\ \frac{C_2 k}{\rho_2(\sigma - kU_2)} \operatorname{sh} kh_2 &= \zeta_0(\sigma - kU_2). \end{aligned}\right\} \tag{6}$$

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* [Translator's note - in the Russian text the minus sign is missing.]

From the two last equations (6) we determine the arbitrary constants C_1 and C_2:

$$\left.\begin{aligned} C_1 &= -\zeta_0 \frac{\rho_1}{k \operatorname{sh} kh_1} (\sigma - kU_1)^2, \\ C_2 &= \zeta_0 \frac{\rho_2}{k \operatorname{sh} kh_2} (\sigma - kU_2)^2. \end{aligned}\right\} \tag{7}$$

Substituting these expressions for C_1 and C_2 in the first equation (6) we obtain an equation determining the frequency of the wave:

$$\rho_1 (\sigma - kU_1)^2 \operatorname{cth} kh_1 + \rho_2 (\sigma - kU_2)^2 \operatorname{cth} kh_2 = gk(\rho_2 - \rho_1). \tag{8}$$

Writing for the sake of brevity

$$\operatorname{cth} kh_1 = \operatorname{cth} \frac{2\pi h_1}{\lambda} = a_1,$$
$$\operatorname{cth} kh_2 = \operatorname{cth} \frac{2\pi h_2}{\lambda} = a_2,$$

we put equation (8) in the following form:

$$\sigma^2 (\rho_1 a_1 + \rho_2 a_2) - 2\sigma k (\rho_1 U_1 a_1 + \rho_2 U_2 a_2) + \\ + k^2 \left\{ \rho_1 U_1^2 a_1 + \rho_2 U_2^2 a_2 - \frac{g}{k} (\rho_2 - \rho_1) \right\} = 0. \tag{9}$$

Solving the quadratic equation (9) we obtain:

$$\sigma = k \frac{\rho_1 U_1 a_1 + \rho_2 U_2 a_2}{\rho_1 a_1 + \rho_2 a_2} \pm \\ \pm k \sqrt{\left(\frac{\rho_1 U_1 a_1 + \rho_2 U_2 a_2}{\rho_1 a_1 + \rho_2 a_2} \right)^2 - \frac{\rho_1 U_1^2 a_1 + \rho_2 U_2^2 a_2}{\rho_1 a_1 + \rho_2 a_2} + \frac{g}{k} \frac{\rho_2 - \rho_1}{\rho_1 a_1 + \rho_2 a_2}}. \tag{10}$$

Or, dividing the two sides of the equation by k (after trivial transformations) we obtain a formula for the wave velocity c:

$$c = \frac{\rho_1 U_1 a_1 + \rho_2 U_2 a_2}{\rho_1 a_1 + \rho_2 a_2} \pm \\ \pm \sqrt{\frac{g\lambda}{2\pi} \frac{\rho_2 - \rho_1}{\rho_1 a_1 + \rho_2 a_2} - \rho_1 \rho_2 a_1 a_2 \left(\frac{U_1 - U_2}{\rho_1 a_1 + \rho_2 a_2} \right)^2}. \tag{11}$$

Formula (22) of the preceding section is a particular case of formula (11). Indeed, putting $\rho_1 = 0$ and omitting the subscript 2, we obtain:

$$a = \operatorname{cth} kh = \frac{e^{kh} + e^{-kh}}{e^{kh} - e^{-kh}} = 1.$$

In analogy with (22) § 4 the first term in the formula (11) will be called the convective velocity of the wave. Obviously, the convective velocity has some intermediate value: it is greater than the velocity of the slow current and smaller than the velocity of the rapid current. When the two layers are of great thickness, we get:

$$c = U \pm \sqrt{\frac{g\lambda}{2\pi} \operatorname{th} \frac{2\pi h}{\lambda}}.$$

The convective velocity c of the wave will then be equal to the mean flow velocity

$$c' = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \tag{12}$$

For layers of great thickness the second term of formula (11) can be rewritten as follows:

$$c'' = \pm \sqrt{\frac{g\lambda}{2\pi}\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} - \rho_1\rho_2\left(\frac{U_1 - U_2}{\rho_1 + \rho_2}\right)^2}. \tag{13}$$

This velocity, in analogy with equation (23) of the preceding section, will be called the dynamic velocity of the wave. As can be seen from (13) the dynamic velocity depends on two factors, namely, the density difference and the velocity difference. To clarify the role of each single factor we shall discuss a few relevant cases.

Let us assume that the oscillations of the interface are executed around a static equilibrium position so that $u_1 = u_2 = 0$. Then we get

$$c = \pm \sqrt{\frac{g\lambda}{2\pi}\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}}. \tag{14}$$

This expression differs from the wave velocity on the free surface by the factor

$$\sqrt{\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}}.$$

For waves on the interface between water and air we may adopt $\rho_2 = 1$; $\rho_1 = 0.0013$. Then we have

$$\sqrt{\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}} = \sqrt{\frac{(\rho_2 - \rho_1)(\rho_2 - \rho_1)}{(\rho_2 + \rho_1)(\rho_2 - \rho_1)}} \approx \frac{\rho_2 - \rho_1}{\rho_2} = 1 - 0.0013.$$

Therefore

$$c = \sqrt{\frac{g\lambda}{2\pi}}(1 - 0.0013).$$

Consequently, the error in c, introduced by disregarding the wave motion of the air does not exceed 0.13%.

However, the wave motions on atmospheric interfaces are carried out under quite different conditions.

To estimate the order of magnitude of this factor for atmospheric conditions it is expedient to use the temperature, determined from the density ρ by Clayperon's equation $\rho_1 = p_1/RT_1$, $\rho_2 = p_2/RT_2$. Noting that the pressure undergoes no discontinuous change at the interface, $p_1 = p_2$; then the following obtains:

$$c = \pm \sqrt{\frac{g\lambda}{2\pi}\frac{T_1 - T_2}{T_1 + T_2}}. \tag{15}$$

For $T_1 = 283$, $T_2 = 273$, we have

$$\sqrt{\frac{T_1 - T_2}{T_1 + T_2}} = \sqrt{\frac{10}{556}} = \sqrt{0.0180} \approx 0.13.$$

We finally obtain:

$$c = 0.13 \sqrt{\frac{g\lambda}{2\pi}}.$$

Waves of the same length therefore propagate much more slowly on an atmospheric interface than on a free surface.

One important point should be stressed.

When $T_1 < T_2$, i.e., when the upper layer is colder than the lower, the wave velocity becomes imaginary. This means that the frequency σ will be imaginary.

Therefore, writing $\sigma = is$, where s is a real number, we obtain for ζ:

$$\zeta = Ae^{s,t} \cos kx.$$

Consequently the disturbance will increase indefinitely with time. Therefore, the fundamental state is considered unstable and wave motion at unstable stratification is impossible.

When the depth of both fluid layers is small in comparison with the wave length, we have

$$a_1 = \operatorname{cth} kh_1 = \frac{1}{kh_1}, \qquad a_2 = \operatorname{cth} kh_2 = \frac{1}{kh_2},$$

and consequently the formula for the velocity of waves on the interface of two resting fluid masses will assume the following form

$$c = \sqrt{\frac{g}{k} \frac{\rho_2 - \rho_1}{\frac{\rho_1}{kh_1} + \frac{\rho_2}{kh_2}}} = \sqrt{\frac{g(\rho_2 - \rho_1) \cdot h_1 h_2}{h_2\rho_1 + h_1\rho_2}}$$

or, introducing the temperature instead of the density:

$$c = \sqrt{\frac{g(T_1 - T_2)\, h_1 h_2}{h_1 T_1 + h_2 T_2}} = \sqrt{gh_2} \cdot \sqrt{\frac{T_1 - T_2}{T_1 + \frac{h_2}{h_1} T_2}}.$$

As in the case of an infinitely thick fluid layer, the presence of the second layer leads to the decrease of the wave velocity. Thus, for $h_1 = h_2$ and for the same values of T_1 and T_2 we have:

$$c = 0.13 \sqrt{gh}.$$

Finally, we examine the case where the thickness of the lower layer is small in comparison with the wave length and the upper layer extends upwards to infinity. Then we get

$$a_2 = \frac{1}{kh_2}; \quad a_1 = 1,$$

and consequently

$$c = \sqrt{\frac{g}{k} \frac{\rho_2 - \rho_1}{\frac{\rho_2}{kh_2} + \rho_1}} = \sqrt{gh_2 \frac{\rho_2 - \rho_1}{\rho_2}} = \sqrt{gh_2 \frac{(T_1 - T_2)}{T_1}}.$$

For the same values of T_1 and T_2 we obtain

$$c = 0.19\sqrt{gh}.$$

We now assume that there is no discontinuity in the density so that $\rho_1 = \rho_2 = \rho$, and at the same time the velocity is discontinuous $U_1 \neq U_2 \neq 0$. Then, for infinitely thick layers we obtain from (11)

$$c = U_1 + U_2 \pm i(U_1 - U_2).$$

Since the dynamic term depends now only upon the velocity difference, such waves may be called "displacement waves". Such waves may be also called inertial waves, since in this case wave motion can develop only at the expense of the kinetic energy of the fundamental motion. However, displacement waves are unstable, as can be seen from (16).

We shall turn now to a more general case, when both the wind velocity and the density are discontinuous on the interface. Then the dynamic velocity will be determined in the following way:

$$c'' = \sqrt{\frac{g\lambda}{2\pi}\frac{\rho_2 - \rho_1}{\rho_2 a_2 + \rho_1 a_1} - \rho_1\rho_2 a_1 a_2 \left(\frac{U_1 - U_2}{\rho_2 a_2 + \rho_1 a_1}\right)^2}. \tag{17}$$

When the expression under the root is positive the fundamental motion is stable, and when this expression is negative the motion is unstable. Therefore, the condition of instability of two currents—and consequently of their interface—is the following:

$$\frac{g\lambda}{2\pi}\frac{\rho_2 - \rho_1}{\rho_2 a_2 + \rho_1 a_1} - \rho_1\rho_2 a_1 a_2 \left(\frac{U_1 - U_2}{\rho_2 a_2 + \rho_1 a_1}\right)^2 < 0. \tag{18}$$

When the wave length and the densities of the two fluids are known, it is more convenient to isolate the velocity difference from the condition of instability

$$(U_1 - U_2)^2 > \frac{g\lambda}{2\pi}(\rho_2 - \rho_1)\left(\frac{1}{\rho_2 a_2} + \frac{1}{\rho_1 a_1}\right). \tag{19}$$

Hence the following conclusion can be drawn at a known wave length and density difference; the fundamental motion is unstable for great velocity differences and stable for small velocity differences of the fundamental motion.

Let us isolate λ from the instability condition (19), regarding the velocity difference as given. The fundamental motion is unstable when the following inequality is fulfilled

$$\lambda < \frac{2\pi}{g(\rho_2 - \rho_1)}\frac{(U_1 - U_2)^2 \rho_1\rho_2 a_1 a_2}{\rho_1 a_1 + \rho_2 a_2}. \tag{20}$$

Thus for given velocity and density differences the fundamental motion is unstable with respect to short waves and stable with respect to long waves.

We shall examine now the importance of the depth of the fluid layer in the stability of motion. For this purpose we introduce into the condition of instability (20) the depth of the layers h_1 and h_2 in the obvious way; then this condition will become:

$$\lambda < \frac{2\pi}{g(\rho_2 - \rho_1)}\frac{(U_1 - U_2)^2 \rho_1\rho_2}{\rho_1 \operatorname{th} kh_2 + \rho_2 \operatorname{th} kh_1}. \tag{21}$$

Consequently, the fundamental motion is stable for deep layers and unstable for shallow ones.

We should note that the dynamic wave velocity vanishes at the limit of stability.

Thus, at the limit of stability the velocity of propagation equals the convective velocity.

Let us write the expression for the critical wave length λ, assuming that the two layers are infinitely thick and introducing the temperature T instead of the density ρ. We then obtain

$$\lambda = \frac{2\pi}{g} \frac{(u_1 - u_2)^2 T_1 T_2}{T_1^2 - T_2^2}.$$

Table 54 gives the critical wave lengths for instability at $T = 273°$. Shorter waves are unstable and longer waves are stable.

Table 54

Critical wave length in meters

Δu m/sec / $\Delta T°C$	4	8	12	16	20
4	352	1408	3668	5616	8789
8	177	709	1595	2829	4426
12	119	475	1073	1901	2974
16	90	359	808	1431	2244
20	73	289	652	1156	1809

We recall that gravitational waves are always stable in regard to the fundamental motion, while the displacement waves are always unstable. Therefore, the force of gravity exerts a stabilizing effect on the fundamental motion, while displacement tends to unsettle it. When the stabilizing effect of the force of gravity prevails over the unsettling effect of the displacement, the motion is stable; in the opposite case it becomes unstable.

§ 6. Small Oscillations of an Interface

The problem of stability of interfaces is closely connected with the generation of cyclones. No generally accepted cyclone theory exists at the present time. Of all existing qualitative theories of cyclogenesis, the most popular is the "frontologic" or wave-cyclone theory. This theory is based on the well known fact that almost all cyclones of middle and polar latitudes have their origin in immense waves, formed on the discontinuity surface between almost resting and rectilinearly moving air masses.

However, not all wave disturbances of stationary fronts develop into cyclones. In some cases the wave disturbance passing the front leaves it intact; in other cases the wave disturbances leads to a vortex motion near the front, whereby the front does not return to the initial position. The wave becomes prominent, and a field of closed isobars and of low pressure is formed, i.e., a cyclone.

In connection with this circumstance a serious problem was raised. Can there be found quantitative criteria permitting to judge when and where a wave will be formed in the front, subsequently developing into a cyclone? The Norwegian school of meteorology identifies the birth of a cyclone wave with the moment the frontal surface loses stability. Evidently the loss of stability can be associated with the operation of mechanical and thermodynamic factors.

In the following sections only mechanical factors will be discussed.

Many attempts to establish criteria of stability did not lead to substantial results. The main difficulty of this problem lies in the fact that the surface of discontinuity is inclined to the rigid surface of the earth. Therefore, the problem of wave propagation on the frontal surface is three-dimensional and consequently very complicated.

Certain foreign investigators considered wave disturbances either on horizontal or on vertical discontinuity surfaces. Bjerknes and his collaborators, as reported in his "Physical hydrodynamics" considered wave disturbances on a sloping frontal surface, dividing two air masses of infinite vertical extension. All these investigations did not yield any results.

The problem of stability of an inclined interface with respect to wave motion was solved for the first time by N. E. Kochin. He offered a theoretical basis for wave-cyclone formation.

Kochin solved this problem by the method of small oscillations, assuming that the vertical accelerations can be omitted from the equation of motion. With the aid of ingenious coordinate transformations Kochin fitted the entire complex of boundary conditions into the equations of motion.

We now apply the equations of small oscillations given in § 2 to derive Kochin's solution of the problem. As fundamental motion we adopt the steady rectilinear motion of two fluid masses parallel to the interface. We assume that the densities of the two fluids are constant and equal to ρ_1 and ρ_2 respectively. We use a right-handed coordinate system (Figure 151). The axis Ox is directed parallel to the motion, and the axis Oz–vertically upwards. The axis Oy is normal to the front, pointing in the direction of the cold mass. With the foregoing assumptions for the fundamental motion we have:

$$\left.\begin{aligned} U_1 &= \text{const.};\ V_1 = 0;\ W_1 = 0;\ Q_1 = \rho_1 = \text{const.};\\ U_2 &= \text{const.};\ V_2 = 0;\ W_2 = 0;\ Q_2 = \rho_2 = \text{const.} \end{aligned}\right\} \tag{1}$$

And

$$X = 0;\ Y = 0;\ Z = -g. \tag{2}$$

From equations (2) § 2 follows that the pressures in the warm and cold masses for the fundamental regime will be:

$$\left.\begin{aligned} P_1 &= -g\rho_1 z - 2\rho_1 U_1 (\omega_z y - \omega_y z) + C,\\ P_2 &= -g\rho_2 z - 2\rho_2 U_2 (\omega_z y - \omega_y z) + C, \end{aligned}\right\} \tag{3}$$

where C is the pressure at the origin of coordinates. The equation of the interface for the fundamental regime, when the condition $P_1 = P_2$, is satisfied, will be:

$$(\rho_2 - \rho_1) gz + 2\omega_y (\rho_1 U_1 - \rho_2 U_2) z - 2\omega_z (\rho_1 U_1 - \rho_2 U_2) y = 0, \tag{4}$$

or

$$z = y \operatorname{tg} \alpha, \tag{5}$$

where

$$\operatorname{tg} \alpha = \frac{2\omega_x (\rho_2 U_2 - \rho_1 U_1)}{g(\rho_1 - \rho_2) + 2\omega_y (\rho_2 U_2 - \rho_1 U_1)}. \tag{6}$$

Consequently, in the fundamental regime the interface is a plane inclined to the horizon at an angle α, determined by Margules' formula (6).

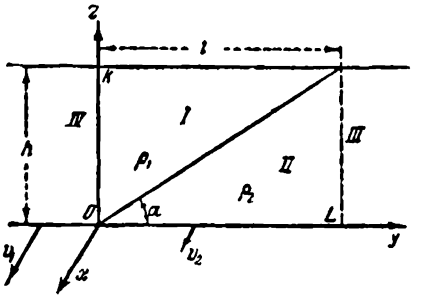

Figure 151. To the derivation of the formulas of small oscillations of an interface

Consider now small oscillations superimposed on the fundamental motion. Let the resulting nonstationary motion be very close to the fundamental stationary motion, so that the components of the velocity and the pressure in the warm and cold masses will be respectively:

$$\begin{gathered} U_1 + u_1,\ v_1,\ w_1,\ P_1 + p_1, \\ U_2 + u_2,\ v_2,\ w_2,\ P_2 + p_2, \end{gathered}$$

and the densities, as assumed before, are constant: $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$ Whereby u, v, w, p are regarded as infinitely small.

According to (13) and (14) § 2, we have the following equations for the determination of the sought functions u, v, w, p

$$\left.\begin{aligned} \frac{\partial u_j}{\partial t} + U_j \frac{\partial u_j}{\partial x} &= -\frac{1}{\rho_j}\frac{\partial p_j}{\partial x} - 2\omega_y w_j + 2\omega_z v_j, \\ \frac{\partial v_j}{\partial t} + U_j \frac{\partial v_j}{\partial x} &= -\frac{1}{\rho_j}\frac{\partial p_j}{\partial y} - 2\omega_z u_j + 2\omega_x w_j, \\ \frac{\partial w_j}{\partial t} + U_j \frac{\partial w_j}{\partial x} &= -\frac{1}{\rho_j}\frac{\partial p_j}{\partial z} - 2\omega_x v_j + 2\omega_y u_j, \\ \frac{\partial u_j}{\partial x} + \frac{\partial v_j}{\partial y} + \frac{\partial w_j}{\partial z} &= 0 \qquad (j = 1, 2). \end{aligned}\right\} \tag{7}$$

Let us further assume that the two masses are bounded by fixed horizontal planes, below by the plane $z = 0$, and above by the plane $z = h$. On these planes the following conditions should hold:

$$\left.\begin{aligned} w_1 &= 0 \quad \text{at} \quad z = h, \\ w_2 &= 0 \quad \text{at} \quad z = 0. \end{aligned}\right\} \tag{8}$$

The equation of the surface of discontinuity for the ensuing nonstationary regime will be written as follows:

$$z = y \operatorname{tg} \alpha + \zeta(x, y, t), \tag{9}$$

where ζ is a very small quantity. According to (20) § 2 we then have for the perturbed motion:

$$\begin{aligned} p_2 - p_1 = P_1 - P_2 = \\ = (\rho_2 - \rho_1) gz + 2(\rho_1 U_1 - \rho_2 U_2) \omega_y z - 2(\rho_1 U_1 - \rho_2 U_2) \omega_z y = \\ = [(\rho_2 - \rho_1) g + 2(\rho_1 U_1 - \rho_2 U_2) \omega_y] (z - y \operatorname{tg} \alpha) \end{aligned} \tag{10}$$

or

$$p_2 - p_1 = [(\rho_2 - \rho_1) g + 2(\rho_1 U_1 - \rho_2 U_2) \omega_y] \zeta \tag{11}$$

on the surface

$$z = y \operatorname{tg} \alpha + \zeta(x, y, t).$$

Besides, for the same surface, according to (18) § 2 the following condition holds:

$$\begin{aligned} &\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} U_j + v_j \operatorname{tg} \alpha - w_j = 0 \qquad (j = 1, 2) \\ &\text{for} \quad z = y \operatorname{tg} \alpha + \zeta, \end{aligned} \tag{12}$$

which is verified by taking:

$$F = y \operatorname{tg} \alpha - z; \quad f = \zeta(x, y, t). \tag{13}$$

At this stage Kochin makes an essential and bold assumption – he omits the vertical accelerations:

$$\frac{\partial w_j}{\partial t} + U_j \frac{\partial w_j}{\partial x}$$

from the equations of motion. The vertical accelerations are neglected frequently in this way in the study of long waves. Such omission is equivalent to the assumption that the vertical pressure distribution obeys the barometric formula. Thus p_1 and p_2 are independent of the coordinate z and are functions of x, y, and t only. Similarly we assume that the perturbations u_1, u_2, v_1, v_2 are functions of x, y and t. These assumptions are justified by the fact that the vertical extent of the surface of discontinuity is many times smaller than the horizontal extent.

We find w_1 and w_2 from the last equation (7):

$$\left.\begin{aligned} w_1 &= \int_z^h \left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right) dz = (h-z)\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_1}{\partial y}\right), \\ w_2 &= -\int_0^z \left(\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}\right) dz = -z\left(\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}\right). \end{aligned}\right\} \qquad (14)$$

Equations (14) show that the vertical velocities $w_j (j = 1, 2)$ are small in comparison with horizontal velocities, for owing to the small slope gradient, the vertical displacement is considerably less than the horizontal. Therefore, in the first two equations (7) the terms containing w_j can be rejected and consequently ω_x and ω_y will disappear from the equations. The terms containing ω_x and ω_y are omitted by Kochin even from the third equation (7). Writing for brevity $\omega_z = \omega$, we obtain, on the basis of the foregoing assumptions, the following seven equations for the determination of the seven functions u_1, v_1, u_2, v_2, p_1, p_2, ζ:

$$\left.\begin{aligned} &\left.\begin{aligned} \frac{\partial u_j}{\partial t}+U_j\frac{\partial u_j}{\partial x} &= -\frac{1}{\rho_j}\frac{\partial p_j}{\partial x}+2\omega v_j, \\ \frac{\partial v_j}{\partial t}+U_j\frac{\partial v_j}{\partial x} &= -\frac{1}{\rho_j}\frac{\partial p_j}{\partial y}-2\omega u_j, \end{aligned}\right\} \quad (j=1,2) \\ &g(\rho_2-\rho_1)\zeta = p_2-p_1, \\ &\frac{\partial \zeta}{\partial t}+U_j\frac{\partial \zeta}{\partial x} = -v_1 \operatorname{tg}\alpha+(h-y\operatorname{tg}\alpha)\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right), \\ &\frac{\partial \zeta}{\partial t}+U_2\frac{\partial \zeta}{\partial x} = -v_2\operatorname{tg}\alpha-y\operatorname{tg}\alpha\left(\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}\right) \end{aligned}\right\} \qquad (15)$$

($y \operatorname{tg}\alpha$ is substituted for z in the last two equations, and the second order magnitudes ζw_1 and ζw_2 are disregarded).

§ 7. Zonal Oscillations of an Interface

We shall now examine oscillations in the direction perpendicular to the front, i.e., the so-called zonal oscillations. As an example of zonal oscillations may serve the oscillations of an interface having the form of a surface of revolution around the terrestrial axis. This surface separates the polar cap of cold air from the warmer tropical air. The zonal oscillation of this interface is performed in the meridian plane and is independent of the longtitude. In the considered case none of the seven unknown functions depends upon the coordinate x therefore, equations (15) of the preceding section are simplified and take the following form:

$$\left.\begin{aligned}
&\frac{\partial u_j}{\partial t} = 2\omega v_j,\\
&\frac{\partial v_j}{\partial t} = -\frac{1}{\rho_j}\frac{\partial p_j}{\partial y} - 2\omega u_j,\\
&p_2 = p_1 = g(\rho_2 - \rho_1)\zeta,\\
&\frac{\partial \zeta}{\partial t} = -v_1 \operatorname{tg}\alpha + (h - y\operatorname{tg}\alpha)\cdot\frac{\partial v_1}{\partial y},\\
&\frac{\partial \zeta}{\partial t} = -v_2 \operatorname{tg}\alpha - y\operatorname{tg}\alpha\cdot\frac{\partial v_2}{\partial y}.
\end{aligned}\right\} \qquad (1)$$

As a first step towards the integration of the system (1) of differential equations of the first order we can first of all reduce by differentiations the number of the equations and the number of the unknown functions, but at the expense of increasing the order of the equations. Differentiating the third equation with respect to y and applying the second equation we obtain:

$$g(\rho_2 - \rho_1)\frac{\partial \zeta}{\partial y} = \frac{\partial p_2}{\partial y} - \frac{\partial p_1}{\partial y} = \rho_1\frac{\partial v_1}{\partial t} - \rho_2\frac{\partial v_2}{\partial t} + 2\omega(\rho_1 u_1 - \rho_2 u_2). \qquad (2)$$

Thus we have eliminated the unknown functions p_1 and p_2, reducing the number of the equations to five. We eliminate another two unknown functions, u_1 and u_2. Differentiating equation (2) with respect to t and applying the third equation (1) we obtain in place of the five equations (1) one equation

$$g(\rho_2 - \rho_1)\frac{\partial^2 \zeta}{\partial y\,\partial t} = \rho_1\frac{\partial^2 v_1}{\partial t^2} - \rho_2\frac{\partial^2 v_2}{\partial t^2} + 4\omega^2(\rho_1 v_1 - \rho_2 v_2). \qquad (3)$$

Finally after elimination of ζ we obtain from the last two equations (1):

$$\frac{\partial[(h - y\operatorname{tg}\alpha)v_1]}{\partial y} = -\frac{\partial[v_2 y\operatorname{tg}\alpha]}{\partial y} \qquad (4)$$

$$(h - y\operatorname{tg}\alpha)v_1 + y\operatorname{tg}\alpha v_2 = f(t). \qquad (5)$$

Let us determine the form of the function $f(t)$. For $y = 0$, we obtain

$$h v_1(0, t) = f(t).$$

Thus the function $f(t)$ is determined by the boundary condition, namely, by specifying the function $v_1(y, t)$ at y = 0*. For simplicity we can put $f(t) = 0$.

Following Kochin, we introduce a new auxiliary function $V(y, t)$ assuming

$$V(y, t) = (h - y\operatorname{tg}\alpha)v_1 = -y\operatorname{tg}\alpha v_2. \qquad (6)$$

Then

$$v_1 = \frac{V}{h - y\operatorname{tg}\alpha}; \quad v_2 = -\frac{V}{y\cdot\operatorname{tg}\alpha}, \qquad (7)$$

---

* [Translator's note - the russian text has f(t) = 0 instead of y = 0, an obvious misprint.]

whence

$$\frac{\partial^2 v_1}{\partial t^2} = \frac{\frac{\partial^2 V}{\partial t^2}}{h - y \operatorname{tg} \alpha}; \quad \frac{\partial^2 v_2}{\partial t^2} = -\frac{\frac{\partial^2 V}{\partial t^2}}{y \cdot \operatorname{tg} \alpha}. \tag{8}$$

Besides, it follows from (6) and (1) that

$$\frac{\partial V}{\partial y} = \frac{\partial \zeta}{\partial t}. \tag{9}$$

Substituting (8) and (9) in (3) we obtain:

$$g(\rho_2 - \rho_1)\frac{\partial^2 V}{\partial y^2} - \left(\frac{\rho_1}{h - y \operatorname{tg} \alpha} + \frac{\rho_2}{y \operatorname{tg} \alpha}\right)\left(\frac{\partial^2 V}{\partial t^2} + 4\omega^2 V\right) = 0. \tag{10}$$

Thus the system of equation (1) is reduced to one differential equation of the second order in one unknown function $V(y, t)$.

Let us determine the boundary conditions for this new unknown function. The discontinuity surface $z = y \operatorname{tg} \alpha$ intersects the upper fixed boundary $z = h$ along the straight line $y = l$; $z = h$ where $l = h \operatorname{ctg} \alpha$ (Figure 151). Therefore we consider equation (10) only within the limits $0 \leqslant y \leqslant l$. At these limits, according to (6), the following should hold:

$$V(0, t) = V(l, t) = 0, \tag{11}$$

providing we accept the natural condition that v_1 and v_2 do not become infinite at $y = 0$ and $y = l$.

The initial conditions for V and $\frac{\partial V}{\partial t}$ are obtained from (6) and (2):

$$V(y, 0) = (h - y \operatorname{tg} \alpha) \cdot v_1(y, 0),$$

$$\left(\frac{\rho_1}{h - y \operatorname{tg} \alpha} + \frac{\rho_2}{y \operatorname{tg} \alpha}\right) \cdot \frac{\partial V(y, 0)}{\partial t} =$$
$$= g(\rho_2 - \rho_1)\frac{\partial \zeta(y, 0)}{\partial y} + 2\omega\left[\rho_2 u_2(y, 0) - \rho_1 u_1(y, 0)\right]. \tag{12}$$

It is obvious that all the initial values for the first mass, u_1, v_1, ζ, and also the initial values of the tangential component of the velocity perturbation of the second mass u_2 can be arbitarily assigned.

In this way Kochin reduces the problem of zonal oscillations of an interface to the integration of the partial differential equation of the second order (10) for the boundary conditions (11) and the initial conditions (12).

Kochin further introduces a new dimensionless variable η, defined by

$$\eta = \frac{2y - l}{l}; \quad y = l\frac{\eta + 1}{2}. \tag{13}$$

Equation (10), after obvious transformations, takes the form:

$$(1+\eta^2)\frac{\partial^2 V}{\partial \eta^2}-\frac{l(\rho_1+\rho_2)}{2g(\rho_2-\rho_1)\operatorname{tg}\alpha}\left[1-\frac{\rho_2-\rho_1}{\rho_1+\rho_2}\cdot\eta\right]\left[\frac{\partial^2 V}{\partial t^2}+4\omega^2 V\right]=0, \quad (14)$$

and the boundary conditions (11) are easily brought to the form

$$V(\pm 1, t)=0. \quad (15)$$

Following Kochin, let us introduce a further simplification. We neglect the quotient in the square brackets $\frac{\rho_2-\rho_1}{\rho_1+\rho_2}$ which is small compared with unity, and introduce the mean density $\rho=\frac{\rho_1+\rho_2}{2}$. Then equation (14) will take the following form

$$(1-\eta^2)\frac{\partial^2 V}{\partial \eta^2}-\frac{l\rho}{g(\rho_2-\rho_1)\operatorname{tg}\alpha}\left[\frac{\partial^2 V}{\partial t^2}+4\omega^2 V\right]=0. \quad (16)$$

Since we consider small oscillations,it is natural to seek the solution of the problem by Fourier's method in the form of a product of a periodic function of the time t and some function of η. Thus a particular solution of (16) is sought in the form

$$V(\eta, t)=e^{i\sigma t}W(\eta). \quad (17)$$

The substitution of (17) in (16) permits the determination of the unknown function $W(\eta)$ by the ordinary second order differential equation

$$(1-\eta^2)\frac{d^2W}{d\eta^2}+\mu W=0, \quad (18)$$

where we write for brevity:

$$\mu=\frac{l\rho(\sigma^2-4\omega^2)}{g(\rho_2-\rho_1)\operatorname{tg}\alpha}. \quad (19)$$

The boundary condition for W, as follows from (15), has the form

$$W(\pm 1, t)=0. \quad (20)$$

Equation (18) has two singular points, $\eta=+1$ and $\eta=-1$. But theorems of the theory of differential equations show that the solution of equation (18) for the boundary condition (20) should be holomorphic (i. e., developable in a power series) at the points $\eta=\pm 1$.

Differentiating equation (18) with respect to η, we obtain for the function $W'(\eta)$ the differential equation

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dW'}{d\eta}\right]+\mu W'=0. \quad (21)$$

This equation has a holomorphic solution at the points $\eta=\pm 1$ for particular values of μ only [eigenvalues], namely:

$$\mu=n(n+1),$$

where $n=0, 1, 2...$ The obtained solutions are rather well known special functions, the so-called Legendre polynomials:

$$W_n'(\eta)=P_n(\eta)=\frac{1}{2^n\cdot n!}\frac{d^n(\eta^2-1)^n}{d\eta^n}. \tag{22}$$

Integrating (22) with respect to η and applying condition (20), we obtain

$$W_n(\eta)=\frac{1}{2^n\cdot n!}\frac{d^{n-1}(\eta^2-1)^n}{d\eta^{n-1}}, \tag{23}$$

the case $n=0$ may now be dropped.

Any of the following polynomials can be taken as the function $W(\eta)$:

$$\left.\begin{aligned} &W_1(\eta)=\frac{\eta^2-1}{2},\\ &W_2(\eta)=\frac{\eta(\eta^2-1)}{2},\\ &W_3(\eta)=6(\eta^2-1)(5\eta^2-1),\\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{24}$$

Thus, we have found a set of particular solutions of equation (16) which satisfy the boundary conditions (15). These particular solutions are

$$V=W_n(\eta)\cos\sigma_n t;\quad V=W_n(\eta)\sin\sigma_n t, \tag{25}$$

whereby σ_n should satisfy the condition

$$\frac{l\rho(\sigma_n^2-4\omega^2)}{g(\rho_2-\rho_1)\operatorname{tg}\alpha}=n(n+1)\quad(n=1, 2, \ldots) \tag{26}$$

or

$$\sigma_n=\sqrt{4\omega^2+\frac{g(\rho_2-\rho_1)\operatorname{tg}\alpha(n+1)n}{l\rho}}\quad(n=1, 2, \ldots). \tag{27}$$

Applying the formula of Margules

$$\operatorname{tg}\alpha=\frac{2\omega(\rho_1U_1-\rho_2U_2)}{g(\rho_2-\rho_1)},$$

we obtain

$$\sigma_n=\sqrt{4\omega^2+\frac{2\omega(\rho_1U_1-\rho_2U_2)(n+1)n}{l\rho}}. \tag{28}$$

The obtained particular solutions of equation (16) give simple periodic oscillations of the discontinuity surface with the period $T=\frac{2\pi}{\sigma_n}$. Assuming $\varphi=30°$, $U_1=-U_2=10$ m sec^{-1}; $l=10^6$ m, we obtain

for $n=1$ $\sigma_n = 9.1 \cdot 10^{-5}\ \text{sec}^{-1}$; T = 0.8 day (24 hours)

" $n=2$ $\sigma_n = 9.85 \cdot 10^{-5}\ \text{sec}^{-1}$; T = 0.74 " " "

" $n=3$ $\sigma_n = 11.78 \cdot 10^{-5}\ \text{sec}^{-1}$; T = 0.62 " " "

" $n=4$ $\sigma_n = 14.12 \cdot 10^{-5}\ \text{sec}^{-1}$; T = 0.52 " " "

i.e., even the largest period does not exceed 24 hours.

It should be stressed that the frequency of oscillation has turned out to be a real number. This means that the interface is stable in regard of all zonal perturbations. Had the frequency of oscillation proved to be imaginary, we would have discovered an instability of motion in one case, namely, when the mass above the discontinuity surface is cooler than that below it.

Having determined the particular solutions of the equation (16) by the method of Fourier it is easy to find the general solution of this equation, satisfying the boundary conditions (10):

$$V(\eta, t) = \sum_{n=1}^{\infty} (A_n \cos \sigma_n t + B_n \sin \sigma_n t)\, W_n(\eta), \tag{29}$$

where A_n and B_n are arbitrary constants. Thus

$$V(\eta, 0) = \sum_{n=1}^{\infty} A_n W_n(\eta); \quad \frac{\partial V(\eta, 0)}{\partial t} = \sum_{n=1}^{\infty} B_n \sigma_n W_n(\eta). \tag{30}$$

But the functions $V(\eta, 0)$ and $\frac{\partial V(\eta, 0)}{\partial t}$ should satisfy the initial conditions (12) having in the new variables the form

$$V(\eta, 0) = \frac{h}{2}(1 - \eta) \cdot v_1(\eta, 0);$$

$$\frac{1}{1-\eta^2} \frac{\partial V(\eta, 0)}{\partial t} = \frac{g(\rho_2 - \rho_1) \operatorname{tg} \alpha}{2\rho} \cdot \frac{\partial \zeta(\eta, 0)}{\partial \eta} + \frac{\omega h}{2} [u_2(\eta, 0) - u_1(\eta, 0)]. \tag{31}$$

We point out that all polynomials $W_k(\eta)$ satisfy the condition of orthogonality:

$$\left.\begin{aligned} &\int_{-1}^{+1} \frac{W_k(\eta)\, W_l(\eta)}{1-\eta^2}\, d\eta = 0 \quad \text{for} \quad k \neq l, \\ &\int_{-1}^{+1} \frac{W_k^2(\eta)}{1-\eta^2}\, d\eta = \frac{2}{k(k+1)(2k+1)}. \end{aligned}\right\} \tag{32}$$

Employing the relationships (30), (31) and (32), all coefficients of the series (30) can be determined. For this purpose it is sufficient to multiply (30) by $W_k(\eta)$ and to integrate the product with respect to η over the interval -1 to $+1$.

We obtain then:

$$\left.\begin{aligned}
&\int_{-1}^{+1} \frac{V(\eta, 0)\cdot W_k(\eta)}{1-\eta^2}\, d\eta = A_k \frac{2}{k(k+1)(2k+1)},\\
&\int_{-1}^{+1} \frac{\frac{\partial V(\eta, 0)}{\partial t}\cdot W_k(\eta)}{1-\eta^2}\, d\eta = B_k \frac{2\sigma_k}{k(k+1)(2k+1)}.
\end{aligned}\right\} \tag{33}$$

The relationships (33) determine all coefficients of the series (30) providing the functions $V(\eta, 0)$ and $\frac{\partial V(\eta, 0)}{\partial t}$, determined by the formulas (31), satisfy the condition of continuity.

With the help of the equation

$$\frac{\partial \zeta}{\partial t} = \frac{\partial V}{\partial \eta}\cdot\frac{2}{l} \tag{34}$$

we can find the function $\zeta(\eta, t)$, i.e., we may determine the form of the disturbed discontinuity surface

$$\frac{\partial \zeta}{\partial t} = \frac{2}{l}\left\{\sum_{n=1}^{\infty}(A_n \cos\sigma_n t + B_n \sin\sigma_n t) P_n(\eta)\right\}, \tag{35}$$

consequently

$$\zeta(\eta, t) = \zeta(\eta, 0) + \frac{2}{l}\sum_{n=1}^{\infty}[A_n \sin\sigma_n t + B_n(1-\cos\sigma_n t)]\frac{P_n(\eta)}{\sigma_n}. \tag{36}$$

From equations (7) we find v_1 and v_2:

$$\left.\begin{aligned}
v_1(\eta, t) &= \frac{2}{1-\eta}\cdot\frac{1}{h}\sum_{n=1}^{\infty}(A_n \cos\sigma_n t + B_n \sin\sigma_n t) W_n(\eta),\\
v_2(\eta, t) &= \frac{2}{1+\eta}\cdot\frac{1}{h}\sum_{n=1}^{\infty}(A_n \cos\sigma_n t + B_n \sin\sigma_n t) W_n(\eta).
\end{aligned}\right\} \tag{37}$$

The functions u_1 and u_2 are found from equations (1):

$$\left.\begin{aligned}
u_1(\eta, t) &= u_1(\eta, 0) -\\
&-\frac{2}{1-\eta}\cdot\frac{2\omega}{h}\sum_{n=1}^{\infty}[A_n \sin\sigma_n t + B_n(1-\cos\sigma_n t)]\frac{W_n(\eta)}{\sigma_n},\\
u_2(\eta, t) &= u_2(\eta, 0) -\\
&-\frac{2}{1+\eta}\cdot\frac{2\omega}{h}\sum_{n=1}^{\infty}[A_n \sin\sigma_n t + B_n(1-\cos\sigma_n t)]\frac{W_n(\eta)}{\sigma_n}.
\end{aligned}\right\} \tag{38}$$

Finally, the functions p_1 and p_2 are also determined from equation (1):

$$\left.\begin{aligned}
p_1(\eta, t) &= p_1(0, t) + l\rho_1\omega\int_0^\eta v_1(\eta, 0)\,d\eta - \\
&- \frac{8\omega^2}{h}\sum_{n=1}^{\infty}\frac{1}{\sigma_n}[A_n\sin\sigma_n t + B_n(1-\cos\sigma_n t)]\int_0^\eta\frac{W_n(\eta)}{1-\eta}\,d\eta - \\
&- \frac{2}{h}\sum_{n=1}^{\infty}\sigma_n[B_n\cos\sigma_n t - A_n\sin\sigma_n t]\int_0^\eta\frac{W_n(\eta)}{1-\eta}\,d\eta; \\
p_2(\eta, t) &= p_2(0, t) + l\rho_2\omega\int_0^\eta v_2(\eta, 0)\,d\eta - \\
&- \frac{8\omega^2}{h}\sum_{n=1}^{\infty}\frac{1}{\sigma_n}[A_n\sin\sigma_n t + B_n(1-\cos\sigma_n t)]\cdot\int_0^\eta\frac{W_n(\eta)}{1+\eta}\,d\eta - \\
&- \frac{2}{h}\sum_{n=1}^{\infty}\sigma_n[B_n\cos\sigma_n t - A_n\sin\sigma_n t]\cdot\int\frac{W_n(\eta)}{1+\eta}\,d\eta.
\end{aligned}\right\} \quad (39)$$

Kochin illustrates the obtained results by the following example. Let us assume that at the initial moment $t=0$ the surface of discontinuity is an inclined plane whose slope angle does not coincide with that determined by the formula of Margules, but differs from it by a small value. Then $\zeta(\eta, 0)$ will be a linear function of η and we may set

$$\zeta(\eta, 0) = a\eta;$$

where a is a constant. We assume as well that there are no other disturbances of the stationary motion of the two given currents, i.e., that

$$u_j(\eta, 0) = v_j(\eta, 0) = p_j(\eta, 0) = 0.$$

Then (31) yields

$$V(\eta, 0) = 0,$$

$$\frac{\partial V(\eta, 0)}{\partial t} = \frac{g(\rho_2-\rho_1)\operatorname{tg}\alpha}{2\rho}\cdot a(1-\eta^2) = -\frac{2\omega}{\rho}(\rho_1U_1 - \rho_2U_2)\,a\cdot W_1(\eta),$$

and, applying (30), we obtain

$$\begin{aligned}
A_k &= 0 \qquad (k = 1, 2, \ldots), \\
B_k &= 0 \qquad (k = 2, 3, \ldots), \\
B_1 &= -\frac{2\omega}{\rho\sigma_1}(\rho_1U_1 - \rho_2U_2)\cdot a,
\end{aligned}$$

where

$$\sigma_1 = \sqrt{4\omega^2 + \frac{4\omega(\rho_1U_1 - \rho_2U_2)}{\rho l}}.$$

Repeating the calculations performed above for the general case, we obtain:

$$V(\eta, t) = \frac{\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho\sigma_1} a \sin\sigma_1 t (1 - \eta^2);$$

$$v_1(\eta, t) = \frac{2\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho\sigma_1 h} a (1 + \eta) \sin\sigma_1 t,$$

$$v_2(\eta, t) = -\frac{2\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho\sigma_1 h} a (1 - \eta) \sin\sigma_1 t,$$

$$u_1(\eta, t) = \frac{4\omega^2(\rho_1 U_1 - \rho_2 U_2)}{\rho\sigma_1^2 h} \cdot a (1 + \eta)(1 - \cos\sigma_1 t),$$

$$u_2(\eta, t) = -\frac{4\omega^2(\rho_1 U_1 - \rho_2 U_2)}{\rho\sigma_1^2 h} a (1 - \eta)(1 - \cos\sigma_1 t),$$

$$p_1(\eta, t) = -\frac{4\omega^3(\rho_1 U_1 - \rho_2 U_2)}{\sigma_1^2 h}\left(\eta + \frac{\eta^2}{2}\right)\left[1 + \frac{\rho_1 U_1 - \rho_1 U_2}{\rho l\omega}\cos\sigma_1 t\right],$$

$$p_2(\eta, t) = \frac{4\omega^3(\rho_1 U_1 - \rho_2 U_2)}{\sigma_1^2 h}\left(\eta - \frac{\eta^2}{2}\right)\left[1 + \frac{\rho_1 U_1 - \rho_2 U_2}{\rho l\omega}\cos\sigma_1 t\right],$$

$$\zeta(\eta, t) = \frac{4\omega^2 a\eta}{\sigma_1^2}\left(1 + \frac{\rho_1 U_1 - \rho_2 U_2}{\rho l\omega}\cos\sigma_1 t\right).$$

The vertical velocities can be determined by formula (14) of the preceding section:

$$w_1 = \frac{2(h - z)}{l} \cdot \frac{\partial v_1}{\partial\eta} = \frac{4\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho\sigma_1 l} a \left(1 - \frac{z}{h}\right)\sin\sigma_1 t,$$

$$w_2 = -\frac{2z}{l} \cdot \frac{\partial v_2}{\partial\eta} = -\frac{4\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho\sigma_1 l} \cdot a \cdot \frac{z}{h}\sin\sigma_1 t.$$

We determine also the components of acceleration

$$\frac{dv_1}{dt} = \frac{2\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho h} a (1 + \eta)\cos\sigma_1 t,$$

$$\frac{dv_2}{dt} = -\frac{2\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho h} a (1 - \eta)\cos\sigma_1 t,$$

$$\frac{du_1}{dt} = \frac{4\omega^3(\rho_1 U_1 - \rho_2 U_2)}{\rho\sigma_1 h} a (1 + \eta)\sin\sigma_1 t,$$

$$\frac{du_2}{dt} = -\frac{4\omega^2(\rho_1 U_1 - \rho_2 U_2)}{\rho\sigma_1 h} a (1 - \eta)\sin\sigma_1 t,$$

$$\frac{dw_1}{dt} = \frac{4\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho l} a \left(1 - \frac{z}{h}\right)\cos\sigma_1 t,$$

$$\frac{dw_2}{dt} = -\frac{4\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho l} a \cdot \frac{z}{h}\sin\sigma_1 t.$$

The analysis of the obtained solution allows to draw the following conclusions:

1) A discontinuity surface whose slope departs from that given by Margules' formula does not remain stationary. It oscillates around some mean position determined by the equation

$$\zeta_0 = \frac{4\omega^2}{\sigma_1^2} \cdot a\eta.$$

This mean position does not coincide with the position of the stationary interface and deviates from this position in the same direction as in the initial moment.

2) The tangential components of the velocity perturbations u_1 and u_2 fluctuate in magnitude but maintain the same sign. Thus, for example, at $a > 0$ $u_1 > 0$ and $u_2 < 0$. Consequently, when the surface of discontinuity is steeper than the stationary interface, the fundamental velocity of the first current U_1 will increase on the average, and the fundamental velocity of the second current U_2 will, on the average, somewhat decrease. Consequently, the average velocity difference at the surface of discontinuity will slightly increase. This increase of the difference in the tangential velocity leads, according to the Margules formula, to the increase of the angle α.

3) The obtained solution shows that an increase in either the velocity difference or the slope angle causes the other quantity to increase.

4) Nonstationary motions, originating at the interface and close to the fundamental motion, are very complex, leading to the development of additional motions in all directions.

Let us explain in greater detail what would happen if the interface were initially steeper than the stationary surface, so that $\alpha > 0$, and there were no other perturbations of the fundamental motion. The formulas derived in this section show that the cold mass immediately begins to slide under the warm mass. This is accompanied by a descending current of cold air, especially marked in the upper part, and by an ascending current of warm air. Thus, the warm air will displace the cold air in the upper region, and the interface will become progressively steeper.

Simultaneously with the emergence of the velocities v_1 and v_2, the Coriolis force will start deflecting the motion to the right, relative to the pressure gradients which have caused these additional velocities. As a result, horizontal velocity components u_1 and u_2 will appear.

However, such velocities cannot build up indefinitely. A moment comes when the process begins to develop in the reverse direction. This moment corresponds to the minimum slope of the surface of discontinuity.

The simplification made by Kochin permits to grasp the basic outlines of the processes taking place near the nonstationary interface in the case of zonal oscillation. However, the real processes are more complex than the examined scheme.

Two factors complicate the zonal oscillations of nonstationary interfaces:

It was assumed until recently that the oscillations of an interface are zonal, i. e., the air flows equally in all planes perpendicular to the front. In reality, however, the interface is subject also to nonzonal oscillation. In addition it undergoes a deformation. Such oscillations will be discussed in the following section.

Friction is the second factor which distorts the given picture of zonal oscillation. Its influence precludes pure oscillations of a discontinuity surface; the periodic process is then less pronounced and competes with aperiodic processes.

§ 8. Interface Stability with Respect to Nonzonal Oscillations

In the preceding section we dealt with the zonal oscillations of an interface. It was shown that the interface cannot lose its stability by zonal oscillations. We will now discuss the more general case of nonzonal interface oscillations, which was also studied by N. E. Kochin.

We start from equations similar to (1) of the preceding section:

$$\left.\begin{aligned}
&\frac{\partial u_j'}{\partial t}+U_j\frac{\partial u_j'}{\partial x}=-\frac{1}{\rho}\frac{\partial p_j'}{\partial x}+2\omega v_j' \qquad (j=1,2),\\
&\frac{\partial v_j'}{\partial t}+U_j\frac{\partial v_j'}{\partial x}=-\frac{1}{\rho}\frac{\partial p_j'}{\partial y}-2\omega u_j' \qquad (j=1,2),\\
&g(\rho_2-\rho_1)\zeta'=p_2'-p_1',\\
&\frac{\partial \zeta'}{\partial t}+U_1\frac{\partial \zeta'}{\partial x}=-v_1'\operatorname{tg}\alpha+(h-y\operatorname{tg}\alpha)\left(\frac{\partial u_1'}{\partial x}+\frac{\partial v_1'}{\partial y}\right),\\
&\frac{\partial \zeta'}{\partial t}+U_2\frac{\partial \zeta'}{\partial x}=-v_2'\operatorname{tg}\alpha-y\operatorname{tg}\alpha\left(\frac{\partial u_2'}{\partial x}+\frac{\partial v_2'}{\partial y}\right).
\end{aligned}\right\} \tag{1}$$

In the two first equations we have set $\rho\approx\rho_1\approx\rho_2$; ω stands for the vertical component of the angular velocity of the earth. The function $\zeta'(x, y, t)$ determines the perturbations of the discontinuity surface. The equation of this surface for the perturbed motion has the following form

$$z=y\cdot\operatorname{tg}\alpha+\zeta'(x, y, t). \tag{2}$$

Kochin considers waves propagating along the front of the discontinuity surface, i. e., along the axis Ox. The solution of the system of seven equations (1) in the seven unknown functions u_1', u_2', v_1', v_2', p_1', p_2', ζ', should accordingly be sought in the form

$$\left.\begin{aligned}
u_j'&=e^{i(kx+\sigma' t)}u_j(y),\\
v_j'&=e^{i(kx+\sigma' t)}v_j(y),\\
p_j'&=e^{i(kx+\sigma' t)}p_j(y),\\
\zeta'&=e^{i(kx+\sigma' t)}\zeta_j(y).
\end{aligned}\right\} \tag{3}$$

In the formulas (3) the functions u_j, v_j, p_j, ζ are complex. However, in the following we shall take only the real part to represent u_j', v_j', p_j', ζ.

Substituting (3) in the first two equations (1), we obtain after trivial transformations

$$\left.\begin{aligned}
i(kU_j+\sigma')u_j-2\omega\cdot v_j&=-\frac{ik}{\rho}\cdot p_j,\\
i(kU_j+\sigma')v_j+2\omega\cdot u_j&=-\frac{1}{\rho}\frac{dp_j}{dy},
\end{aligned}\right\} \tag{4}$$

where

$$\left.\begin{aligned} u_j &= \frac{k(kU_j+\sigma')p_j - 2\omega\frac{dv_j}{dy}}{\rho[4\omega^2-(kU_j+\sigma')^2]}, \\ v_j &= i\,\frac{2\omega k p_j - (kU_j+\sigma')\frac{dp_j}{dy}}{\rho[4\omega^2-(kU_j+\sigma')^2]}. \end{aligned}\right\} \tag{5}$$

Eliminating u_j', v_j' and ζ from the equations (1) we obtain the following system of differential equations for the functions p_1 and p_2:

$$\left.\begin{aligned} &\frac{d}{dy}\left(y\frac{dv_2}{dy}\right) - \left(k^2 y + \frac{2k\omega}{kU_2+\sigma'}\right)\cdot p_2 = \\ &\qquad = \frac{\rho[4\omega^2-(kU_2+\sigma')^2]}{g(\rho_1-\rho_2)\operatorname{tg}\alpha}\cdot(p_1-p_2), \\ &\frac{d}{dy}\left[(y-l)\frac{dp_1}{dy}\right] - \left[k^2(y-l)+\frac{2k\omega}{kU_1+\sigma'}\right]p_1 = \\ &\qquad = \frac{\rho[4\omega^2-(kU_1+\sigma')^2]}{g(\rho_1-\rho_2)\operatorname{tg}\alpha}\cdot(p_1-p_2). \end{aligned}\right\} \tag{6}$$

The points $y=0$ and $y=l$ are the singular points of this system. We require that the solution remain finite at these points. Then p_1 and p_2 will be entire functions of y.

The solution of the system (6) must agree with the following boundary conditions

$$\frac{dp_1}{dy} = kp_1 \quad \text{at} \quad y=0, \tag{7}$$

$$\frac{dp_2}{dy} = -kp_2 \quad \text{at} \quad y=l. \tag{8}$$

To this end Kochin proceeds as follows. The motion in the region $y<0$ is considered. From the equation of continuity we obtain

$$w_1' = -z\left(\frac{\partial u_1'}{\partial x} + \frac{\partial v_1'}{\partial y}\right),$$

since at $z=0$ $w_1'=0$. But in the region $y<0$ at $z=h$ also $w_1'=0$. Consequently,

$$\frac{\partial u_1'}{\partial x} + \frac{\partial v_1'}{\partial y} = 0.$$

Applying (3) we obtain:

$$iku_1 + \frac{dv_1}{dy} = 0 \quad \text{for} \quad y<0.$$

Inserting in this equation the values of u_1 and v_1 from (5), we obtain:

$$\frac{d^2p_1}{dy^2} - k^2 p_1 = 0.$$

The general integral of this differential equation has the form

$$p_1 = C_0 e^{ky} + C_1 e^{-ky},$$

where we must put $C_1 = 0$, since as $y \to -\infty$ p_1 should remain finite.

Consequently

$$p_1 = C_0 e^{ky} \text{ for } y < 0,$$

whence

$$\frac{dp_1}{dy} = kp_1 \quad \text{for} \quad y < 0. \tag{9}$$

It is obvious that upon transition from region I into region IV (Figure 151), u_1 and v_1 should change continuously. Consequently, according to (5) the functions p_1 and $\frac{dp_1}{dy}$ should also change continuously. Therefore, the condition (9) should be fulfilled even at $y = 0$. Thus, the boundary condition (7) is imposed on the solution of the system (6). The second boundary condition (8) is treated in exactly the same way.

We introduce again the dimensionless variable

$$\eta = \frac{2y - l}{l}; \quad y = l\frac{1+\eta}{2}. \tag{10}$$

To the interval $0 \leqslant y \leqslant l$ will then correspond the interval $-1 \leqslant \eta \leqslant +1$.

In addition, we introduce the velocity

$$U = \frac{U_1 - U_2}{2}. \tag{11}$$

In this case the formula of Margules can be approximated by the expression

$$\operatorname{tg}\alpha = \frac{2\omega(\rho_1 U_1 - \rho_2 U_2)}{g(\rho_2 - \rho_1)} \approx \frac{4\omega\rho U}{g(\rho_2 - \rho_1)}. \tag{12}$$

We introduce a new variable σ' instead of the frequency σ:

$$\sigma = \sigma' + \frac{k(U_1 + U_2)}{2} \tag{13}$$

Obviously

$$\sigma' + kU_1 = \sigma + kU, \quad \sigma' + kU_2 = \sigma - kU, \tag{14}$$

and the system of equations (6) takes the form

$$\left.\begin{aligned} \frac{d}{d\eta}\left[(1+\eta)\frac{dp_2}{d\eta}\right] - \left[\frac{k^2l^2}{4}(1+\eta) + \frac{k\omega l}{\sigma - kU}\right]p_2 = \\ = \frac{l[4\omega^2 - (\sigma - kU)^2]}{8\omega U}(p_2 - p_1), \\ \frac{d}{d\eta}\left[(1-\eta)\frac{dp_1}{d\eta}\right] - \left[\frac{k^2l^2}{4}(1-\eta) - \frac{k\omega l}{\sigma + kU}\right]p_1 = \\ = \frac{l[4\omega^2 - (\sigma + kU)^2]}{8\omega U}(p_1 - p_2), \end{aligned}\right\} \tag{15}$$

the boundary conditions will assume the form

$$\left.\begin{aligned} \frac{dp_1}{d\eta} &= \frac{kl}{2}\cdot p_1 \quad \text{at} \quad \eta = -1, \\ \frac{dp_2}{d\eta} &= -\frac{kl}{2}\cdot p_2 \quad \text{at} \quad \eta = +1. \end{aligned}\right\} \tag{16}$$

Three dimesionless parameters enter in equations (15)

$$\frac{kU}{2\omega} = \beta; \quad \frac{l\omega}{U} = \alpha; \quad \frac{\sigma}{kU} = \tau. \tag{17}$$

Introducing this abbreviated notation into equation (15) we finally obtain the following equation:

$$\left.\begin{aligned} \frac{d}{d\eta}\left[(1+\eta)\frac{dp_2}{d\eta}\right] - \left[\alpha^2\beta^2(1+\eta) + \frac{\alpha}{\tau-1}\right]p_2 = \\ = \frac{\alpha}{2}\left[1-\beta^2(\tau-1)^2\right](p_2-p_1), \\ \frac{d}{d\eta}\left[(1-\eta)\frac{dp_1}{d\eta}\right] - \left[\alpha^2\beta^2(1-\eta) - \frac{\alpha}{\tau+1}\right]p_1 = \\ = \frac{\alpha}{2}\left[1-\beta^2(\tau+1)^2\right](p_1-p_2). \end{aligned}\right\} \tag{18}$$

The boundary conditions appear in the following form

$$\left.\begin{aligned} \frac{dp_1}{d\eta} - \alpha\beta p_1 = 0 \quad \text{at} \quad \eta = -1, \\ \frac{dp_2}{d\eta} + \alpha\beta p_2 = 0 \quad \text{at} \quad \eta = +1. \end{aligned}\right\} \tag{19}$$

In this way Kochin reduced the problem of nonzonal interface oscillations to the investigation of the system of equations (18). The parameters α, β, τ, appearing in these equations have the following physical meaning. The parameter

$$\alpha = \frac{l\omega}{U} = \frac{2h\omega \operatorname{ctg}\alpha}{U_1 - U_2} \tag{20}$$

is determined according to the known elements of the fundamental motion.

The parameter

$$\beta = \frac{kU}{2\omega} = \frac{\pi U}{\lambda\omega} = \frac{\pi(U_1-U_2)}{2\omega\lambda} \tag{21}$$

characterizes the length of the waves moving along the front. And the parameter

$$\tau = \frac{\sigma}{kU} = \frac{\sigma' + \frac{k(U_1+U_2)}{2}}{k\frac{(U_1-U_2)}{2}} \tag{21'}$$

characterizes the oscillation frequency, which is unknown and subject to determination by the foregoing equations.

Thus for the given α and β those values of τ should be determined, which upon solution of the system (18) will fulfill the boundary conditions (19), while the functions $p_1(\eta)$ and $p_2(\eta)$ should be holomorphic at the points $\eta = +1$ and $\eta = -1$ and consequently they are entire functions.

In connection with the problem of cyclone formation it is indispensable to discuss the possible loss of stability of the interface with respect to very long waves, when β is very small and consequently the wave length λ is large relative to $c\frac{U}{\omega}$.

Kochin points out that at a first glance the case of very long waves may appear to be reducible to the case of infinitely long waves, discussed in the previous section, when the perturbations are independent of the coordinate x and the interface is subject to zonal oscillations. But for such disturbances, close to the zonal oscillations considered above σ is finite and hence

$$\tau = \frac{\sigma}{kU} = \frac{\sigma}{2\omega\beta}$$

is large for small values of β.

Kochin showed that, apart from disturbances close to the zonal oscillations, other types of long wave disturbances occur for which τ remains finite. The frequencies σ (and consequently σ') of these disturbances are very small and their periods are very large.

Kochin formulated the extreme case of these disturbances by taking in equations (18) and in the boundary conditions (19) $\beta = 0$.

For this extreme case we obtain

$$\left.\begin{aligned} \frac{d}{d\eta}\left[(1+\eta)\frac{dp_2}{d\eta}\right] - \frac{\alpha}{\tau-1}\cdot p_2 &= \frac{\alpha}{2}(p_2 - p_1), \\ \frac{d}{d\eta}\left[(1-\eta)\frac{dp_1}{d\eta}\right] + \frac{\alpha}{\tau+1}\cdot p_1 &= \frac{\alpha}{2}(p_1 - p_2), \end{aligned}\right\} \qquad (22)$$

$$\left.\begin{aligned} \frac{dp_1}{d\eta} &= 0 \quad \text{at} \quad \eta = -1, \\ \frac{dp_2}{d\eta} &= 0 \quad \text{at} \quad \eta = +1. \end{aligned}\right\} \qquad (23)$$

We multiply the first equation (22) by $(\tau - 1)$, the second by $(\tau + 1)$ and add them together. Integrating the obtained equation we get:

$$(\tau-1)(1+\eta)\frac{dp_2}{d\eta} + (\tau+1)(1-\eta)\frac{dp_1}{d\eta} = \text{const.} = 0. \qquad (24)$$

The constant in equation (24) should be adjusted to the boundary condition (23).

Defining the auxiliary function

$$\Phi(\eta) = (1-\tau)(1+\eta)\frac{dp_2}{d\eta} = (1+\tau)(1-\eta)\frac{dp_1}{d\eta} \qquad (25)$$

and introducing it into equations (22) we obtain the differential equation

$$\Phi''(\eta) = \frac{a}{2}\left[\frac{\tau+1}{(\tau-1)(1+\eta)} + \frac{\tau-1}{(\tau+1)(1-\eta)}\right]\Phi(\eta), \tag{26}$$

from which $\Phi(\eta)$ can be determined. The boundary condition for the function $\Phi(\eta)$ appears as:

$$\Phi(\pm 1) = 0. \tag{27}$$

The equation (26) can be brought to the form

$$(1-\eta^2)\,\Phi''(\eta) + (r+s\eta)\,\Phi(\eta) = 0, \tag{28}$$

where

$$r = a\frac{1+\tau^2}{1-\tau^2}; \quad s = -\frac{2a\tau}{1-\tau^2} \tag{29}$$

and consequently

$$\tau = -\frac{s}{r+a}, \tag{30}$$

whereby, identically

$$r^2 - s^2 = a^2. \tag{31}$$

Equation (28) has the same sign as equation (21) of the previous section. Therefore the solution of equation (28) is to be sought as a series of polynomials W_n, as in the integration of equation (21) § 7:

$$\Phi(\eta) = a_1 W_1(\eta) + a_2 W_2(\eta) + \ldots, \tag{32}$$

whereby the polynomials are connected by the recurrence formula:

$$(2n+1)\cdot\eta W_n = (n-1)\,W_{n-1} + (n+2)W_{n+1} \quad (n = 1,2,\ldots). \tag{33}$$

Inserting (32) into equation (28) and applying (33) and (34),

$$(1-\eta^2)\,W_n'' + n(n+1)\,W_n = 0 \qquad (n = 1,2,\ldots), \tag{34}$$

we obtain the following relations between the coefficients:

$$\begin{aligned}
&a_1[r - 1\cdot 2] + \tfrac{1}{5}sa_2 = 0,\\
&a_2[r - 2\cdot 3] + \tfrac{2}{7}sa_3 + \tfrac{3}{3}s\cdot a_1 = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_k[r - k(k+1)] + \frac{k}{2k+3}\cdot sa_{k+1} + \frac{k+1}{2k-1}\cdot s\cdot a_{k-1} = 0\\
&\qquad\qquad (k = 2,3,\ldots).
\end{aligned} \tag{35}$$

whence we find

$$\frac{a_1}{a_2} = \frac{\frac{1}{5}\cdot s}{r - 1\cdot 2},$$

$$s\cdot\frac{a_1}{a_2} = 2\cdot 3 - r - \frac{\delta_2 \cdot s^2}{\frac{4}{5}\, s\cdot\frac{a_2}{a_3}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (36)$$

$$\frac{k+1}{2k-1}\cdot s\,\frac{a_{k-1}}{a_k} = k(k+1) - r - \frac{\delta_k s^2}{\frac{k+2}{2k+1}\, s\cdot\frac{a_k}{a_{k+1}}},$$

where

$$\delta_k = \frac{k(k+2)}{(2k+1)(2k+3)}. \qquad (37)$$

Combining successive equalities of (36) and equating the expressions for $\frac{a_1}{a_2}$ the following relation between r and s is obtained, expressed by a continued fraction:

$$\frac{\delta_1 \cdot s^2}{1\cdot 2 - r} = 2\cdot 3 - r - \cfrac{\delta_2 \cdot s^2}{3\cdot 4 - r - \cfrac{\delta_3 \cdot s^2}{4\cdot 5 - r - \ldots}}. \qquad (38)$$

The relations (38) and (31) fully determine r and s and consequently τ, in view of (30). At $s = 0$ we obtain from (38)

$$r_1 = 1\cdot 2; \quad r_2 = 2\cdot 3 \ldots \quad r_n = n(n+1),$$

and from (31)

$$r = \alpha.$$

Consequently, for $\alpha = n(n+1)$, one of the solutions equations (38) and (31) is

$$r = n(n+1); \quad s = 0. \qquad (39)$$

In particular, for $\alpha = 2$

$$r = 2 \text{ and } s = 0.$$

Kochin regards the values α, as close to 2. Then, for $r \approx 2$ and $s \approx 0$, we obtain from (38)

$$r \approx 2 - \frac{\delta_1 s^2}{4},$$

and from (31)

$$4 - \delta_1 s^2 - s^2 + \ldots = \alpha^2$$

or, since

$$\delta_1 = \frac{1}{5},$$

we get

$$4 - \frac{6}{5}\, s^2 + \ldots = \alpha^2.$$

Consequently

$$s^2 \approx \frac{5}{6}(4 - \alpha^2). \tag{40}$$

It follows from (40) that when α is somewhat greater than 2, $s^2 < 0$ and s is an imaginary number. Then τ and σ will be imaginary too, and the motion will be unstable.

Thus, for $\alpha > 2$ the motion of Margules is unstable with respect to very long waves. Kochin also showed that on the other hand the motion of Margules is stable in this case for $0 < \alpha < 2$.

We shall now write the condition of stability in an explicit form, inserting the expression for α :

$$\alpha = \frac{l\omega}{U} = \frac{2h\omega \operatorname{ctg} \alpha}{U_1 - U_2} < 2.$$

By (12), the condition of stability of the Margules interface with respect to long waves, can be put in the form

$$U > \sqrt{\frac{gh(\rho_2 - \rho_1)}{8\rho}}. \tag{41}$$

Thus, the interface determined by the Margules formula is more stable with respect to very long waves the higher the relative velocity of the fundamental currents and the smaller their depth, and also the slighter the discontinuity of the density.

At $h = 8$ km, $\frac{\rho_1 - \rho_2}{4\rho} = \frac{1}{40}$ (i. e., at a temperature difference $\approx 7°$) the fundamental motion is stable at $U > 16$ m/sec.

A similar result is obtained by Kochin for waves of a finite length. To each value of β corresponds a critical value $\alpha_0(\beta)$. For $0 < \alpha < \alpha_0(\beta)$ the fundamental motion is stable with respect to waves of length $\lambda = \frac{\pi U}{\omega \beta}$ and unstable for $\alpha > \alpha_0(\beta)$.

Figure 152 shows a curve $\alpha_0(\beta)$, which separates the region of stability from the region of instability. From this figure it can be seen that the frontal surfaces are unstable with respect to short waves (large β). However, frontal surfaces are stable for long waves. They lose their stability again for very long waves (of the order of 500 - 1000 km). These very long waves lead to the formation of cyclones.

Thus, in spite of the schematization of the studied process, Kochin demonstrated the possibility of atmospheric conditions under which long wave perturbations of an interface are unstable. Such instability leads to a deformation of the interface, which swells and surges. As a result, vortexes develop, which can be identified with the cyclones observed in the atmosphere.

We should point out that certain of the employed simplifications brought objections from some foreign meteorologists. So, for example, Godske insisted that in neglecting the vertical accelerations Kochin has omitted a definite type of waves. However, Blinova (1935) solved the problem of interface zonal oscillations without resorting to the elimination of the vertical accelerations and her result was close to the result of Kochin. This justifies Kochin's procedures.

The work of Kochin served as a stimulus to a number of investigations of this problem under more general conditions. So, for example, Dorodnitsyn (1936)

generalized Kochin's results on the zonal oscillations for a barotropic medium. Yudin (1937), and afterwards Blinova (1939), employing another method, have solved the problem of nonzonal interface oscillations for a baroclinic fluid.

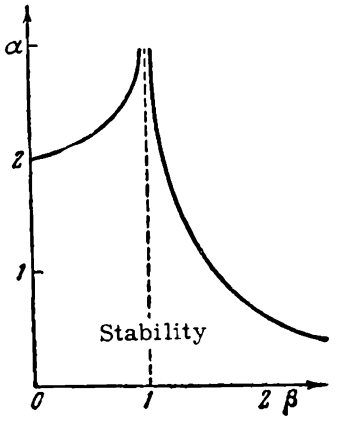

Figure 152. The dependence of interface stability upon the wave length

Chapter XX

I. A. KIBEL'S THEORETICAL METHOD OF WEATHER FORECAST

§ 1. The Boundary Layer and the Free Atmosphere

Dynamic meteorology is now being successfully applied to weather forecasting by talented representatives of the Soviet school of dynamic meteorology, founded by A. A. Fridman.

Particularly remarkable successes in the application of dynamic meteorology to weather forecasting have recently been achieved by the representatives of this school, following the publication of I. A. Kibel's work "Meteorological applications of the equations of baroclinic fluid mechanics". In this work Kibel' showed, that the predetermination of the fields of pressure and temperature can be effected with sufficient accuracy through the use of two approximations. It turned out that even simplified solution permits to obtain satisfactory results. Since this work is of fundamental importance we shall discuss its content as fully as possible.

In the previous chapters we have pointed out the fundamental differences between processes occurring in the so-called boundary layer and beyond it, in the free atmosphere.

Eddy viscosity and eddy heat conductivity are particularly decisive in the presence of large gradients of velocity and temperature. Such conditions prevail in the planetary surface layer treated before, which extends from the earth surface up to about 1 km. Similar conditions exist at the tropopause, permanently separating the troposphere from the stratosphere. In the troposphere internal boundary layers, separating two air masses of different properties, are also often encountered. The eddy viscosity and the eddy heat conductivity play a fundamental role in these boundary layers, while outside them their effects may be neglected.

According to Chapter IX the two equations of hydrodynamics must have the following form within the boundary layer:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}= \\
&\qquad =-\frac{1}{\rho}\frac{\partial p}{\partial x}+lv+\frac{\partial}{\partial z}\left(\frac{\mu}{\rho}\frac{\partial u}{\partial z}\right)+\frac{\mu'}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}= \\
&\qquad =-\frac{1}{\rho}\frac{\partial p}{\partial y}-lu+\frac{\partial}{\partial z}\left(\frac{\mu}{\rho}\frac{\partial v}{\partial z}\right)+\frac{\mu'}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right).
\end{aligned}\right\} \quad (1)$$

The third equation of hydrodynamics can be replaced, to a high degree of accuracy, by the hydrostatic equation:

$$0=-g-\frac{1}{\rho}\frac{\partial p}{\partial z}. \quad (2)$$

The equation of continuity will be written in the usual form

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0. \quad (3)$$

The same applies to the equation of state

$$p = \rho RT. \tag{4}$$

Following Kibel' we use the equation of energy balance in the following form:

$$\frac{dT}{dt} - \frac{R}{c_p}\frac{T}{p}\frac{dp}{dt} = \frac{\partial}{\partial z}\lambda\frac{\partial T}{\partial z} + \frac{k}{c_p}\cdot F. \tag{5}$$

In equations (1)-(5) the usual notation is adopted; u, v, w are the velocity components along the x, y, z axes; t - the time, p - the pressure, ρ-density, T - temperature, g-the acceleration of gravity, $l = 2\omega \sin\varphi$ - double the vertical component of the angular velocity of the earth, R-the gas constant, c_p - specific heat at constant pressure, μ-the coefficient of eddy viscosity, μ'-coefficient of lateral mixing $k\rho$-the coefficient of absorption, $k\rho F$-the radiant heat influx, λ-the coefficient of eddy heat conductivity.

Attention should be drawn to the fact that the air humidity is not accounted for in these equations.

In the free atmosphere, outside the boundary layer, the equations of motion and of energy can be simplified, since the terms expressing eddy viscosity and eddy heat conductivity may be omitted, i.e., the terms containing μ and λ. Thus, Kibel' takes the atmosphere at first as a perfect fluid. He seeks such solutions of equations (1)-(5) that would satisfy the boundary conditions on the earth and on the tropopause. Solutions obtained in this way should then be corrected at the boundary layer, for the effect of the eddy viscosity and eddy heat conductivity. Kibel' points out that the recommended method of solution is similar to this employed in aerodynamics for the solution of problems concerning the boundary layer formed around airfoils. The boundary layer, in aerodynamics as well as here, distorts the pressure field only slightly; therefore, the solution obtained for the pressure may be adopted without further corrections. The boundary layer, however, exercises a significant influence on the fields of velocity and temperature.

Relegating the wind velocity and temperature variations in the boundary layer to a later separate examination, which this problem requires, Kibel' centers his attention on the precalculation of the pressure and temperature fields outside the boundary layer.

§ 2. Determination of u and v from the Equations of Motion

We have to integrate the equations of motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + lv, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - lu \tag{2}$$

together with the equation of continuity and the equation of heat balance.

For the estimation of the order of the terms entering in these equations, Kibel' introduces the following characteristic magnitudes: the characteristic horizontal dimension of the phenomenon L ($L \approx 10^8$) the characteristic horizontal velocity U ($U \approx 10^3$ cm/sec), the characteristic altitude h ($h \approx 10^6$ cm/sec) the characteristic vertical velocity W ($W \approx 10^0$ cm/sec) the characteristic time τ ($\tau \approx 10^5$ sec).

Beside these, Kibel' introduces dimensionless lengths and velocities x, y, z, u, v, w according to the following formulas

$$x = L\bar{x};\ y = L\bar{y};\ z = h\bar{z};\ u = U\bar{u};\ v = V\bar{v};\ w = W\bar{w}, \qquad (3)$$

and also $t = \tau \cdot \bar{t}$.

The equations of motion assume the following form:

$$\left.\begin{aligned} \frac{1}{l\tau}\frac{\partial \bar{u}}{\partial \bar{t}} + \frac{U}{lL}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) + \frac{W}{lh}\cdot\bar{w}\frac{\partial \bar{u}}{\partial \bar{z}} &= -\frac{1}{lU}\frac{1}{\rho}\frac{\partial p}{\partial x} + \bar{v}, \\ \frac{1}{l\tau}\frac{\partial \bar{v}}{\partial \bar{t}} + \frac{U}{lL}\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) + \frac{W}{lh}\cdot w\frac{\partial \bar{v}}{\partial \bar{z}} &= \frac{1}{lU}\frac{1}{\rho}\frac{\partial p}{\partial y} + \bar{u}. \end{aligned}\right\} \qquad (4)$$

Since, by definition of the characteristic magnitudes, $\bar{u}, \bar{v}, \bar{w}, \bar{x}, \bar{y}, \bar{z}$, and consequently $\frac{\partial \bar{u}}{\partial \bar{x}} \cdots \frac{\partial \bar{w}}{\partial \bar{z}}$ are of the order 1, the orders of terms entering in the equations will be determined by the order of the coefficients of the corresponding derivatives. Since in the middle latitudes $l = 2\omega \sin\varphi \approx 10^{-4}$ sec, the first three terms will be of the order 10^{-1} and the fourth, containing W, of the order 10^{-2}. Finally since $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \approx 10^{-4}$ g/cm^2 sec^2 and $\rho \approx 10^{-3}\ \frac{\text{g}}{\text{cm}^3}$, we have $\frac{1}{\rho}\frac{\partial p}{\partial x} \approx 10^{-1}$ cm/sec^2; consequently the first terms on the right side of both equations are of order 1.

This evaluation of the order of the terms of the equations of motion shows that for a scale of motion determined by the characteristic magnitudes we have adopted, the horizontal velocities equal the geostrophic velocities to a first approximation.

Introducing $\varepsilon = \frac{U}{lL} = \frac{1}{l\tau} \approx 10^{-1} < 1$ and $\varepsilon' = \frac{W}{U}\frac{L}{h} = 10^{-2}$, we can write the differential equations of motion in the following dimensionless form:

$$\left.\begin{aligned} \varepsilon\left(\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \varepsilon'\bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}\right) &= -\frac{1}{lU}\frac{1}{\rho}\frac{\partial p}{\partial \bar{x}} + \bar{v}, \\ \varepsilon\left(\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} + \varepsilon'\bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}\right) &= -\frac{1}{lU}\frac{1}{\rho}\frac{\partial p}{\partial \bar{y}} - \bar{u}. \end{aligned}\right\} \qquad (5)$$

When $\bar{w}$, $\frac{1}{p}\frac{\partial p}{\partial \bar{x}}$, $\frac{1}{\rho}\frac{\partial p}{\partial \bar{y}}$ are given, we have a system (5) of partial differential equations for the two unknown functions $\bar{u}$ and $\bar{v}$.

Once the initial distribution of $\bar{u}$ and $\bar{v}$ is given, we may find the subsequent distribution of $\bar{u}$ and $\bar{v}$ at any time by means of equations (5). We have repeatedly stated that the ensuing motion differs only slightly from the geostrophic wind. Therefore, we can adopt, for the initial velocity, a distribution which corresponds to the geostrophic wind. In this case we may neglect the initial conditions and seek the particular solution of the system in the form of a development of $\bar{u}$ and $\bar{v}$ in powers of the small parameter ε, whereby we take the geostrophic wind as a first term.

Limiting ourselves to the terms containing ε to the first power we write the solution in the form

$$\left.\begin{aligned} \bar{u} &= -\frac{1}{lU}\frac{1}{\rho}\frac{\partial p}{\partial \bar{y}} + \varepsilon\bar{u}', \\ \bar{v} &= -\frac{1}{lU}\frac{1}{\rho}\frac{\partial p}{\partial \bar{x}} + \varepsilon\bar{v}'. \end{aligned}\right\} \qquad (6)$$

§ 3. Determination of w from the Equation of Continuity

We shall now evaluate the orders of the terms entering the equation of continuity.

$$\frac{\partial \rho}{\partial t} \approx 10^{-10},$$

$$\frac{\partial (\rho w)}{\partial z} \approx 10^{-9}, \frac{\partial \rho u}{\partial x}, \frac{\partial \rho v}{\partial y} \approx 10^{-8}$$

(Employing consistently the same units).

Therefore, to a first approximation the term $\frac{\partial \rho}{\partial t}$ may be omitted from the equation of continuity.

Then the equation of continuity can be written as follows:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0, \tag{1}$$

or, passing to dimensionless variables,

$$\frac{\partial \rho \bar{w}}{\partial \bar{z}} = -\frac{1}{\varepsilon'}\left(\frac{\partial \rho \bar{u}}{\partial \bar{x}} + \frac{\partial \rho \bar{v}}{\partial \bar{y}}\right). \tag{2}$$

Substituting in (2) instead of $\bar{u}$ and $\bar{v}$ their equivalent expressions from (6) § 2, we obtain:

$$\frac{\partial \rho \bar{w}}{\partial \bar{z}} = -\frac{\varepsilon}{\varepsilon'}\left(\frac{\partial \rho u'}{\partial x} + \frac{\partial \rho v'}{\partial y}\right). \tag{3}$$

In particular when $u' = v' = 0$, i.e., when the wind is geostrophic, $w = 0$.

As is seen from (3) the value $\rho\bar{w}$ contains the factor $\frac{\varepsilon}{\varepsilon'}$. Introducing w into equations (5) § 2, we see that the fourth term on the left side will contain the factor ε^2. If the linear term in ε is sufficient, the terms of the equations of motion which contain w may be omitted.

For the determination of the quantity ρw, Kibel' uses the equation of atmospheric motion in equation (15) § 1 Chapter IX.

Deleting the terms containing w, we obtain:

$$\left.\begin{aligned} \varepsilon\left[\rho\frac{\partial \bar{u}}{\partial \bar{t}} + \rho\frac{\partial}{\partial \bar{x}}\frac{\bar{u}^2+\bar{v}^2}{2} - \rho\bar{v}\left(\frac{\partial \bar{v}}{\partial \bar{x}} - \frac{\partial \bar{u}}{\partial \bar{y}}\right)\right] &= \frac{1}{lLU}\frac{\partial p}{\partial \bar{x}} + \rho\bar{v}, \\ \varepsilon\left[\rho\frac{\partial \bar{v}}{\partial \bar{t}} + \rho\frac{\partial}{\partial \bar{y}}\frac{\bar{u}^2+\bar{v}^2}{2} + \rho\bar{u}\left(\frac{\partial \bar{v}}{\partial \bar{x}} - \frac{\partial \bar{u}}{\partial \bar{y}}\right)\right] &= -\frac{1}{lLU}\frac{\partial p}{\partial y} - \rho\bar{u}. \end{aligned}\right\} \tag{4}$$

The density ρ can be included in the differentiated functions, on the assumption

$$\rho\frac{\partial}{\partial \bar{x}}\frac{\bar{u}^2+\bar{v}^2}{2} \approx \frac{\partial}{\partial \bar{x}}\rho\frac{\bar{u}^2+\bar{v}^2}{2}; \quad \rho\frac{\partial}{\partial \bar{y}}\frac{\bar{u}^2+\bar{v}^2}{2} \approx \frac{\partial}{\partial \bar{y}}\rho\frac{\bar{u}^2+\bar{v}^2}{2},$$

since according to Fridman and Hesselberg $u^2\frac{\partial \ln \rho}{\partial x} \approx 10^{-6} - 10^{-5}$ and

$$\frac{\partial \bar{u}^2}{\partial \bar{x}} \approx 10^{-4} - 10^{-3}.$$

Then equations (4) can be rewritten as follows:

$$\left.\begin{aligned} \rho\bar{u} &= -\frac{1}{1+\varepsilon\Omega}\cdot\frac{\partial}{\partial \bar{y}}\left(\frac{p}{lLU} + \rho\varepsilon\frac{\bar{u}^2+\bar{v}^2}{2}\right) - \varepsilon\frac{\rho}{1+\varepsilon\Omega}\frac{\partial \bar{v}}{\partial \bar{t}}, \\ \rho\bar{v} &= \frac{1}{1+\varepsilon\Omega}\frac{\partial}{\partial \bar{x}}\left(\frac{p}{lLU} + \rho\varepsilon\frac{\bar{u}^2+\bar{v}^2}{2}\right) + \varepsilon\frac{\rho}{1+\varepsilon\Omega}\frac{\partial \bar{u}}{\partial t}, \end{aligned}\right\} \tag{5}$$

where

$$\Omega = \frac{\partial \bar{v}}{\partial \bar{x}} - \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{1}{lLU}\left(\frac{\partial}{\partial \bar{x}}\frac{1}{\rho}\frac{\partial p}{\partial \bar{x}} + \frac{\partial}{\partial \bar{y}}\frac{1}{\rho}\frac{\partial p}{\partial \bar{y}}\right) + \varepsilon\left(\frac{\partial v'}{\partial \bar{x}} - \frac{\partial u'}{\partial \bar{y}}\right). \tag{6}$$

Differentiating the first equation (5) with respect to $\bar{x}$ and the second with respect to y and adding them together we obtain

* [In the Russian text the factor $\frac{1}{lLU}$, which pertains to p only, stands in front of the brackets in both equations (5) – Translator's note]

$$\frac{\partial \rho \bar{u}}{\partial x}+\frac{\partial \rho \bar{v}}{\partial y}=\frac{\varepsilon}{lLU}\left[\frac{\partial \Omega}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \Omega}{\partial y}\frac{\partial p}{\partial x}\right]-\varepsilon\rho\frac{\partial \Omega}{\partial t}, \tag{7}$$

whereby we have omitted all terms containing ε^2.

Upon the calculation of Ω the last term containing ε, can also be omitted. Consequently

$$\frac{\partial \rho w}{\partial z}=-\frac{1}{lLU}\frac{\varepsilon}{\varepsilon'}\left(\frac{\partial \Omega}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \Omega}{\partial y}\frac{\partial p}{\partial x}\right)+\frac{\varepsilon}{\varepsilon'}\rho\frac{\partial \Omega}{\partial t}. \tag{8}$$

§ 4. Summary of Formulas for the Determination of u, v, w to the First and Second Approximations

We shall now compile a summary of formulas determining u, v, w to a first and second approximation, denoting the first approximation by the subscript 1, and the second by the subscript 2. Then we have

$$u_1=-\frac{1}{l\rho}\frac{\partial p}{\partial y}, \tag{1}$$

$$v_1=\frac{1}{l\rho}\frac{\partial p}{\partial x}, \tag{2}$$

$$w_1=0, \tag{3}$$

$$u_2=-\frac{1}{l\rho}\frac{\partial p}{\partial y}+\frac{1}{l^2}\left[-\frac{\partial}{\partial t}\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{l}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\partial}{\partial y}\frac{1}{\rho}\frac{\partial p}{\partial x}\right)\right], \tag{4}$$

$$v_2=\frac{1}{l\rho}\frac{\partial p}{\partial x}+\frac{1}{l^2}\left[-\frac{\partial}{\partial t}\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{1}{l}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial p}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\partial}{\partial y}\frac{1}{\rho}\frac{\partial p}{\partial y}\right)\right], \tag{5}$$

$$w_2=\frac{1}{l^2\rho}\int_0^z\left(\frac{\partial p}{\partial x}\frac{\partial \Omega}{\partial y}-\frac{\partial p}{\partial y}\frac{\partial \Omega}{\partial x}\right)dz+\frac{1}{l\rho}\int_0^z\rho\frac{\partial \Omega}{\partial t}dz, \tag{6}$$

where

$$\Omega=\frac{1}{l}\left[\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\frac{1}{\rho}\frac{\partial p}{\partial y}\right]. \tag{7}$$

Thus Kibel' showed that the components of wind velocity in the first and second approximations are determined by magnitudes $\frac{1}{\rho}\frac{\partial p}{\partial x}$ and $\frac{1}{\rho}\frac{\partial p}{\partial y}$ or by the vector

$$\frac{1}{\rho}\operatorname{grad}_2 p. \tag{8}$$

Consequently, forecasting the wind velocity is reduced to forecasting the vector field (8).

The vector field (8) can be expressed by the pressure or temperature field at the earth's surface. To this end, Kibel applies the barometric formula in the most general form:

$$p(x, y, z, t)=p_0(x, y, t)\,e^{-\frac{g}{R}\int_0^z\frac{dz}{T(x, y, z, t)}}, \tag{9}$$

where $p(x, y, z, t)$ and $T(x, y, z, t)$ are the corresponding pressure and temperature at the altitude z, and $p_0(x, y, t)$-the pressure at ground level.

Logarithmically differentiating the barometric formula (9) with respect to x and y separately and applying Clapeyron's equation, we obtain:

$$\left.\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial x}&=RT\left[\frac{1}{p_0}\frac{\partial p_0}{\partial x}+\frac{g}{R}\int_0^z\frac{1}{T^2}\frac{\partial T}{\partial x}dz\right],\\ \frac{1}{\rho}\frac{\partial p}{\partial y}&=RT\left[\frac{1}{p_0}\frac{\partial p_0}{\partial y}+\frac{g}{R}\int_0^z\frac{1}{T^2}\frac{\partial T}{\partial y}dz\right].\end{aligned}\right\}\qquad(10)$$

Thus, according to (10) the determination of the vector (8) is reduced to that of the pressure distribution at ground level and the spatial distribution of the temperature, i.e., finding the functions $p_0(x, y, t)$ and $T(x, y, z, t)$.

§ 5. Evaluation of the Dependence of Temperature on Altitude

Prior to the derivation of the equation determining the functions T and p_0, Kibel' brilliantly determined the form of the dependence of T upon z, employing the equation of heat balance.

As we have seen earlier, the function F enters in the equation of heat balance. This function expresses the radiational heat influx. It can be approximately determined in the following way:
It should satisfy the equations

$$\frac{1}{k\rho}\frac{\partial}{\partial z}\frac{1}{k\rho}\frac{\partial F}{\partial z}=F-\frac{1}{k\rho}\frac{\partial}{\partial z}\frac{1}{k\rho}\frac{\partial E}{\partial z},\qquad(1)$$

where $k\rho$ is the absorption coefficient, and

$$E=\sigma T^4 f(T)\qquad(2)$$

(σ is the Stefan-Boltzmann constant; $f(T)<1$ is the empirical function introduced to account for the difference between water vapor and black body emissivities).

In formula (2) we replace T by its mean altitude distribution. From (1) we then obtain a second order differential equation for the unknown function F, which yields F upon its integration.

The solution of equation (1) should satisfy certain boundary conditions. Kibel' showed that for these we may adopt the following:

$$\left.\begin{aligned}&F=0 \quad \text{as} \quad z\to\infty,\\ &F=\sigma k T_0^4[0.97+0.145\sqrt{e}-2f(T_0)] \quad \text{as} \quad z\to 0.\end{aligned}\right\}\qquad(3)$$

This is the Simpson-Brunt empirical formula, where e is the water vapor pressure in mm Hg. The investigations of Kibel' have shown that the function F found in this way is close to zero everywhere outside the boundary layer, and reaches appreciable magnitudes only with boundary layers. At the tropopause the function F has a considerably negative extremum since it is negative in the upper part of the troposphere and in the stratosphere. In the planetary boundary layer F may assume either positive or negative values.

We should note that the sign of F indicates the sign of the curvature of the stratification * curve, for a resting atmosphere and for stationary radiation conditions. Under these conditions

$$u=v=w=0,$$

$$\frac{\partial p}{\partial t}=\frac{\partial T}{\partial t}=0,$$

and the equation of heat balance takes the following form

$$\frac{1}{\rho}\frac{\partial}{\partial z}\lambda\frac{\partial T}{\partial z}=-\frac{k}{c_p}\cdot F.\qquad(4)$$

---

* [Translator's note. The graph of temperature vs. altitude.]

Consequently, outside the boundary layer, where $F=0$,

$$\frac{\partial T}{\partial z}=\text{const.},$$

i.e., the altitude distribution of the temperature should be practically linear.

On these grounds Kibel' concludes that in the general case we should look for a temperature distribution outside the boundary layer of the form:

$$T(x, y, z, t)=T_0(x, y, t)-\gamma z+\delta(x, y, z, t), \tag{5}$$

where δ is some small quantity and γ is the lapse rate, independent of position and fixed in time. Since the departure from linearity of the temperature distribution in altitude depends upon the heat radiation function F as well as upon the atmospheric motion and nonstationary conditions, the quantity δ can be arbitrarily split into two parts

$$\delta=\delta'+\delta'',$$

where δ' is the departure from linearity due to radiative processes, and δ'' the departure due to atmospheric motion.

Thus, assuming the following equality to be approximately valid

$$\frac{1}{\rho}\frac{\partial}{\partial z}\lambda\frac{\partial\delta'}{\partial z}=-\frac{k}{c_p}\cdot F, \tag{6}$$

the equation of heat balance will have the form of an adiabatic equation outside the boundary layer

$$\frac{dT}{dt}-\frac{R}{c_p}\cdot\frac{T}{p}\cdot\frac{dp}{dt}=0. \tag{7}$$

§ 6. Kibel's First Fundamental Equation

In this way, Kibel' has shown that the distribution of temperature should have the form

$$T(x, y, z, t)=T_0(x, y, t)-\gamma z+\delta(x, y, z, t). \tag{1}$$

Therefore formulas (10) § 4 for the components of the vector $a\,\text{grad}\,p$, will assume the form

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{RT}{p_0}\frac{\partial p_0}{\partial x}+Tg\int_0^z\frac{1}{(T_0-\gamma z+\delta)^2}\left(\frac{\partial T_0}{\partial x}+\frac{\partial\delta}{\partial x}\right)dz. \tag{2}$$

Since $\delta\ll T_0$ and $\gamma z\ll T$ the equations can be written to a high accuracy as

$$\left.\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial x}&=\frac{RT_0}{p_0}\frac{\partial p_0}{\partial x}+\left(\frac{g}{T_0}\frac{\partial T_0}{\partial x}-\frac{R\gamma}{p_0}\frac{\partial p_0}{\partial x}\right)z+\frac{g}{T_0}\int_0^z\frac{\partial\delta}{\partial x}dz,\\ \frac{1}{\rho}\frac{\partial p}{\partial y}&=\frac{RT_0}{p_0}\frac{\partial p_0}{\partial y}+\left(\frac{g}{T_0}\frac{\partial T_0}{\partial y}-\frac{R\gamma}{p_0}\frac{\partial p_0}{\partial y}\right)z+\frac{g}{T_0}\int_0^z\frac{\partial\delta}{\partial y}dz.\end{aligned}\right\} \tag{3}$$

For $\delta=0$, i.e., when the temperature varies linearly with altitude and the lapse rate is constant, equations (3) coincide with those already discussed in

Chapter XI. In this case the problem is solved according to a "simplified scheme" (Kibel's terminology). When δ is different from zero, the components of $\frac{1}{\rho}\operatorname{grad} p$ will contain an additional term. In these cases the problem is solved according to the "full scheme".

We now write the dry-adiabatic equation in full:

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}-\frac{R}{c_p}\frac{T}{p}\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)=0. \quad (4)$$

We introduce the following expressions into (4):

$$u=-\frac{1}{l\rho}\frac{\partial p}{\partial y}+u',$$
$$v=\frac{1}{l\rho}\frac{\partial p}{\partial x}+v', \quad (5)$$
$$-\frac{\partial T}{\partial z}=\gamma;\ -\frac{R}{c_p}\frac{T}{p}\frac{\partial p}{\partial z}=\frac{g}{c_p}=\gamma_a.$$

The dry-adiabatic equation will then assume the following form

$$\frac{\partial T}{\partial t}+\left(-\frac{1}{l\rho}\frac{\partial p}{\partial y}+u'\right)\frac{\partial T}{\partial x}+\left(\frac{1}{l\rho}\frac{\partial p}{\partial x}+v'\right)\frac{\partial T}{\partial y}+w(\gamma_a-\gamma)-$$
$$-\frac{R}{c_p}\frac{T}{p}\left[\frac{\partial p}{\partial t}+u'\frac{\partial p}{\partial x}+v'\frac{\partial p}{\partial y}\right]=0. \quad (6)$$

Replacing T in (6) by its expression (5) § 5 and $\frac{1}{\rho}\frac{\partial p}{\partial x}$ and $\frac{1}{\rho}\frac{\partial p}{\partial y}$ by their corresponding expressions from (3) § 6, we obtain:

$$\frac{\partial T_0}{\partial t}+\frac{\partial\delta}{\partial t}+\left[-\frac{1}{l}\left(\frac{RT_0}{p_0}\frac{\partial p_0}{\partial y}+\frac{gz}{T_0}\frac{\partial T_0}{\partial y}-\frac{R\gamma z}{p_0}\frac{\partial p_0}{\partial y}+\frac{g}{T_0}\int_0^z\frac{\partial\delta}{\partial y}dz\right)+u'\right]\times$$
$$\times\left(\frac{\partial T_0}{\partial x}+\frac{\partial\delta}{\partial x}\right)+\left[\frac{1}{l}\left(\frac{RT_0}{p_0}\frac{\partial p_0}{\partial x}+\frac{gz}{T_0}\frac{\partial T_0}{ox}-\frac{R\gamma z}{p_0}\frac{\partial p_0}{\partial x}+\frac{g}{T_0}\int_0^z\frac{\partial\delta}{\partial x}dz\right)+v'\right]\times$$
$$\times\left(\frac{\partial T_0}{\partial y}+\frac{\partial\delta}{\partial y}\right)+w\left(\gamma_n-\gamma+\frac{\partial\delta}{\partial z}\right)-\frac{R}{c_p}\left[\frac{T_0}{p}\frac{\partial p_0}{\partial t}+\frac{gz}{RT_0}\frac{\partial T_0}{\partial t}-\frac{\gamma z}{p_0}\frac{\partial p_0}{\partial t}+\right.$$
$$\left.+\frac{g}{RT_0}\int_0^z\frac{\partial\delta}{\partial t}dz\right]-\frac{R}{c_p}\frac{T}{p}\left(u'\frac{\partial p}{\partial x}+v'\frac{\partial p}{\partial y}\right)=0, \quad (7)$$

where we have used the equality

$$\frac{T}{p}\frac{\partial p}{\partial t}=\frac{1}{R}\frac{1}{\rho}\frac{\partial p}{\partial t}=\frac{T_0}{p_0}\frac{\partial p_0}{\partial t}+\frac{gz}{RT_0}\frac{\partial T_0}{\partial t}-\frac{\gamma z}{p_0}\frac{\partial p_0}{\partial t}+\frac{g}{RT_0}\int_0^z\frac{\partial\delta}{\partial t}dz, \quad (8)$$

derived analogously to formulas (3).

We introduce for brevity the following notation

$$\{A,\ B\}=\frac{\partial A}{\partial x}\frac{\partial B}{\partial y}-\frac{\partial A}{\partial y}\frac{\partial B}{\partial x}=\begin{vmatrix}\frac{\partial A}{\partial x}&\frac{\partial A}{\partial y}\\ \frac{\partial B}{\partial x}&\frac{\partial B}{\partial y}\end{vmatrix} \quad (9)$$

This functional determinant is known as the Jacobian. Equation (9) can be written in vector notation as the vector product of grad A and grad B, both lying in the x, y plane.

$$\{A,\ B\}=[\operatorname{grad} A,\ \operatorname{grad} B]. \quad (10)$$

We note that

$$\{B,\ A\}=-\{A,\ B\}. \quad (11)$$

Applying (9) and rearranging the terms containing u', v', w, δ, or z, we bring (7) to the form

$$\frac{\partial T_0}{\partial t}-\frac{R}{c_p}\frac{T_0}{p_0}\frac{\partial p_0}{\partial t}-\frac{RT_0}{lp_0}\{T_0, p_0\}-\gamma_a z\left(\frac{1}{T_0}\frac{\partial T_0}{\partial t}-\frac{R\gamma}{g}\frac{1}{p_0}\frac{\partial p_0}{\partial t}\right)+$$
$$+\frac{R\gamma z}{lp_0}\{T_0, p_0\}+\frac{\partial\delta}{\partial t}-\frac{\gamma_a}{T_0}\int_0^z\frac{\partial\delta}{\partial t}dz-\frac{1}{l\rho}\{\delta, p_0\}+\frac{g}{lT_0}\left\{\int_0^z\delta\,dz, T_0\right\}=$$
$$=-u'\left(\frac{\partial T_0}{\partial x}-\frac{R}{c_p}\frac{T}{p}\frac{\partial p}{\partial x}\right)-v'\left(\frac{\partial T_0}{\partial y}-\frac{R}{c_p}\frac{T}{p}\frac{\partial p}{\partial y}\right)-$$
$$-w\left(\gamma_a-\gamma+\frac{\partial\delta}{\partial z}\right). \qquad (12)$$

We have thus obtained the fundamental equation of the problem. It can be solved by successive approximations. We have shown above that the first approximation will be obtained assuming

$$u'=v'=w=0,$$

i.e., considering the wind to be geostrophic.

In that case the terms on the right side of equation (12) will vanish and equation (12) will assume the following form

$$\frac{\partial T_0}{\partial t}-\frac{R}{c_p}\frac{T_0}{p_0}\frac{\partial p_0}{\partial t}-\frac{RT_0}{lp_0}\{T_0, p_0\}-\gamma_a z\left(\frac{1}{T_0}\frac{\partial T_0}{\partial t}-\frac{R\gamma}{g}\frac{1}{p_0}\frac{\partial p_0}{\partial t}\right)+$$
$$+\frac{R\gamma z}{lp_0}\{T_0, p_0\}+\frac{\partial\delta}{\partial t}-\frac{\gamma_a}{T_0}\int_0^z\frac{\partial\delta}{\partial t}dz-\frac{1}{l\rho}\{\delta, p_0\}+\frac{g}{lT_0}\left\{\int_0^z\delta\,dz, T\right\}=0. \qquad (13)$$

This approximate solution of the problem contains three undetermined functions p_0, T_0, δ. Three equations are required for their determination. The lacking two equations were obtained by Kibel' from the boundary conditions. Assuming that at $z=0$ $\delta(x, y, 0, t)=0$, i.e., that T_0 is the air temperature at ground level, we obtain:

$$\frac{\partial T_0}{\partial t}-\frac{R}{c_p}\frac{T_0}{p_0}\frac{\partial p_0}{\partial t}-\frac{RT_0}{lp_0}\{T_0, p_0\}=0. \qquad (14)$$

Applying (14), equation (13) may be written in a shorter form:

$$\frac{\partial\delta}{\partial t}-\frac{\gamma_a}{T_0}\int_0^z\frac{\partial\delta}{\partial t}dz=\frac{1}{l\rho}\{\delta, p_0\}-\frac{g}{lT_0}\left\{\int_0^z\delta dz, T\right\}+\gamma_a z\left(\frac{1}{T_0}\frac{\partial T_0}{\partial t}-\frac{R\gamma}{g}\frac{1}{p_0}\frac{\partial p_0}{\partial t}\right)-\frac{R\gamma z}{lp_0}\{T_0, p_0\}. \qquad (15)$$

The third necessary equation was obtained by Kibel' from boundary conditions at the tropopause.

§ 7. Boundary Conditions at the Tropopause

The problem of the formulation of an additional boundary condition is of utmost importance, since the obtained solution depends on the nature of this condition. Prior to Kibel's investigations in dynamic meteorology, the corresponding conditions were often taken as infinite, which is meaningless in the context of the present problem. The frequently employed condition $\rho\to 0$, $p\to 0$ as $z\to\infty$ leads to indeterminate expressions since all equations contain terms of the form $\frac{1}{\rho}\frac{\partial p}{\partial t}$, $\frac{1}{\rho}\frac{\partial p}{\partial x}$ etc.

It is more rational to consider the tropopause as a natural interface, and to employ the boundary conditions which hold at the tropopause for air flows in the troposphere.

Such approach encounters serious difficulties, associated with the selection of a suitable hypothesis as to the nature of the motion at the tropopause. The simplest (and roughest) assumption is to consider the tropopause as stationary.

Sometimes the tropopause is considered as a free surface in the hydrodynamic sense, assuming the external pressure on the tropopause to be known.

I. A. Kibel' lays down different conditions for the tropopause. He assumes that the tropopause consists always of the same air particles. This assumption is quite natural and, as we have seen in Chapter XIII, equivalent to the stipulation that the tropopause should move at a velocity considerably lower than that of

sound.

Denoting the altitude of the tropopause by H, we notice that this altitude changes from place to place and with time too, i.e., that

$$H = H(x, y, t). \tag{1}$$

We shall denote the temperature of the air constituting the tropopause by $T_H(x, y, t)$ and consider it as known from observation; the same will be applied to the altitude H of the tropopause. Then the equation of the tropopause may be written in the form

$$T_0(x, y, t) - \gamma z + \delta(x, y, z, t) = T_H(x, y, t). \tag{2}$$

Further Kibel' assumes that the temperature, in its altitude variation, has a minimum at the tropopause, i. e., he assumes that the temperature in the troposphere decreases up to the lower boundary of the stratosphere, and then slowly increases.

Thus in the vicinity of the tropopause δ should fulfil

$$\frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = H(x, y, t).$$

Consequently, differentiating (2) with respect to z, we obtain

$$\frac{\partial \delta}{\partial z} = \gamma \quad \text{at} \quad z = H. \tag{4}$$

For δ to fulfil condition (4) it should have the following form at the tropopause

$$\delta(x, y, z, t) = \delta_H + \gamma(z - H) + (z - H)^2 \tau(x, y, t), \tag{5}$$

where τ is a function, finite at $z = H$, and δ_H stands for the radiation effects.

Substituting (5) in (2) we obtain the equation of the tropopause in the form

$$T_0(x, y, t) - \gamma z + \delta_H + \gamma(z - H) + (z - H)^2 \tau(x, y, t) = T_H(x, y, t)$$

or

$$T_0(x, y, t) - \gamma H + \delta_H + (z - H)^2 \tau(x, y, t) = T_H(x, y, t). \tag{6}$$

Since the tropopause permanently consists of the same particles,

$$\frac{d}{dt}[T_0 - \gamma H + \delta_H + (z - H)^2 \tau - T_H]_{z=H} = 0 \tag{7}$$

or, written in full

$$\frac{\partial}{\partial t}(T_0 - \gamma H - T_H) + u_H \frac{\partial (T_0 - \gamma H - T_H)}{\partial x} + v_H \frac{\partial (T_0 - \gamma H - T_H)}{\partial y} = 0, \tag{8}$$

where u_H, v_H are the components of wind velocity at the tropopause.

In equation (8) all terms are of the same order, but Kibel' assumes that

$$\left.\begin{aligned}\frac{\partial T_H}{\partial t}+\gamma\frac{\partial H}{\partial t}&\ll\frac{\partial T_0}{\partial t},\\ \frac{\partial T_H}{\partial x}+\gamma\frac{\partial H}{\partial x}&\ll\frac{\partial T_0}{\partial x},\\ \frac{\partial T_H}{\partial y}+\gamma\frac{\partial H}{\partial y}&\ll\frac{\partial T_0}{\partial y}.\end{aligned}\right\}\tag{9}$$

Indeed, observations show that a high tropopause is cool and a low tropopause is relatively warm. Therefore, the terms on the left sides of the inequalities (9) often possess different signs and their sum is considerably smaller than the right side. Therefore, the equation of the tropopause can be written in the following form:

$$\frac{\partial T_0}{\partial t}+u_H\frac{\partial T_0}{\partial x}+v_H\frac{\partial T_0}{\partial y}=0.\tag{10}$$

It remains now only to insert in (10) instead of u_H and v_H their equivalent expressions

$$\left.\begin{aligned}u_H&=-\frac{1}{l\rho_H}\frac{\partial p_H}{\partial y}+u'_H,\\ v_H&=\frac{1}{l\rho_H}\frac{\partial p_H}{\partial x}+v'_H.\end{aligned}\right\}\tag{11}$$

Repeating the calculations of § 7 we obtain

$$\frac{\partial T_0}{\partial t}-\frac{R(T_0-\gamma H)}{lp_0}\{T_0,\ p_0\}-\frac{g}{lT_0}\left\{T_0,\ \int_0^H\delta\,dz\right\}=-u'_H\frac{\partial T_0}{\partial x}-v'_H\frac{\partial T_0}{\partial y},\tag{12}$$

where the negligible terms

$$\left(\frac{\partial\delta}{\partial t}\right)_{z=H}+u_H\left(\frac{\partial\delta}{\partial x}\right)_{z=H}+v_H\left(\frac{\partial\delta}{\partial y}\right)_{z=H},$$

are omitted. Thus we obtain the missing equation for the determination of T_0, p_0 and δ.

For the first approximation this equation takes the form

$$\frac{\partial T_0}{\partial t}=\frac{R(T_0-\gamma H)}{lp_0}\{T_0,\ p_0\}+\frac{g}{lT_0}\left\{T_0,\ \int_0^H\delta\,dz\right\}.\tag{13}$$

If the mean value of the integral $\int_0^H\delta\,dz$ and its derivatives can be neglected we obtain an equation for the "simplified scheme"

$$\frac{\partial T_0}{\partial t}=\frac{RT_H}{lp_0}\{T_0,\ p_0\},\tag{14}$$

where T_H is the (given) temperature of the tropopause.

Thus, by our equations (14) § 6 and (14) § 7 we have solved to a first approximation, the problem of the determination of local derivatives of pressure and temperature, i.e., $\frac{\partial T_0}{\partial t}$ and $\frac{\partial p_0}{\partial t}$ according to the simplified scheme. They may be combined and written as follows:

$$\left(\frac{\partial p_0}{\partial t}\right)=-\alpha\{T_0,\ p_0\},\tag{I}$$

$$\left(\frac{\partial T_0}{\partial t}\right)=\beta\{T_0,\ p_0\},\tag{II}$$

whereby

$$\alpha=\frac{c_p}{l}\left(1-\frac{T_H}{T_0}\right),\tag{15}$$

$$\beta=\frac{RT_H}{lp_0}.\tag{16}$$

The values of α and β for a given latitude exhibit only slight diurnal and spatial variation. The magnitude β is less subject to variation as formula (16) shows.

I. A. Kibel' assumes that to β can almost always be assigned the value 5.10^6 m^3/sec ton, at latitudes in the range 55°–65°.

The magnitude α undergoes greater seasonal variation. As can be seen from (15), α is smaller in winter than in summer, since in winter T_0 is small and T_H-large.

Thus, at latitude 60°, in winter for $T_0 = 255°$, $T_H = 225°$, $\alpha \approx 0.9 \cdot 10^6 m^2$/deg sec and in summer for $T_0 = 285°$, $T_H = 215°$, $\alpha \approx 1.6 \cdot 10^6 m^2$/deg sec.

The changes of temperature and pressure over the prognostic period can be obtained by calculating the second derivatives of the temperature and the pressure

$$\frac{\partial^2 T_0}{\partial t^2} = -\alpha \frac{\partial}{\partial t}\{T_0, p_0\}, \tag{17}$$

$$\frac{\partial^2 p_0}{\partial t^2} = \beta \frac{\partial}{\partial t}\{T_0, p_0\} \tag{18}$$

expanding Δp_0 and ΔT_0 in a Taylor series up to terms quadratic in Δt:

$$\Delta p_0 = \frac{\partial p_0}{\partial t}\Delta t + \frac{\partial^2 p_0}{\partial t^2} \cdot \frac{(\Delta t)^2}{2}, \tag{19}$$

$$\Delta T_0 = \frac{\partial T_0}{\partial t}\Delta t + \frac{\partial^2 T_0}{\partial t^2} \cdot \frac{(\Delta t)^2}{2}. \tag{20}$$

§ 8. The Solution of the Problem According to the Simplified Scheme

The precalculation of weather elements is considerably simplified by assuming that the temperature varies strictly linearly with altitude, i. e., that $\delta \equiv 0$.

Equations (12) and (13) § 7 will then assume the form

$$\frac{\partial T_0}{\partial t} - \frac{R}{c_p}\frac{T_0}{p_0}\frac{\partial p_0}{\partial t} - \frac{RT_0}{c_p}\{T_0, p_0\} - \gamma_a z\left(\frac{1}{T_0}\frac{\partial T_0}{\partial t} - \frac{R\gamma}{g}\frac{1}{p_0}\frac{\partial p_0}{\partial t}\right) + \frac{R\gamma z}{lp_0}\{T_0, p_0\} = -u'\left(\frac{\partial T_0}{\partial x} - \frac{R}{c_p}\frac{T}{p}\frac{\partial p}{\partial x}\right) - v'\left(\frac{\partial T_0}{\partial y} - \frac{R}{c_p}\frac{T}{p}\frac{\partial p}{\partial y}\right) - w(\gamma_a - \gamma), \tag{1}$$

$$\frac{\partial T_0}{\partial t} - \frac{R(T_0 - \gamma H)}{lp_0}\cdot\{T_0, p_0\} = -u'_H\frac{\partial T_0}{\partial x} - v'_H\frac{\partial T_0}{\partial y}. \tag{2}$$

Equation (1) refers to an air particle located at an arbitrary altitude z. We shall write it now for a particle located at ground level, i. e., for $z = 0$:

$$\frac{\partial T_0}{\partial t} - \frac{R}{c_p}\frac{T_0}{p_0}\frac{\partial p_0}{\partial t} - \frac{RT_0}{lp_0}\{T_0, p_0\} = u'_0\frac{\partial T_0}{\partial x} - v'_0\frac{\partial T_0}{\partial y} + \frac{R}{c_p}\frac{T_0}{p_0}\left(u'_0\frac{\partial p_0}{\partial x} + v'_0\frac{\partial p_0}{\partial y}\right). \tag{3}$$

In equation (3) the order of the terms $u'_0\frac{\partial p_0}{\partial x}$, $v'_0\frac{\partial p_0}{\partial y}$ is about $10^{-7} - 10^{-6}$ and hence the last terms in this equation may be ignored. The equations for the local pressure and temperature changes may then be written in the following form:

$$\left.\begin{aligned}\frac{\partial T_0}{\partial t} &= \frac{RT_H}{lp_0}\{T_0, p_0\} - u'_H\frac{\partial T_0}{\partial x} - v'_H\frac{\partial T_0}{\partial y},\\ \frac{\partial p_0}{\partial t} &= \frac{c_p(T_H - T_0)}{l}\frac{}{T_0}\{T_0, p_0\} + \frac{c_p}{R}\frac{p_0}{T_0}\left[(u'_0 - u'_H)\frac{\partial T_0}{\partial x} + (v_0 - v'_H)\frac{\partial T_0}{\partial y}\right].\end{aligned}\right\} \tag{4}$$

We shall use equations (4) in the form

$$\left.\begin{aligned}\frac{\partial T_0}{\partial t} &= \left(\frac{\partial T_0}{\partial t}\right)_a + \left(\frac{\partial T_0}{\partial t}\right)_g,\\ \frac{\partial p_0}{\partial t} &= \left(\frac{\partial p_0}{\partial t}\right)_a + \left(\frac{\partial p_0}{\partial t}\right)_g,\end{aligned}\right\} \tag{5}$$

where

$$\left(\frac{\partial T_0}{\partial t}\right)_a = \beta\{T_0, p_0\}. \tag{6}$$

$$\left(\frac{\partial p_0}{\partial t}\right)_a = -\alpha\{T_0, p_0\}. \tag{7}$$

$$\left(\frac{\partial T_0}{\partial t}\right)_g = -u'_H\frac{\partial T_0}{\partial x} - v'_H\frac{\partial T_0}{\partial y}, \tag{8}$$

$$\left(\frac{\partial p_0}{\partial t}\right)_g = \frac{c_p}{R}\frac{p_0}{T_0}\left[(u'_0 - u'_H)\frac{\partial T_0}{\partial x} + (v'_0 - v'_H)\frac{\partial T_0}{\partial y}\right]. \tag{9}$$

The formulas (6) and (7) give the first approximation to the solution of the problem, i.e., such local changes of temperature and pressure which occur in the presence of a geostrophic wind. These changes may be called advective, following L.H. Tabarovskii.

Formulas (8) and (9) contain corrections for the ageostrophic deviations accounted for in the second approximation. Since the ageostrophic deviations are connected with accelerations they may be called dynamic, following Tabarovskii.

§ 9. The Graphical Method of Prognosis in the First Approximation

The formulas obtained in § 7 permit the calculation, to a first approximation, of the temperature and pressure changes at a given point. For the precalculation of the expected distribution of temperature and pressure on a synoptic chart comprising vast regions, the graphical method of prognosis in the first approximation, elaborated by Kibel', is more expedient. This method is based on the following considerations:

Multiplying equation (I) § 7 by β, and equation (II) by α, and adding them together, we obtain:

$$\alpha\frac{\partial T_0}{\partial t} + \beta\frac{\partial p_0}{\partial t} = 0. \tag{1}$$

Since α and β change very little during the prognostic period, we may regard them as constant, and equation (1) can be rewritten as follows:

$$\frac{\partial(\alpha T_0 + \beta p_0)}{\partial t} = 0. \tag{2}$$

Integrating this equation we obtain

$$\alpha T_0(x, y, t) + \beta p_0(x, y, t) = \theta(x, y). \tag{3}$$

Equation (3) shows that the first approximation applies only to cases, in which a pressure drop at the earth's surface is accompanied by a temperature rise. Though such cases are actually quite frequent, they by no means are the rule. The determination of the pressure change in the first approximation will then yield highly unsatisfactory results. Therefore, the second approximation is indispensable in this case.

Let us now introduce the function θ into Kibel's prognostic equations (I) and (II) § 7. We differentiate (3) with respect to x and y:

$$\left.\begin{aligned} \alpha \frac{\partial T_0}{\partial x} &= \frac{\partial \theta}{\partial x} - \beta \frac{\partial p_0}{\partial x}, \\ \alpha \frac{\partial T_0}{\partial y} &= \frac{\partial \theta}{\partial y} - \beta \frac{\partial p_0}{\partial y}. \end{aligned}\right\} \tag{4}$$

Multiplying the first of equations (4) by $\frac{\partial p_0}{\partial y}$ and the second by $\frac{\partial p_0}{\partial x}$ and subtracting the second from the first, we obtain:

$$\alpha\left(\frac{\partial T_0}{\partial x}\cdot\frac{\partial p_0}{\partial y} - \frac{\partial T_0}{\partial y}\cdot\frac{\partial p_0}{\partial x}\right) = \frac{\partial \theta}{\partial x}\cdot\frac{\partial p_0}{\partial y} - \frac{\partial \theta}{\partial y}\cdot\frac{\partial p_0}{\partial x}$$

or

$$\alpha\{T_0, p_0\} = \{\theta, p_0\}.$$

Eliminating the Jacobian $\{T_0, p_0\}$, between (5) and (1) § 7, we get

$$\frac{\partial p_0}{\partial t} = -\{\theta, p_0\}. \tag{I}$$

In the same way an analogous relationship is obtained for the temperature T_0:

$$\frac{\partial T_0}{\partial t} = -\{\theta, T_0\}. \tag{II}$$

Thus, p_0 and T_0 satisfy the same first order partial differential equation. It is therefore, sufficient to discuss one of the equations, (I), say. For this purpose we write it out as follows:

$$\frac{\partial p_0}{\partial t} - \frac{\partial \theta}{\partial y}\cdot\frac{\partial p_0}{\partial x} + \frac{\partial \theta}{\partial x}\cdot\frac{\partial p_0}{\partial y} = 0. \tag{5}$$

By a well known result from the theory of differential equations we construct a system of ordinary differential equations, which is equivalent to the partial differential equation (5):

$$\frac{dt}{1} = \frac{dx}{-\frac{\partial \theta}{\partial y}} = \frac{dy}{\frac{\partial \theta}{\partial x}}. \tag{6}$$

Evidently, when "isothets", i.e., lines of equal values of θ, are plotted on the synoptic chart the prognosis of the fields of pressure and temperature can be graphically performed quite rapidly.

Indeed, let us recall the differential equations of the trajectory for plane motion

$$\frac{dt}{1} = \frac{dx}{u} = \frac{dy}{v} \tag{7}$$

and introduce the stream function ψ into equation (7). This function is connected with the components of the particle velocity by the relations

$$u = -\frac{\partial \psi}{\partial y}; \quad v = \frac{\partial \psi}{\partial x}. \tag{8}$$

The equation of trajectory will then take the form

$$\frac{dt}{1} = \frac{dx}{-\frac{\partial \psi}{\partial y}} = \frac{dy}{\frac{\partial \psi}{\partial x}}. \tag{9}$$

A glance at equations (6) and (9) shows they are identical. This means that at the given pressure p_0 the motion of a point in the scalar field $\vartheta(x, y)$ is described by the same equations, as the motion of an individual particle in the stream field $\psi(x, y)$. Consequently the isotherms are analogues of the streamlines.

Since ϑ depends only on x and y and is independent of the time, the isotherms are equivalent to the stationary streamlines, but in the case of stationary motion, the streamlines coincide with the trajectories. Therefore, at a given pressure (or temperature) the motion of a point in the ϑ field can be plotted graphically, as the motion of a point in a stationary flow field is plotted (Figure 153). For a given p_0 (or T_0) the point moves along the line $\vartheta = \text{const}$ with a velocity $|c| = |\text{grad}\,\vartheta|$, whose components are

$$c_x = -\frac{\partial \vartheta}{\partial y}, \quad c_y = \frac{\partial \vartheta}{\partial x}. \tag{10}$$

The region of lower ϑ values remains thereby to the left of the motion.

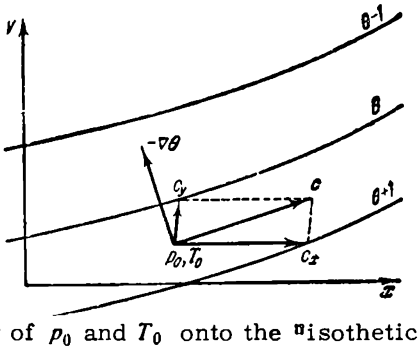

Figure 153. Mapping of p_0 and T_0 onto the "isothetic" field

The method outlined above can be applied to the precalculation of the migration of centers of cyclones and anticyclones, whose trajectories are the isotherms passing through these pressure centers. Apparently in the majority of cases even the first approximation of the simplified scheme suffices for the accurate precalculation of the position of pressure centers for 12–36 hours in advance. However, in the first approximation the pressure center moving along the isotherms does not change its intensity.

Indeed, at a pressure center

$$\frac{\partial p_0}{\partial x} = \frac{\partial p_0}{\partial y} = 0.$$

Consequently in the first approximation, according to (I)

$$\frac{\partial p_0}{\partial t} = 0.$$

Therefore, weakening or build-up of pressure centers can be precalculated only by the more accurate second approximation.

§ 10. Isotherms and the Leading Stream

N. P. Byzov drew attention to the fact that the isotherms very often coincide with the isohypses plotted on pressure charts for the level p = 700 mb. This permits us to re-examine the fruitful hypothesis of the "leading stream", first formulated by the Soviet meteorologist S. I. Troitskii. This hypothesis stems directly from Kibel's prognostic equations. To show it we shall connect the function ϑ with the geopotential Φ (this relationship was obtained by B. I. Izvekov).

It has been shown (Chapter X) that the following relation exists between the components of the pressure gradient and the geopotential:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=\left(\frac{\partial \Phi}{\partial x}\right)_p;\quad \frac{1}{\rho}\frac{\partial p}{\partial y}=\left(\frac{\partial \Phi}{\partial y}\right)_p, \tag{1}$$

where the subscript p indicates that the components of the geopotential gradient are taken at the isobaric surface. Introducing the equivalent expressions for $\frac{1}{\rho}\frac{\partial p}{\partial x}$ and $\frac{1}{\rho}\frac{\partial p}{\partial y}$ from (1) and (3) § 6 and omitting the terms containing δ, we obtain

$$\left.\begin{aligned}\left(\frac{\partial \Phi}{\partial x}\right)_p&=\frac{gz}{T_0}\frac{\partial T_0}{\partial x}+\frac{R(T_0-\gamma z)}{T_0}\frac{\partial p_0}{\partial x},\\ \left(\frac{\partial \Phi}{\partial y}\right)_p&=\frac{gz}{T_0}\frac{\partial T_0}{\partial y}+\frac{R(T_0-\gamma z)}{T_0}\frac{\partial p_0}{\partial y}.\end{aligned}\right\} \tag{2}$$

On the other hand, we have from formula (3) § 9

$$\left.\begin{aligned}\frac{\partial \theta}{\partial x}&=\alpha\frac{\partial T_0}{\partial x}+\beta\frac{\partial p_0}{\partial x},\\ \frac{\partial \theta}{\partial y}&=\alpha\frac{\partial T_0}{\partial x}+\beta\frac{\partial T_0}{\partial y}.\end{aligned}\right\} \tag{3}$$

Comparing equations (2) and (3) we see that the relation

$$\left(\frac{\partial \Phi}{\partial x}\right)_p=m\frac{\partial \theta}{\partial x};\quad \left(\frac{\partial \Phi}{\partial y}\right)_p=m\frac{\partial \theta}{\partial y}, \tag{4}$$

exists between the gradients of the functions Φ and θ where m is a yet undetermined parameter. For the determination of m we shall use the equations

$$\frac{gz}{T_0}=m\alpha,\quad \frac{R(T_0-\gamma z)}{p_0}=m\beta$$

or

$$gz=\alpha T_0 m;\quad RT_0-R\gamma z=\beta p_0 m. \tag{5}$$

Solving equation (5), we obtain:

$$m=\frac{g}{\alpha\gamma+\beta g\rho_0};\quad z=\frac{\alpha T_0}{\alpha\gamma+\beta g\rho_0}. \tag{6}$$

Now relation (4) can be rewritten as follows:

$$\left(\frac{\partial \theta}{\partial x}\right)=\frac{\alpha\gamma+\beta g\rho_0}{g}\left(\frac{\partial \Phi}{\partial x}\right)_p;\quad \frac{\partial \theta}{\partial v}=\frac{\alpha\gamma+\beta g\rho_0}{g}\left(\frac{\partial \Phi}{\partial y}\right)_p. \tag{7}$$

The formulas (7) assume a simpler form by passing from the dynamic altitude [geopotential] to the ordinary altitude. Since

$$d\Phi=g\,dz,$$

we have

$$\left.\begin{aligned}\frac{\partial \theta}{\partial t}&=(\alpha\gamma+\beta g\rho_0)\left(\frac{\partial z}{\partial x}\right)_p,\\ \frac{\partial \theta}{\partial y}&=(\alpha\gamma+\beta g\rho_0)\left(\frac{\partial z}{\partial y}\right)_p.\end{aligned}\right\} \tag{8}$$

Let us recall the relations we have derived

$$c_x=-\frac{\partial \theta}{\partial y};\quad c_y=\frac{\partial \theta}{\partial x},$$

$$u_g=-\frac{g}{l}\left(\frac{\partial z}{\partial y}\right)_p;\quad v_g=\frac{g}{l}\left(\frac{\partial z}{\partial x}\right)_p.$$

Substituting these values of c_x and c_y in formulas (8), we obtain

$$\left.\begin{aligned}c_x&=\frac{(\alpha\gamma+\beta g\rho_0)\,l}{g}u_g,\\ c_y&=\frac{(\alpha\gamma+\beta g\rho_0)\,l}{g}v_g\end{aligned}\right\} \tag{9}$$

or

$$\left.\begin{aligned} c_x &= ku_g, \\ c_y &= kv_g, \end{aligned}\right\} \tag{10}$$

where

$$k = \frac{\alpha\gamma + \beta g\rho_0}{g} \cdot l. \tag{11}$$

Thus, plotting a pressure contour map corresponding to an altitude of the order of

$$z = \frac{\alpha T_0}{\alpha\gamma + \beta g\rho_0}, \tag{12}$$

ground level air particles having a certain pressure and temperature will be displaced in the direction of the geostrophic wind at the corresponding altitude or, in other words, along the isohypses, with a velocity equal to k times the velocity of the geostrophic wind.

Inserting now the values of α and β from (15) and (16) § 7 into the formulas for k and z we obtain, after some transformations

$$k = \frac{1 + \tau\varepsilon}{1 + \varepsilon}, \tag{13}$$

$$z = \frac{H}{1 + \tau\varepsilon}. \tag{14}$$

where we have put for brevity

$$\tau = \frac{T_H}{T_0}; \quad \varepsilon = \frac{\gamma_a - \gamma}{\gamma}, \tag{15}$$

and H is the altitude of the tropopause.

Let us evaluate k and z in two extreme cases, namely for average conditions in summer and winter.

In winter, for $T_0 = 255^\circ$, $T_H = 225^\circ$, $\gamma = 0.4^\circ/100$ m $= 0.004^\circ/$m, we obtain: $H \approx 7$km, $\tau = 0.9$, $\varepsilon = 1.5$. Consequently, $z \approx 3$ km, $k \approx 0.9$. In summer for $T_0 = 285^\circ$, $T_H = 215^\circ$, $\gamma = 0.6^\circ$ $\tau = 0.7$, $\varepsilon \approx 0.7$, $H = 11$ km. Consequently $z = 7$ km, $k \approx 0.8$.

Thus, the altitude of the leading stream may change from season to season, within wide limits, from 3 to 7 km. In the winter the altitude is lower than in summer.

The factor k is always less than unity since $\tau = \frac{T_H}{T_0} < 1$ whereby k is larger in winter than in summer.

With regard to the center of a cyclone, this means that the center moves at a lower velocity than the leading stream, whereby at the same velocity of the leading stream the center of the cyclone will move more rapidly in winter than in summer. These results are in accord with empirical facts established long ago.

§ 11. The Dynamic Change of Pressure

The evaluation of the order of the terms in equations (4) § 8 shows that the dynamic change of temperature can be neglected. This means that the temperature changes forecasted by the first approximation cannot be improved by employing the second approximation. It does not mean, however, that the first approximation gives always a correct temperature forecast, but only that outside the boundary layer, where the processes are approximately adiabatic, the local changes of

temperature are mainly connected with the geostrophic mass advection and not with ageostrophic deviations.

In contrast, for the forecast of the evolution of the pressure field and not only its motion, the second approximation is indispensable. By evolution we mean the contribution of the acceleration of the air motion to the change of pressure, i. e., the ageostrophic deviations.

Let us consider formula (9) § 8 which expresses the dynamic pressure change:

$$\left(\frac{\partial p_0}{\partial t}\right)_g = \frac{c_p}{R}\frac{p_0}{T_0}\left[(u'_0 - u'_H)\frac{\partial T_0}{\partial x} + (v'_0 - v'_H)\frac{\partial T_0}{\partial y}\right]. \tag{1}$$

But according to formulas (4) and (5) § 4,

$$\left.\begin{aligned} u' &= \frac{1}{l^2}\left[-\frac{\partial}{\partial t}\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{l}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\partial}{\partial y}\frac{1}{\rho}\frac{\partial p}{\partial x}\right)\right], \\ v' &= \frac{1}{l^2}\left[-\frac{\partial}{\partial t}\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{l}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\partial}{\partial y}\frac{1}{\rho}\frac{\partial p}{\partial x}\right)\right]. \end{aligned}\right\} \tag{2}$$

Substituting (2) in (1), after cumbersome transformations in the course of which a number of simplifying assumptions were made concerning the derivatives, Kibel' obtained the following expression, which represents a correction of the first approximation in order to obtain the second approximation:

$$\left(\frac{\partial p_1}{\partial t}\right)_g = A\left[\frac{\partial T_0}{\partial x}\left\{T_0, \frac{\partial \tilde{\theta}}{\partial x}\right\} + \frac{\partial T_0}{\partial y}\left\{T_0, \frac{\partial \tilde{\theta}}{\partial y}\right\}\right], \tag{3}$$

where

$$\tilde{\theta} = \tilde{\alpha}T_0 + \beta p_0, \tag{4}$$

$$\tilde{\alpha} = \frac{\left(\frac{\gamma_a}{\gamma}\right)}{\frac{2\gamma_a}{\gamma} + \frac{T_0}{T_H} - 1}\cdot a, \tag{5}$$

$$A = \frac{c_p}{R}\cdot\frac{p_0}{T_0}\cdot\frac{a}{l}\left[2\cdot\frac{\gamma_a}{\gamma} + \frac{T_0}{T_H} - 1\right]. \tag{6}$$

As can be seen from formula (5) $\tilde{\alpha}$ differs only slightly from a. Therefore the field $\tilde{\theta}(x, y, t)$ in a given moment differs little from the field $\theta(x, y)$. The difference between $\tilde{\theta}$ and θ is only in that $\tilde{\theta}$ depends on the time while θ is time independent. Let us explain the reason for the time variation of $\tilde{\theta}$. Differentiating (4) with respect to t, we obtain:

$$\frac{\partial \tilde{\theta}}{\partial t} = \beta\frac{\partial p_0}{\partial t} + \tilde{\alpha}\frac{\partial T_0}{\partial t}.$$

But

$$\frac{\partial p_0}{\partial t} = \left(\frac{\partial p_0}{\partial t}\right)_a + \left(\frac{\partial p_0}{\partial t}\right)_g; \quad \frac{\partial T_0}{\partial t} = \left(\frac{\partial T_0}{\partial t}\right)_a.$$

Consequently

$$\frac{\partial \theta}{\partial t} = \beta\left(\frac{\partial p_0}{\partial t}\right)_a + \tilde{\alpha}\left(\frac{\partial T_0}{\partial t}\right)_a + \beta\left(\frac{\partial p_0}{\partial t}\right)_g.$$

But

$$\beta\left(\frac{\partial p_0}{\partial t}\right)_a + \tilde{\alpha}\left(\frac{\partial T_0}{\partial t}\right)_a = -a\beta\{T_0, p_0\} + \tilde{\alpha}\beta\{T_0, p_0\} = 0.$$

Consequently

$$\frac{\partial \tilde{\theta}}{\partial t} = \beta\left(\frac{\partial p_0}{\partial t}\right)_g.$$

Thus the $\tilde{\theta}$ field changes with time under the influence of dynamic factors which cause changes in pressure. However, we shall continue to ignore the differences between $\tilde{\theta}$ and θ and consistently use $\tilde{\theta}'$ in place of θ.

§ 12. The Connection between the Dynamic Pressure Variation and the Geometrical Properties of the Fields of θ and T_0.

Formula (3) § 11 shows that the dynamic pressure change depends on the geometrical properties of the scalar fields θ and T_0. In order to express this dependence more clearly, we first transform formula (3) § 11 as follows:

$$\left(\frac{\partial p_0}{\partial t}\right)_g = A\left[\frac{\partial T_0}{\partial x}\cdot\frac{\partial T_0}{\partial y}\left(\frac{\partial^2\theta}{\partial y^2}-\frac{\partial^2\theta}{\partial x^2}\right)+\frac{\partial^2\theta}{\partial x\,\partial y}\left\{\left(\frac{\partial T_0}{\partial x}\right)^2-\left(\frac{\partial T_0}{\partial y}\right)^2\right\}\right]. \qquad (1)$$

Let us now write equation (1) in intrinsic coordinates s, n.

For this purpose we direct the x-axis along the tangent to the curve θ = const., so that looking in its positive direction the region of higher θ values will lie to the right of the axis. We direct the y-axis along the normal to the curve θ = const., in the direction of decreasing θ. Thus the coordinate x is identified with s, and the coordinate y with (−n)(Figure 154).

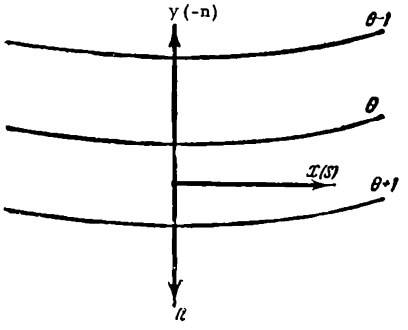

Figure 154. Introduction of the intrinsic co-ordinates s, n.

Denoting the line element by ds, and the normal element by dn, we note the following obvious equalities:

$$\frac{\partial\theta}{\partial x}=\frac{\partial\theta}{\partial s}=0;\quad \frac{\partial\theta}{\partial y}=-\frac{\partial\theta}{\partial n};\quad \frac{\partial^2\theta}{\partial y^2}=\frac{\partial^2\theta}{\partial n^2};\quad \frac{\partial^2\theta}{\partial x\,\partial y}=-\frac{\partial^2\theta}{\partial s\,\partial n}. \qquad (2)$$

The following relation is less obvious:

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{r}\frac{\partial\theta}{\partial n}, \qquad (3)$$

where r is the radius of curvature of the curve θ = const. at the origin of coordinates O.

We shall now prove this equality.

The curvature of the curve y = f(x) is expressed by the formula

where
$$K=\frac{1}{r}=\frac{y''}{[1+y'^2]^{3/2}},$$

$$y'=\frac{df}{dx};\qquad y''=\frac{d^2f}{dx^2}.$$

Consider now the equation of some isothet

$$\theta(x,\ y)=\text{const.}$$

Regarding x as the independent variable, we obtain

$$\frac{\partial\theta}{\partial x}+\frac{\partial\theta}{\partial y}y'=0,$$

$$\frac{\partial^2\theta}{\partial x^2}+2\frac{\partial^2\theta}{\partial x\partial y}y'+\frac{\partial^2\theta}{\partial y^2}y'^2+\frac{\partial\theta}{\partial y}y''=0.$$

Directing now the x-axis along the tangent to the isothet, we obtain:

$$\frac{\partial\theta}{\partial x}=\frac{\partial\theta}{\partial s}=0.$$

Consequently

$$y'=-\frac{\frac{\partial\theta}{\partial x}}{\frac{\partial\theta}{\partial y}}=0;$$

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial\theta}{\partial y}y''=0,$$

whence

$$K=\frac{1}{r}=y''=-\frac{\frac{\partial^2\theta}{\partial x^2}}{\frac{\partial\theta}{\partial y}}$$

and thus

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{r}\frac{\partial\theta}{\partial n}.$$

We shall denote by φ the angle between the isotherm T_0 = const. and the isothet θ = const., at the considered field point.

The angle φ is taken as positive when the shortest turn from θ = const. to T_0 = const. is in anticlockwise. The curves θ = const. and T_0 = const. are oriented so that advance in the positive direction leaves the higher values of θ and T_0 to the right of the corresponding curve.

We then obtain

$$\frac{\partial T_0}{\partial x}=\frac{\partial T_0}{\partial \nu}\sin\varphi;\qquad \frac{\partial T_0}{\partial y}=-\frac{\partial T_0}{\partial \nu}\cos\varphi, \tag{4}$$

where ν is the normal to the isotherm T_0. Consequently,

$$\frac{\partial T_0}{\partial x}\cdot\frac{\partial T_0}{\partial y}=-\frac{1}{2}\left(\frac{\partial T_0}{\partial \nu}\right)^2\sin 2\varphi;\quad \left(\frac{\partial T_0}{\partial x}\right)^2-\left(\frac{\partial T_0}{\partial y}\right)^2=-\left(\frac{\partial T_0}{\partial \nu}\right)^2\cos 2\varphi. \tag{5}$$

Inserting the obtained expressions for $\frac{\partial T_0}{\partial x}$, $\frac{\partial T_0}{\partial y}$, $\frac{\partial^2\theta}{\partial y^2}$, $\frac{\partial^2\theta}{\partial x^2}$, $\frac{\partial^2\theta}{\partial x\partial y}$ we obtain the dynamic change of pressure in the new coordinate system:

$$\left(\frac{\partial p_0}{\partial t}\right)_g=\frac{A}{2}\left(\frac{\partial T_0}{\partial \nu}\right)^2\left[2\cos 2\varphi\frac{\partial^2\theta}{\partial s\partial n}-\sin 2\varphi\frac{\partial^2\theta}{\partial n^2}+\sin 2\varphi\,\frac{1}{r}\frac{\partial\theta}{\partial n}\right]. \tag{6}$$

Formula (6), due to Kibel', determines all dynamic pressure variations from the given temperature and θ scalar fields, or from the equivalent field of isohypses at the altitude of about 3.5 km.

Apart from purely dynamic factors, the pressure also changes under the influence of advective factors. We will express this variation also in intrinsic coordinates.

It is obvious that

$$\frac{\partial p_0}{\partial t}=-a\{T_0,\ p_0\}=-\frac{a}{\beta}\frac{\partial T_0}{\partial t}=\frac{a}{\beta}\{\theta,\ T_0\}=\frac{a}{\beta}\left(\frac{\partial\theta}{\partial x}\cdot\frac{\partial T_0}{\partial y}-\frac{\partial\theta}{\partial y}\cdot\frac{\partial T_0}{\partial x}\right).$$

Applying equations (2) and (4), we obtain

$$\left(\frac{\partial p_0}{\partial t}\right)_a=\frac{a}{\beta}\frac{\partial\theta}{\partial n}\frac{\partial T_0}{\partial x}=\frac{a}{\beta}\frac{\partial\theta}{\partial n}\frac{\partial T_0}{\partial \nu}\sin\varphi. \tag{7}$$

Formula (6) shows that the dynamic pressure change is proportional to the square of the temperature gradient $\left(\frac{\partial T_\theta}{\partial \nu}\right)^2$. Consequently, the considerable pressure variations accompanying the fronts are due not only to advection of air of different temperature but also to dynamic factors.

The dynamic pressure change is further influenced by the geometry of the θ field. Finally, this variation is also connected with the relative position of the field of temperature and of the θ field.

Let us now examine in greater detail the influence of each of these factors.

We denote the distance between the individual isotherms by h_T and the distance between the isothets by h_θ. Then formulas (6) and (7) can be rewritten as follows:

$$\left(\frac{\partial p_0}{\partial t}\right)_a = \frac{a}{\beta}\frac{\sin\varphi}{h_\theta \cdot h_T}, \tag{8}$$

$$\left(\frac{\partial p_0}{\partial t}\right)_g = \frac{A}{2}\cdot\frac{1}{h_T^2}\left[-\frac{2\cos 2\varphi}{h_\theta^2}\frac{\partial h_\theta}{\partial s} + \frac{\sin 2\varphi}{h_\theta^2}\frac{\partial h_\theta}{\partial n} + \frac{\sin 2\varphi}{h_\theta}\cdot\frac{1}{r}\right]. \tag{9}$$

1) Isothetic divergence.

Let us consider the first term of equation(9):

$$\left(\frac{\partial p_0}{\partial t}\right)'_g = -A\cdot\frac{\cos 2\varphi}{h_T^2 h_\theta^2}\frac{\partial h_\theta}{\partial s}. \tag{10}$$

We notice that at the given density of isothets and isotherms the dynamic changes of pressure due to the divergence of isothets attains a maximum when $\varphi = 0$ and when there is no advection.

Equation (10) shows that the pressure change due to isothetic divergence can be either positive or negative, depending on the relative position of the isotherms and the isothets.

Table 55 gives the sign of the pressure change corresponding to various relative positions of isotherms and isothets.

Thus, when the isothets diverge in the direction of the motion the pressure drops for $|\varphi| < \frac{\pi}{4}$ (Figures 155 and 156). Attention should be drawn to the fact that the dynamic change of pressure due to isothet divergence may be opposite to the advective change. Thus, for example, at a positive isothet divergence the pressure may drop, in spite of advection of cold air (Figure 155).

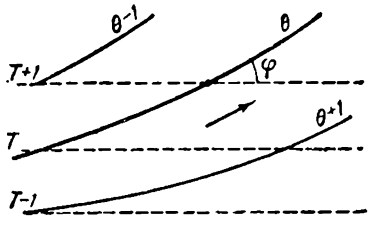

Figure 155. Isothet divergence. Advection of cold air. Increase of advection and dynamic drop of pressure.

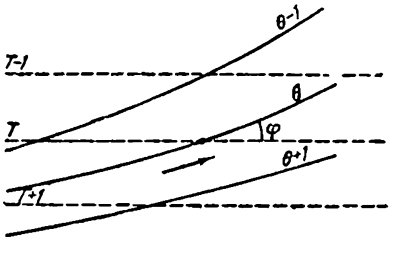

Figure 156. Isothet divergence. Advection of hot air. Advective and dynamic drop of pressure ($|\varphi| < \pi/4$).

Wherever φ is close to $\frac{\pi}{4}$, the effect of divergence can be neglected.

We should note that for $|\varphi| > \frac{\pi}{4}$ the pressure variation changes its sign and an isothet divergence is accompanied by a rise in pressure (Figure 157), and a convergence, by a drop in pressure. Such conditions are not encountered at considerable temperature gradients. Therefore, the possible change of pressure is not great.

2) Drawing together of isothets.

The next factor affecting the dynamic change of pressure is increasing isothet density in the direction normal to the isothets

$$\left(\frac{\partial p_0}{\partial t}\right)''_g = \frac{A}{2} \frac{\sin 2\varphi}{h_T^2 h_\theta^2} \frac{\partial h_\theta}{\partial n}. \qquad (11)$$

This factor, however, plays but a minor role since it reaches maximum effect for $\varphi = \frac{\pi}{4}$, i.e., when the horizontal temperature gradient cannot be large. On the other hand, at high horizontal temperature gradients the angle φ is small and consequently the pressure change is slight.

The sign of the pressure change due to isothets drawing together can be obtained in the following way (Table 55). When the isothets disperse in the direction of increasing θ, $\frac{\partial h_\theta}{\partial n} > 0$ and for $\varphi > 0$, i.e., for advection of cold air, dynamic increase in pressure will take place.

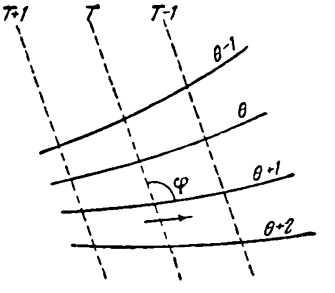

Figure 157. Isothet divergence Advection of hot air. Advective and dynamic drop of pressure ($|\varphi| > \pi/4$).

3) Isothet curvature.

Let us finally consider the effect of curvature on the dynamic change of pressure:

$$\left(\frac{\partial p_0}{\partial t}\right)'''_g = \frac{A}{2} \frac{\sin 2\varphi}{h_T^2 h_\theta^2} \cdot \frac{1}{r}. \qquad (12)$$

This factor is most effective for $\varphi = \frac{\pi}{4}$ but at such large angles the horizontal temperature gradient can not be considerable; therefore, the isothet curvature cannot compare with isothet divergence as a cause of dynamic pressure variation.

For cyclonic curvature $\left(\frac{1}{r} > 0\right)$ the sign of the dynamic pressure change coincides with that of the advective change. For anticyclonic curvature $\left(\frac{1}{r} < 0\right)$ these signs are opposite.

Thus, the separate examination of the terms determining the dynamic change pf pressure leads to the conclusion, that isothet approach and isothet curvature have no significant effect on the pressure change, the more so since their signs coincide with those of the advectional changes. Therefore, in quantitative analysis they may be taken to be amalgamated with the advective changes. The divergence of isothets, however, may cause considerable pressure changes and therefore special attention should be paid to it in quantitative analysis.

The important effect of isothet divergence on the pressure change in the second approximation can be shown in the following way.

Let us consider the formula for the local pressure change in the second approximation:

Table 55

Dynamic and advective change of pressure (13)

	$\varphi < \pi/4$	$\varphi = \pi/4$	$0 < \varphi < \pi/4$	$\varphi = 0$	$-\pi/4 < \varphi < 0$	$\varphi = -\pi/4$	$-\pi/2 < \varphi < -\pi/4$
	$\left(\frac{\partial p_0}{\partial t}\right)_a < 0$			0	$\left(\frac{\partial p_0}{\partial t}\right)_a > 0$		
$\frac{\partial h_\theta}{\partial s} > 0$	$\left(\frac{\partial p_0}{\partial t}\right)'_g > 0$	0	$\left(\frac{\partial p_0}{\partial t}\right)'_g < 0$			0	$\left(\frac{\partial p_0}{\partial t}\right)'_g > 0$
$\frac{\partial h_\theta}{\partial s} < 0$	$\left(\frac{\partial p_0}{\partial t}\right)'_g < 0$	0	$\left(\frac{\partial p_0}{\partial t}\right)'_g > 0$			0	$\left(\frac{\partial p_0}{\partial t}\right)'_g < 0$
$\frac{\partial h_\theta}{\partial n} > 0$	$\left(\frac{\partial p_0}{\partial t}\right)''_g < 0$			0	$\left(\frac{\partial p_0}{\partial t}\right)''_g > 0$		
$\frac{\partial h_\theta}{\partial n} < 0$	$\left(\frac{\partial p_0}{\partial t}\right)''_g > 0$			0	$\left(\frac{\partial p_0}{\partial t}\right)''_g < 0$		
$\frac{1}{r} > 0$	$\left(\frac{\partial p_0}{\partial t}\right)'''_g < 0$			0	$\left(\frac{\partial p_0}{\partial t}\right)'''_g > 0$		
$\frac{1}{r} < 0$	$\left(\frac{\partial p_0}{\partial t}\right)'''_g > 0$			0	$\left(\frac{\partial p_0}{\partial t}\right)'''_g < 0$		

$$\frac{\partial p_0}{\partial t} = -\alpha\{T_0,\ p_0\} + A\left[\frac{\partial T_0}{\partial x}\left\{T_0, \frac{\partial \theta}{\partial x}\right\} + \frac{\partial T_0}{\partial y}\left\{T_0, \frac{\partial \theta}{\partial y}\right\}\right], \tag{13}$$

and apply this formula to such region of the pressure field where

$$\frac{\partial p_0}{\partial x} \approx 0; \quad \frac{\partial p_0}{\partial y} \approx 0.$$

This corresponds to pressure centers and also to diffuse fields.
Since

$$\theta = \alpha T_0 + \beta p_0,$$

we have

$$\frac{\partial T_0}{\partial x} \approx \frac{1}{\alpha}\frac{\partial \theta}{\partial x}; \quad \frac{\partial T_0}{\partial y} \approx \frac{1}{\alpha}\frac{\partial \theta}{\partial y},$$

and consequently

$$\frac{\partial T_0}{\partial x}\left\{T_0, \frac{\partial \theta}{\partial x}\right\} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial x}\left\{\theta, \frac{\partial \theta}{\partial x}\right\} = \frac{1}{\alpha^2}\left(\frac{\partial \theta}{\partial x}\cdot\frac{\partial^2 \theta}{\partial x \partial y} - \frac{\partial \theta}{\partial y}\cdot\frac{\partial^2 \theta}{\partial x^2}\right)\frac{\partial \theta}{\partial x} =$$

$$= \frac{1}{2\alpha^2}\left[\frac{\partial \theta}{\partial x}\,2\,\frac{\partial \theta}{\partial x}\cdot\frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial x}\right) - \frac{\partial \theta}{\partial y}\,2\,\frac{\partial \theta}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right)\right] = \frac{1}{2\alpha^2}\left\{0, \left(\frac{\partial \theta}{\partial x}\right)^2\right\}.$$

Analogously

$$\frac{\partial T_0}{\partial y}\left\{T_0, \frac{\partial \theta}{\partial y}\right\} = \frac{1}{2a^2}\left\{\theta, \left(\frac{\partial \theta}{\partial y}\right)^2\right\}.$$

Therefore, formula (13) may be rewritten as follows:

$$\frac{\partial p_0}{\partial t} = -a\{T_0, p_0\} + \frac{A}{2a^2}\left\{\theta, \left[\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2\right]\right\}$$

or

$$\frac{\partial p_0}{\partial t} = -\left\{\theta, p_0 - \frac{A}{2a^2}\left[\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2\right]\right\}.$$

Assuming $\frac{\partial \theta}{\partial t} \approx 0$, $\frac{\partial p_0}{\partial t}$ can be replaced by $\frac{\partial \tilde{p}_0}{\partial t}$, in the left side of the last equation, where

$$\tilde{p}_0 = p_0 - \frac{A}{2a^2}\left[\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2\right] = p_0 - \frac{A}{2a}|\operatorname{grad}\theta|^2.$$

Then the formula for the solution in the second approximation assumes the form

$$\frac{\partial \tilde{p}_0}{\partial t} = -\{\theta, \tilde{p}_0\}.$$

Thus, not the pressure p_0 is transferred along the isothets, but the combination

$$p_0 - \frac{A}{2a^2}|\operatorname{grad}\theta|^2.$$

Let us consider now two points located on the same isothet in a region of isothet divergence. The following quantity is transferred along the isothet:

$$\tilde{p}_0 = p_0 - \frac{A}{2a^2}|\operatorname{grad}\theta|^2.$$

Consequently at these points the values of $\tilde{p}_0$ should be identical and

$$p_{01} - \frac{A}{2a^2}|\operatorname{grad}\theta|_1^2 = p_{02} - \frac{A}{2a^2}|\operatorname{grad}\theta|_2^2$$

or

$$p_{02} = p_{01} + \frac{A}{2a^2}\{|\operatorname{grad}\theta|_2^2 - |\operatorname{grad}\theta|_1^2\}.$$

When the transfer proceeds in the direction of divergency we have

$$|\operatorname{grad}\theta|_2^2 < |\operatorname{grad}\theta|_1^2$$

and consequently

$$p_{01} < p_{02},$$

i. e., in a region of isothet divergence, a pressure drop is to be expected. In a region of isothet convergence, on the other hand, the pressure should rise.

Chapter XXI

INTEGRATION OF THE WEATHER EQUATIONS

§ 1. The System of Weather Equations

We have shown in Chapter VIII that a complete system of weather equations, describing the simultaneous action of all atmospheric processes has not yet been formulated.

A drastic simplification of the relevant processes has been unavoidable from the outstart. The greatest advance till now has been the formulation of a closed system of hydromechanical, thermodynamical and radiation equations which accounts for the transfer of water vapor (in the absence of evaporation or condensation), and also of condensation products and other radiationally active admixtures (carbon dioxide, ozone, dust).

The obtained equations cannot, however, be applied to the atmosphere, since while accounting for molecular viscosity, heat conductivity and diffusivity, they do not include the effects of the atmospheric turbulence mechanism – eddy internal friction, eddy heat conduction and eddy diffusion.

We shall now write down consecutively the equations which together make up the system of weather equations.

This system includes, first of all, the three equations of averaged atmospheric motion in the gravity and Coriolis fields:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\left[\omega_z v - \omega_y w\right] + \frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(A'\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A'\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A\frac{\partial u}{\partial z}\right)\right], \tag{1}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + 2\left[\omega_x w - \omega_z u\right] + \frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(A'\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A'\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A\frac{\partial v}{\partial z}\right)\right], \tag{2}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\left[\omega_y u - \omega_x v\right] + \frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(A'\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A'\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(A\frac{\partial w}{\partial z}\right)\right]. \tag{3}$$

Next comes the equation of continuity in its usual form

$$\frac{d\rho}{dt} + \rho \operatorname{div} \boldsymbol{v} = 0. \tag{4}$$

Clapeyron's equation introduces only a particular transformation of the variables

$$\frac{p}{\rho} = RT. \tag{5}$$

The equation of heat balance accounts for the eddy flux of heat and radiative heat transfer, disregarding the negligible contribution from the dissipation of mechanical energy:

$$\frac{\partial}{\partial x}\left(K'\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K'\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)+\rho\int_0^\infty \alpha_\nu d\nu\int I_\nu d\omega=\rho c_v\frac{dT}{dt}+$$

$$+p\,\mathrm{div}\,v-4\pi\rho\int_0^\infty \eta_\nu d\nu. \quad (6)$$

Radiative transfer is determined by the equation

$$\frac{1}{\rho}\frac{\partial I_\nu}{\partial s}=\eta_\nu+\frac{\sigma_\nu}{4\pi}\int I_\nu(P,\ r')\,\gamma_\nu(t,\ P,\ r',\ r)\,d\omega'-(\alpha_\nu+\sigma_\nu)\,I_\nu. \quad (7)$$

The distribution of radiation follows the law

$$\eta_\nu=\alpha_\nu\cdot\frac{2h\nu^3}{c^2}\,\frac{1}{e^{\frac{h\nu}{kT}}-1}, \quad (8)$$

obtained by combining the laws of Kirchhoff and Planck.

In equations (6), (7), (8) the absorption coefficient is a function of the specific humidity s, so that

$$\alpha_\nu=\alpha'_\nu\cdot s.$$

The transfer of water vapor is governed by the equation of eddy diffusion

$$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}+w\frac{\partial s}{\partial z}=\frac{\partial}{\partial x}\left(D'\frac{\partial s}{\partial x}\right)+\frac{\partial}{\partial y}\left(D'\frac{\partial s}{\partial y}\right)+\frac{\partial}{\partial z}\left(D\frac{\partial s}{\partial z}\right). \quad (9)$$

Equations (1)—(9) constitute the system of weather equations. These 9 equations contain 9 unknown functions: 3: u, v, w, p, ρ, T, I_ν, η_ν, s. In addition to the constant or almost constant physical parameters ω_x, ω_y, ω_z, R, c_p, c_v, h, ν, c, k, the equations include the variable parameters: A', A, K', K, D', D, σ_ν, γ_ν, α'_ν.

This system of weather equations is so complicated as to defy solution without preliminary simplifications.

Two different approaches are possible to the simplification of the weather equations.

First, we may schematize the physical processes involved. This approach does not impair the rigor of our equations; the correspondence of such processes with reality might however, be questioned.

Alternatively, we may simplify the system of weather equations without altering the processes themselves. We might assume that not all the terms entering in the equations are equally important. Accordingly, those terms which are small in comparison with the others may be ignored. In this case it may happen that an expression omitted from one equation is retained in another.

The simplified equations thus obtained are not strictly rigorous, but approximate. Such simplification must be carried out with caution, we should remember that the concerned equations are differential, and have yet to be integrated after simplification.

Let us first examine some instances of simplification of the first type.

1. We assume the air to be absolutely dry. Then s = 0, and the equation of vapor transfer is redundant. Assuming further that emission and absorption are effected exclusively by water vapor, we find that the considered medium (dry

air) neither emits nor absorbs radiation but only scatters it. Then $a_\nu = \eta_\nu = 0$. In this case the radiative terms do not appear in the equation of heat balance, and the hydromechanical and thermodynamic equations will form a separate subsystem, independent of radiative processes. The Kirchhoff-Planck equation disappears from the radiation equations, and the equation of radiative transfer assumes the form

$$\frac{1}{\rho}\frac{\partial I_\nu}{\partial s} = \frac{\sigma_\nu}{4\pi}\int I_\nu(P, r')\,\gamma_\nu(t, P, r', r)\,d\omega' - \sigma_\nu I_\nu. \qquad (7')$$

This equation includes a single undetermined function, which must be found from the hydromechanical and thermodynamic equations: the density ρ of the scattering medium. When ρ is given, we obtain a single integro-differential equation for I_ν.

Let us note that the splitting of the weather equations into two independent subsystems in the case of pure scattering could have been expected, since radiation is not converted into heat in the process of scattering.

The scattering of radiation is in itself of no interest for dynamic meteorology. Hence equation (7') is not utilized there. This example serves only to show that for dry air the system of weather equations includes equations (1)—(5) in unaltered form and the equation of heat balance in the following form

$$\frac{\partial}{\partial x}\left(K'\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K'\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right) = \rho c_v \frac{dT}{dt} + p\,\mathrm{div}\,v, \qquad (6')$$

the remaining equations drop altogether.

2. Let us consider another particular case which is of certain practical importance. We assume that the medium both emits and absorbs radiation, but does not scatter it, i.e., that $\sigma_\nu = 0$.

Under this condition all weather equations retain their original form, with the exception of the equation of radiative energy transfer, which becomes:

$$\frac{1}{\rho}\frac{\partial I_\nu}{\partial s} = \eta_\nu - a_\nu I_\nu. \qquad (7'')$$

Still another simplification, which has important applications and is commonly used in texts on dynamic meteorology, is effected by assuming the medium to be gray. This means that emission and absorption processes in this medium do not depend on the radiation frequency.

This simplification allows to do away with the independent variable ν by integrating the equation of radiative transfer (7'') with respect to ν and introducing the new functions:

$$\eta = \int_0^\infty \eta_\nu d\nu; \quad I = \int_0^\infty I_\nu d\nu, \qquad (10)$$

The equation of radiative transfer then assumes the form

$$\frac{1}{\rho}\frac{\partial I}{\partial s} = \eta - aI, \qquad (7''')$$

where a is the absorption coefficient, which has the same value over the entire spectrum.

For this purpose the Kirchhoff–Planck equation is replaced by the Stefan-Boltzmann formula. Indeed, integrating (8) with respect to ν and applying (10), we find:

$$\eta = \alpha \int_0^\infty \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} d\nu.$$

According to radiation theory, the last integral equals $\sigma T^4/\pi$; consequently

$$\eta = \alpha \cdot \frac{\sigma}{\pi} \cdot T^4, \qquad (8')$$

where σ — is the Stefan–Boltzmann constant.

However, we usually do not stop at these simplifications. In problems of dynamic meteorology, the radiational energy flux is assumed to be directed vertically, consisting of the downward energy flux, composed of gray short–wave solar radiation S and gray long–wave atmospheric emission A, and an upward energy flux, composed of gray long–wave terrestrial and atmospheric radiation B.

Thus, a unit volume of humid air will receive the following quantity of heat: by absorption of long–wave radiation $\alpha\rho_w (A + B)$; by absorption of short-wave solar radiation $\alpha\beta\rho_w S$. However, this volume unit of humid air will also lose heat by emission. The quantity of heat lost by radiation upwards and downwards is $2\alpha\rho_w fE$, where α is the absorption coefficient for the gray long-wave radiation, $\alpha\beta$ coefficient for the gray short-wave radiation, f - a factor which allows for the selectivity of emission by water vapor, and E is the emission of an ideal black body according to the Stefan-Boltzmann law $E = \sigma T^4$.

Upon these assumptions the equation of heat balance may be written as follows:

$$\rho c_v \frac{dT}{dt} + p \operatorname{div} v = \frac{\partial}{\partial x}\left(K' \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K' \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial T}{\partial z}\right) + \alpha\rho_w (A + B) + \alpha\beta\rho_w S - 2\alpha\rho_w fE. \qquad (6^{\text{II}})$$

The equation of radiative transfer splits into three equations for the three gray components of the radiation flux

$$\frac{\partial A}{\partial z} = \alpha\rho_w (A - fE); \quad \frac{\partial B}{\partial z} = \alpha\rho_w (fE - B); \quad \frac{\partial S}{\partial z} = -\alpha\beta\rho_w S. \qquad (7^{\text{IV}})$$

The simplified equations obtained are strictly rigorous, if we accept the mentioned radiation processes as real. However, some serious doubts rise against this supposition. In fact, the radiation energy is not only absorbed and transmitted in the atmosphere, but is scattered too. Morever, the radiant energy flux is by no means vertical and isotropic. Such a schematization of radiative processes in the atmosphere is rather crude. However, the considerable simplification of the system of weather equations is very attractive. Comparison of theoretical results with observation shows that a substantial simplification of atmospheric radiative processes is permissible in problems of dynamic meteorology.

3. We now assume that turbulent exchange develops only in a vertical direction. In this case the terms representing the horizontal turbulent exchange disappear from the weather equations, and the system becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2(\omega_z v - \omega_y w) + \frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right), \qquad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + 2(\omega_x w - 2\omega_z u) + \frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right), \qquad (2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2(\omega_y u - \omega_x v) + \frac{\partial}{\partial z}\left(K\frac{\partial w}{\partial z}\right), \qquad (3)$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0, \tag{4}$$

$$\frac{p}{\rho}=RT, \tag{5}$$

$$\rho c_v\frac{dT}{dt}+p\,\mathrm{div}\,v=\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)+a\rho_w(A+B)+a\beta\rho_w S-2a\rho_w fE, \tag{6}$$

$$\frac{\partial A}{\partial z}=a\rho_w(A-fE);\quad \frac{\partial B}{\partial z}=a\rho_w(fE-B);\quad \frac{\partial S}{\partial z}=-a\beta\rho_w S;\quad E=\sigma T^4, \tag{7}$$

$$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}+w\frac{\partial s}{\partial z}=\frac{\partial}{\partial z}\left(K\frac{\partial S}{\partial z}\right). \tag{8}$$

These are the simplified weather equations, in the form usually applied to problems of dynamic meteorology.

However, as we have seen before, still further simplification of the weather equations is possible, based not on schematization of the atmospheric processes, but on evaluation of the orders of the terms appearing in the equations.

In this chapter we shall consider some of the simplest problems connected with integration of the approximate weather equations, which have been solved by Soviet meteorologists.

§ 2. Temperature Distribution in the Terrestrial Atmosphere as Due to Radiation and Eddy Transfer of Heat

The theory of radiative equilibrium, which explains the stratospheric isothermality in a satisfactory manner, gives exaggerated temperature gradients in the troposphere. A. A. Fridman has already shown, that the appreciable discrepancy between observed gradients and their values as computed according to the theory of radiative equilibrium might be explained by vertical motions.

I. A. Kibel' has obtained results in good agreement with observation from his theoretical temperature distribution in the atmosphere, as due to radiation and turbulent heat exchange.

Kibel' proceeds from the equation of heat balance, which he applies in the following form:

$$\varepsilon_1+\varepsilon_2=c_p\rho\frac{dT}{dt}-\frac{dp}{dt}, \tag{1}$$

where ε_1 is the heat influx (per unit volume and unit time) due to radiation, ε_2 the heat influx due to eddy flux of heat. The influx of heat caused by condensation and evaporation processes is not taken into consideration at all.

For the solution of the stationary problem of the mean annual temperature distribution, I. A. Kibel' makes some simplifications. In this case the velocity field may be disregarded and the atmosphere assumed to be calm. The individual derivative $\frac{dT}{dt}$ may be replaced by the local derivative $\frac{\partial T}{\partial t}$, which vanishes in the stationary case; consequently $\frac{dT}{dt}=0$. In addition, the pressure may also be assumed to be time-independent, hence $\frac{dp}{dt}=0$.

The radiational heat influx is computed by Kibel' according to a simplified scheme. He assumes that there are two vertical components of radiation energy flux:

1) the downwards flux, composed of gray short-wave solar radiation S and of gray long-wave atmospheric radiation A.

2) the upward flux, composed of gray long-wave terrestrial and atmospheric radiation, B.

Thus

$$\varepsilon_1 = \alpha\rho_w (A + B) + \alpha\beta\rho_w S - 2\alpha\rho_w fE, \tag{2}$$

where α is the absorption coefficient for gray long-wave radiation; $\alpha\beta$ — the absorption coefficient for gray short-wave radiation; f – absorptivity factor of water vapor; ρ_w — the density of water vapor which is the active substance in radiation exchange; E – the radiation of an ideal black body according to the Stefan–Boltzmann law:

$$E = \sigma T^4. \tag{3}$$

The eddy flux of heat is determined by the formula

$$\varepsilon_2 = \frac{\partial}{\partial x}\left(K' \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K' \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial T}{\partial z}\right).$$

Considering only vertical mixing, Kibel' simplifies this equation, obtaining

$$\varepsilon_2 = \frac{\partial}{\partial z}\left(K \frac{\partial T}{\partial z}\right). \tag{4}$$

In virtue of all the assumptions made, Kibel' obtains the equation of heat balance in the following form

$$\frac{d}{dz}\left(K \frac{dT}{dz}\right) + \alpha\rho_w (A + B - 2fE + \beta S) = 0. \tag{5}$$

Thus, equation (5) includes 5 unknown functions: T, E, A, B, S, only one of which is of real interest to us, in view of the nature of the problem, i.e., the temperature T; the remaining functions are auxiliary.

In order to complete this system of equations, which contains five unknown functions, Kibel' adds to equation (5) equation (3) and also three other equations which determine the radiative energy transfer in the atmosphere:

$$\left.\begin{aligned} \frac{dA}{dz} &= \alpha\rho_w (A - fE), \\ \frac{dB}{dz} &= \alpha\rho_w (fE - B), \\ \frac{dS}{dz} &= \alpha\beta\rho_w \cdot S. \end{aligned}\right\} \tag{6}$$

Kibel' proceeds to define the following boundary conditions.

The upper boundary of the atmosphere receives only short-wave solar radiation or rather – strictly speaking – a part of it W, since the other part is reflected back from the atmosphere, the surface of the earth, and from the clouds. In consequence the average albedo of the earth is taken into consideration. In addition, it is assumed that at its upper boundary the atmosphere radiates back into outer space the same amount of energy it receives. Thus the conditions at the upper boundary may be set as follows:

$$A = 0; \quad S = W; \quad B = W \text{ (for } z = \infty). \tag{7}$$

Kibel' applies the equation of heat balance at the earth's surface

$$-K\frac{dT}{dz} + K^*\frac{dT^*}{dz} = A + S - B \text{ (for } z = 0), \tag{8}$$

whereby we may write for the upwards heat flux:

$$B = q \cdot E \text{ (for } z = 0), \tag{9}$$

where q is the emissivity factor of the earth's surface relative to an ideal black body. However, the boundary condition (8) introduces a new unknown function – the soil temperature T* (the coefficient of heat conductivity of the soil K* is considered an unknown physical parameter). The introduction of the new function T* necessitates an additional equation for its determination. However, Kibel' believes that in his determination of the mean annual temperature distribution, the heat flux emanating from the soil may be disregarded. Then the boundary condition (8) is simplified and assumes the form

$$-K\frac{dT}{dz}=A+S-B \quad (\text{for } z=0). \tag{10}$$

Kibel' introduces a new independent variable instead of the coordinate z – the "optical thickness" τ for gray radiation, defined by the formula

$$\tau=\int_z^\infty a\rho_w\,dz \text{ and } d\tau=-a\rho_w dz. \tag{11}$$

Kibel' excludes the temperature T from the equations, expressing it by E. Since

$$\frac{dE}{d\tau}=4\sigma T^3\frac{dT}{d\tau}=-\frac{4\sigma T^3}{a\rho_w}\cdot\frac{dT}{dz},$$

equation (5) may be brought to the form

$$\frac{d}{d\tau}\left(\frac{Ka\rho_w}{4\sigma T^3}\frac{dE}{d\tau}\right)+A+B-2fE+\beta S=0. \tag{12}$$

Equations (6) in the new variables will now have the form

$$\left.\begin{aligned}\frac{dA}{d\tau}&=fE-A,\\ \frac{dB}{d\tau}&=B-fE,\end{aligned}\right\} \tag{13}$$

$$\frac{dS}{d\tau}=-\beta S. \tag{14}$$

The boundary conditions must be taken in the following form:

$$A=0;\quad S=W;\quad B=W \qquad (\text{for } \tau=0), \tag{15}$$

$$\frac{Ka\rho_w}{4\sigma T^3}\cdot\frac{dE}{d\tau}=A+S-qE \qquad \left(\text{for } \tau=\tau_0=\int_0^\infty a\cdot\rho_w\,dz\right). \tag{16}$$

Equation (14) may be directly integrated. Considering the boundary condition (15) we find:

$$S=W\cdot e^{-\beta\tau}. \tag{17}$$

We have thus obtained three equations (12) and (13) for three undetermined functions A, B, E. In order to solve these equations, Kibel' introduces dimensionless functions a, y, ε, defined by the equalities:

$$B+A=W\cdot a;\quad B-A=W\cdot y;\quad fE=W\cdot\varepsilon. \tag{18}$$

By means of equations (13) it is easy to obtain formulas which express two of the new auxiliary functions by the third, for instance by y. Since

$$\frac{d}{d\tau}(A+B)=B-A, \tag{19}$$

we have

$$\frac{da}{d\tau}=y. \tag{20}$$

Integrating (20), we obtain

$$a(\tau) = a(0) + \int_0^\tau y\, d\tau,$$

or

$$a = 1 + \int_0^\tau y\, d\tau, \tag{21}$$

since

$$a(0) = \frac{B(0) + A(0)}{W} = 1.$$

From (13) and (18) we also have

$$\varepsilon = \frac{f \cdot E}{W} = \frac{1}{2}\left\{\frac{A+B}{W} - \frac{d}{d\tau}\left(\frac{B-A}{W}\right)\right\} = \frac{1}{2}\left\{a - \frac{dy}{d\tau}\right\}$$

or, applying (21):

$$\varepsilon = \frac{1}{2}\left\{1 + \int_0^\tau y\, d\tau - \frac{dy}{dt}\right\}. \tag{22}$$

In order to determine y a differential equation is formed in the following way. Since according to (13),

$$A + B - 2fE = \frac{d}{d\tau}(B - A), \tag{23}$$

we see, by applying (14) and (23), that equation (12) may be rewritten in the following form:

$$\frac{d}{d\tau}\left(\frac{K\sigma\rho_w}{4\sigma T^3} \cdot \frac{dE}{d\tau}\right) + \frac{d}{d\tau}(B - A) - \frac{dS}{d\tau} = 0. \tag{24}$$

and we get:

$$\frac{K\sigma\rho_w}{4\sigma T^3} \cdot \frac{dE}{d\tau} + B - A = W \cdot e^{-\beta\tau}. \tag{25}$$

But

$$\frac{dE}{d\tau} = \frac{W}{f}\frac{d\varepsilon}{d\tau} = \frac{W}{2f}\left(y - \frac{d^2v}{d\tau^2}\right)$$

and

$$B - A = yW. \tag{26}$$

Consequently, the equation for y assumes the form

$$\frac{d^2y}{d\tau^2} - \left(1 + \frac{4\sigma T^3}{K\alpha\rho_w} \cdot 2f\right)y = -\frac{4\sigma T^3}{K\sigma\rho_w} \cdot 2f \cdot e^{-\beta\tau}. \tag{27}$$

The boundary conditions for the function y are obviously the following:

$$y = 1 \quad (\text{at } \tau = 0), \tag{28}$$

$$y + \frac{q}{f}\frac{dy}{d\tau} = \left(\frac{q}{f} - 1\right)\left(1 + \int_0^\tau y\, d\tau\right) \quad (\text{at } \tau = \tau_0). \tag{29}$$

Kibel' believes that although each of the values T, ρ_w, K, α changes with altitude, the expression $fT^3/K\alpha\rho_w$ keeps an almost constant value. Assuming

$$\frac{4\sigma T^3 f}{K\alpha\rho_w} = \frac{m^2 - 1}{2} = \text{const.}, \tag{30}$$

we obtain an ordinary differential equation of the second order with constant coefficients

$$\frac{d^2y}{d\tau^2} - m^2 y = (m^2 - 1)e^{-\beta\tau}, \tag{31}$$

This equation is easily integrated. Its general integral has the form

$$y = C_1 e^{m\tau} + C_2 e^{-m\tau} + \frac{m^2 - 1}{m^2 - \beta^2} e^{-\beta\tau}. \tag{32}$$

The arbitrary constants C_1 and C_2 are determined from the boundary conditions (28) and (29). Kibel' indicates that further simplifications which would still facilitate the calculation are possible. Since $\tau_0 \gg 1$, the terms containing $e^{-m\tau_0}$, may be ignored, being small with respect to unity. Then we get

$$C_1 \approx \frac{m^2 - 1}{m^2 - \beta^2} \frac{\left[(q-1)\frac{1 - e^{-\beta\tau}}{\beta} - (1 - q\beta)e^{-\beta\tau_0}\right] + (q-1)\left(1 + \frac{1 - \beta^2}{m^2 - \beta^2}\frac{1}{m}\right)}{1 + m \cdot q + \frac{1 - q}{m}}, \tag{33}$$

$$C_2 \approx \frac{1 - \beta^2}{m^2 - \beta^2}. \tag{34}$$

Finally, Kibel' obtains the following solution of the initial problem:

$$\varepsilon = \frac{1}{2}\frac{m^2 - 1}{m^2 - \beta^2}\left\{\frac{m^2 - \beta^2}{m^2 - 1} + \frac{1 - \beta^2}{m^2 - \beta^2} \cdot \frac{1}{m} + \beta e^{-\beta\tau} + \frac{1 - e^{-\beta\tau}}{\beta} + \right.$$
$$\left. + \frac{1 - \beta^2}{m} \cdot e^{-m\tau} - \frac{m^2 - \beta^2}{m} \cdot C_1 e^{m\tau}\right\}. \tag{35}$$

Thus Kibel' succeeded in obtaining a generalization of Emden's results for the theory of radiative equilibrium. Emden's solution is a particular case of the solution of Kibel'. It corresponds to the absence of turbulence, when K = 0.

Kibel' carried out the calculations for the latitude 42°, adopting the values m = 1.75; q/f = 1.15; W = 0.138 cal/cm^2 min. For the altitude distribution of water vapor Kibel' used the empirical formula

$$\rho_w = \rho_{w_0} \cdot 10^{-\frac{z}{l}},$$

where

$$l = 8 \cdot 10^5 \text{ cm}$$

Then

$$\tau = \int_z^\infty \frac{a\rho_{w_0} T_0 \cdot 10^{-\frac{z}{l}}}{T} dz.$$

The value of T in the integrand was estimated by a formula due to Emden:

$$\frac{T}{T_\infty} = (1 + \tau)^{1/4}.$$

Thus, the following formula was actually used in the calculations:

$$\frac{4}{5}\left[(1 + \tau)^{5/4} - 1\right] = a\rho_{w_0}\frac{T_0}{T_\infty}\frac{l}{\ln 10} \cdot 10^{-\frac{z}{l}},$$

The following values were adopted:

$$\rho_{w_0} = 6.2 \cdot 10^{-6}$$
$$a = 7.25 \text{ cm}^2 \text{ g}^{-1}$$
$$T_0 : T_\infty = 280 : 220,$$

then $\tau_0 = 12.6$.

For

$$\sigma f T^4 = 0.047\left[6.80 - 4.80 \cdot e^{-0.2\tau} + 0.57 \cdot e^{-1.75\tau} - 0.36e^{1.75(\tau - 12.6)}\right].$$

the following expression was obtained:

$$W\varepsilon = \sigma f T^4$$

This distribution of temperature in altitude is described in Table 56. In this table are also given the mean temperatures, calculated from radiosonde measurements conducted over a period of 3 years (1938, 1939, 1940) at the Omaha station (USA), where $\varphi \approx 60°$.

Table 56

Temperature distribution in altitude

Z km	0.3	0.5	1.0	2.0	3	4	5	6	7	8	9	10	11	12	13
τ	12.6	12.0	10.6	8.4	6.5	5.1	4.0	3.2	2.5	1.9	1.45	1.1	0.8	0.6	0.4
T_{calc}	8	9	9	4.5	−0.5	−6.5	−13.0	−19.5	−26.5	−33.5	−40.0	−45.0	−49.0	−52.0	−53.0
T_{obs}	8.1	9.1	8.9	5.2	−0.1	−6.1	−12.2	−19.5	−26.4	−33.7	−40.8	−47.2	−52.3	−55.7	−55.7

As is to be seen from the table, this theory yields results in very close agreement with observation.

Despite the good agreement of Kibel's results with observations, N. R. Malkin critized his theory on the following grounds.

According to Kibel's theory the coefficients of his equation (27) are constant. Notwithstanding, these coefficients indefinitely increase with altitude, since ρ_w (the water vapor density) which stands in their denominators tends to zero with increasing altitude. Malkin points out that over the layer extending from 1 to 10 km ρ_w decreases 160 times. The conformity of Kibel's results with experimental data is to be attributed, in Malkin's opinion, to a fortuitous coincidence. As the given example shows, the obtained results depend greatly upon the choice of the values of the physical parameters, and these are not known to any great accuracy.

Malkin himself solved the same problem more rigorously (by means of integral equations), assuming the coefficients of equation (31) to be variable. Unfortunately Malkin does not give even one numerical example to illustrate his results. Consequently, it is not yet clear to what extent does Kibel's approximation of the coefficients by constant values distort the actual situation.

In connection with Malkin's objections, we may mention that the agreement of Kibel's results with observation may be explained by the hypothesis that ρ_w expresses not the water vapor density, but the total density of all substances which are radiationally active (i. e., water vapor + ozone + carbon dioxide).

§ 3. The Annual Mean Temperature Distribution in the Terrestrial Atmosphere

Kibel's work on the dependence of the mean temperature on altitude has substantially contributed to the theoretical solution of this problem. The example worked out by Kibel' shows how closely do the theoretically obtained mean temperatures fit the observed values. However, such a good accord of theory with observation is not obtained at all latitudes.

This is due to the presence, in addition to the vertical exchange, of horizontal turbulent exchange on a considerable scale which has an appreciable effect on the dependence of temperature on altitude. The horizontal exchange coefficient turns out to be maximal at low latitudes on one hand, and at polar latitudes on the other hand, whereas in the middle latitudes its effect is insignificant. We recall that Kibel' has compared his results with observations carried out at a latitude of 40°, where the horizontal exchange is smallest.

We conclude, that in order to determine the distribution in altitude of the annual mean temperature at any point on the globe we must take into consideration not only radiation effects and vertical eddy flux of heat but also the large scale horizontal exchange.

E. N. Blinova made the first attempt at a theoretical determination of the mean annual temperature distribution in the terrestrial atmosphere, allowing for the influence of continents and oceans, and for the effects of horizontal large-scale turbulent exchange.

She started from the same equation of heat balance used by Kibel':

$$c_p\rho\frac{dT}{dt}-\frac{dp}{dt}=\varepsilon_1+\varepsilon_2, \tag{1}$$

where ε_1 is the heat influx due to radiation and ε_2 —that due to turbulence. The heat influx due to phase changes of water is disregarded.

For the radiational heat transfer Blinova also adopts the equation used by Kibel':

$$\varepsilon_1=a\rho_w(A+B)+a\beta\rho_w S-2a\rho_w E, \tag{2}$$

since she makes the same simplifying assumptions on the nature of radiant energy flux in the atmosphere, as Kibel' made.

Regarding the earth as a sphere of radius a_0, Blinova introduces spherical coordinates: r – the distance from the earth's center, θ – the complementary latitude, and ψ – the longitude of the locality, measured from the zero meridian. The expression for the eddy flux of heat in spherical coordinates will be written in the following form:

$$\varepsilon_2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2K\frac{\partial T}{\partial r}\right)+\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}K'\frac{\partial T}{\partial\theta}\right)+\frac{\partial}{\partial\psi}\left(\frac{K'}{r\sin\theta}\frac{\partial T}{\partial\psi}\right)\right], \tag{3}$$

where K is the coefficient of vertical eddy heat conductivity, and K' – the coefficient of horizontal eddy heat conductivity.

Instead of the coordinate r, the coordinate $z = r-a_0$ is introduced; equation (3) then assumes the form

$$\varepsilon_2=\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)+\frac{1}{a_0^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\cdot K'\frac{\partial T}{\partial\theta}\right)+\frac{\partial}{\partial\psi}\left(\frac{K'}{\sin\theta}\cdot\frac{\partial T}{\partial\psi}\right)\right]. \tag{4}$$

E. N. Blinova supposes that the annual mean temperature distribution is stationary. In addition, she assumes that the mean velocities vanish. These assumptions imply, in effect, that steady heat transfer is absent and the total horizontal heat exchange is due to horizontal large-scale turbulence.

The relevant equations are then:

$$a\rho_w(A+B+\beta S-2E)+\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)+\frac{1}{a_0^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(K'\sin\theta\frac{\partial T}{\partial\theta}\right)+\right.$$
$$\left.+\frac{\partial}{\partial\psi}\left(\frac{K'}{\sin\theta}\frac{\partial T}{\partial\psi}\right)\right]=0, \tag{5}$$

$$\frac{\partial A}{\partial z}=a\rho_w(A-E), \tag{6}$$

$$\frac{\partial B}{\partial z} = \alpha\rho_w (E - B), \tag{7}$$

$$\frac{\partial S}{\partial z} = \alpha\beta\rho_w S, \tag{8}$$

$$E = f\sigma T^4. \tag{9}$$

The function sought is the temperature T; the remaining functions play a merely accessory role.

The boundary conditions are the following:

1. Only short-wave radiation is incident on the upper boundary of the atmosphere, hence

$$A = 0 \quad \text{for} \quad z \to \infty. \tag{10}$$

2. Not all the incident short-wave insolational radiation enters the atmosphere. A part of it is returned to outer space as a result of reflection at the earth's surface, air molecues [sic] and clouds. Blinova accounts for this circumstance by formulating the following boundary condition:

$$S = [1 - \Gamma(\theta, \psi)] \cdot w(\theta) = W(\theta, \psi) \quad (\text{for } z \to \infty), \tag{11}$$

where Γ is the albedo of the earth, determined mainly by the nature of the terrestrial surface, and also by cloudiness; $w(\theta)$ is the mean insolation per unit surface and unit time at the upper boundary of the atmosphere. This magnitude depends only upon the latitude φ.

3. It is assumed, that at the upper boundary of the atmosphere the temperature is horizontally constant, hence

$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial \psi} = 0 \quad (\text{at} \quad z = \infty). \tag{12}$$

4. It is also assumed, that the total long-wave radiation outgoing from the atmosphere equals the total influx of short-wave radiation, i.e.,

$$\iint\limits_{(S)} B\,ds = \iint\limits_{(S)} W ds \quad (\text{at} \quad z = \infty), \tag{13}$$

where the integral is taken over the surface which is assumed to represent the upper boundary of the atmosphere.

5. At the earth's surface the condition of heat balance holds:

$$-K\frac{\partial T}{\partial z} + K^*\frac{\partial T^*}{\partial z} = A + S - B \quad (\text{at} \quad z = 0), \tag{14}$$

where $K^*\frac{\partial T^*}{\partial z}$ is the heat flux from the soil. However, Blinova believes that for the annual mean temperature distribution in the terrestrial atmosphere, this heat flux may be disregarded.

6. Finally,

$$B = qE = qf\sigma T^4 \quad (\text{at} \quad z = 0), \tag{15}$$

where q·f — is the grayness factor of the soil — a coefficient (close to unity) which shows that the soil does not radiate as an ideal black body, but somewhat less.

Introducing instead of z the above defined optical thickness x:

$$x=\frac{1}{\tau_0}\int_z^\infty a\rho_w\,dz,\ \text{where}\ \tau_0=\int_0^\infty a\rho_w\,dz, \tag{16}$$

and adopting Kibel's notation A + B = a, B − A = y, Blinova brings the equation of heat balance to the following form:

$$a+\beta S-2E+\frac{1}{\tau_0^2}\frac{\partial}{\partial x}\left(\frac{2}{m^2-1}\frac{\partial E}{\partial x}\right)+ \\ +\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(M\sin\theta\frac{\partial E}{\partial\theta}\right)+\frac{\partial}{\partial\psi}\left(\frac{M}{\sin\theta}\frac{\partial E}{\partial\psi}\right)\right]=0, \tag{17}$$

where

$$\frac{2}{m^2-1}=\frac{Ka\rho_w}{4f\sigma T^3};\ M=\frac{K'}{4f\sigma T^3a_0^2a\rho_w}. \tag{18}$$

The magnitudes m and M are regarded by Blinova as constants.

From equation (17) follows, that

$$a=2E-\beta S-\frac{2}{\tau_0^2(m^2-1)}\frac{\partial^2E}{\partial x^2}- \\ -M\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2E}{\partial\psi^2}\right]. \tag{19}$$

As was shown in § 2, equations (6) and (7) imply

$$\frac{da}{dx}=\tau_0 y, \tag{20}$$

$$\frac{\partial y}{\partial x}=\tau_0(a-2E), \tag{21}$$

applying (8) and (11) we find:

$$S=We^{-\tau_0\beta x}. \tag{22}$$

Therefore from equations (19) and (20) we obtain for y:

$$y=\beta^2S+\frac{2}{\tau_0}\frac{\partial E}{\partial x}-\frac{2}{\tau_0^3(m^2-1)}\frac{\partial^3E}{\partial x^3}-\frac{M}{\tau_0}\frac{\partial}{\partial x}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E}{\partial\theta}+\right.\right. \\ \left.\left.+\frac{1}{\sin^2\theta}\frac{\partial^2E}{\partial\psi^2}\right)\right]. \tag{23}$$

Replacing now a and y in (21) by their equivalent expressions from (19) and (23), we obtain, after some transformations, one equation for the single unknown function E:

$$\frac{2}{m^2-1}\frac{\partial^4E}{\partial x^4}-\frac{2m^2\tau_0^2}{m^2-1}\frac{\partial^2E}{\partial x^2}+\frac{M\tau_0^2}{\sin\theta}\left[\frac{\partial^3}{\partial x^2\partial\theta}\left(\sin\theta\frac{\partial E}{\partial\theta}\right)+\frac{\partial^3}{\partial x^2\partial\psi}\left(\frac{1}{\sin\theta}\frac{\partial E}{\partial\psi}\right)\right]- \\ -\tau_0^2\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E}{\partial\theta}\right)-\tau_0^2\frac{\partial}{\partial\psi}\left(\frac{1}{\sin\theta}\frac{\partial E}{\partial\psi}\right)=\beta(1-\beta^2)\tau_0^4e^{-\tau_0\beta x}W. \tag{24}$$

The boundary conditions for the new functions with respect to the new independent variable x assume the form

$$a - y = 0 \qquad \text{at} \quad x = 0, \tag{25}$$

$$\frac{\partial E}{\partial \theta} = \frac{\partial E}{\partial \psi} = 0 \qquad \text{at} \quad x = 0, \tag{26}$$

$$\iint_S (a - y)\, ds = 2 \iint_s W\, ds \qquad \text{at} \quad x = 0, \tag{27}$$

$$y + \frac{2}{(m^2 - 1)\tau_0} \cdot \frac{\partial E}{\partial x} = S \qquad \text{at} \quad x = 1, \tag{28}$$

$$y + \frac{1}{\tau_0} \frac{\partial y}{\partial x} - 2(q - 1)E = 0 \qquad \text{at} \quad x = 1. \tag{29}$$

Equation (24) shows, that consideration of the horizontal large-scale exchange considerably complicates the problem. Indeed, when only vertical exchange was taken into account, the determination of the temperature dependence on altitude was reduced to performing two quadratures and solving an ordinary differential equation of the second order. But when horizontal exchange effects are included, we are faced with a partial differential equation of the fourth order.

Many problems relating to spherical surfaces admit of a comparatively simple solution in terms of a well studied class of special functions — the spherical harmonics, which play there the same role as the trigonometric functions in plane problems involving circles. Many functions defined on spherical surfaces may be represented by an infinite series in terms of spherical harmonics — in analogy to the developability of many other functions in trigonometric series. This method was used by Blinova for the integration of equation (24).

The function $W(\theta, \psi)$ which expresses the global distribution of solar radiation influx in the atmosphere may be expanded in an infinite series in terms of spherical harmonics

$$W(\theta, \psi) = \sum_{n=0}^{\infty} \left[W_n^0 P_n(\cos\theta) + \sum_{h=1}^{n} (W_n^h \cos h\psi + W_n'^h \sin h\psi) P_n^h(\cos\theta) \right]. \tag{30}$$

where W_n^0, W_n^h, $W_n'^h$ are appropriate constant coefficients, $P_n(\cos\theta)$ is the so-called Legendre polynomials, determined by the formula

$$P_n(y) = \frac{1}{2^n \cdot n!} \frac{d^n (y^2 - 1)^n}{dy^n} \tag{31}$$

and satisfying the equation

$$(1 - y)^2 \frac{d^2 P_n(y)}{dy^2} - 2y \frac{dP_n(y)}{dy} + n(n + 1) P_n(y) = 0, \tag{32}$$

and $P_n^h(y)$ are spherical harmonics, determined by

$$P_n^h(y) = (1 - y^2)^{h/2} \frac{d^h P_n^h(y)}{dy^h} \tag{33}$$

and satisfying the equation

$$\frac{d}{dy}\left[(1 - y^2) \frac{dP_n^h(y)}{dy}\right] + \left(\lambda_n - \frac{h^2}{1 - y^2}\right) P_n^h(y) = 0, \tag{34}$$

where λ_n is a parameter.

Blinova looks for a solution of equation (24) in a form of an expansion in spherical harmonics

$$E(x,\theta,\psi)=\sum_{n=0}^{\infty}\Big\{E_n^0(x)P_n(\cos\theta)+\sum_{h=1}^{n}[E_n^h(x)\cos h\psi+ \\ +E_n'^h(x)\sin h\psi]P_n^h(\cos\theta)\Big\}. \tag{35}$$

We substitute the series (30) and (35) in equation (34). Using the functional equations satisfied by spherical harmonics and equating the coefficients of corresponding terms, we obtain the following equations for the determination of the coefficients of the series (35):

$$\frac{2}{m^2-1}\frac{d^4E_n^h}{dx^4}-\tau_0^2\left[\frac{2m^2}{m^2-1}+M\cdot n(n+1)\right]\frac{d^2E_n^h}{dx^2}+ \\ +M\tau_0^4\cdot n(n+1)E_n^h=\beta(1-\beta^2)\cdot\tau_0^4\cdot e^{-\tau_0\beta x}\cdot W_n^h, \tag{36}$$

where n runs through all integers from 0 to ∞, and h assumes all integral values from 0 to n. For the determination of E'^h_n an analogous equation is derived, differing from (36) in that W^h_n is replaced by W'^h_n, whereby h assumes all integral values from 1 to n.

The solution of (36) for all E^h_n (except E^0_0) has the following form

$$E_n^h=(C_1)_n^h e^{k_n x}+(C_2)_n^h e^{\tilde{k}_n x}+(C_3)_n^h e^{-k_n x}+(C_4)_n^h e^{-\tilde{k}_n x}+A_n^h e^{-\tau_0\beta x}, \tag{37}$$

where $(C_1)^h_n$, $(C_2)^h_n$, $(C_3)^h_n$, $(C_4)^h_n$ are arbitrary constants to be determined from the boundary conditions (25)–(29), and k_n and $\tilde{k}_n$ are given by the formulas

$$\left.\begin{aligned}k_n&=\frac{\tau_0\cdot m}{\sqrt{2}}\sqrt{1+\frac{D_n}{m^2}+\sqrt{1+\frac{2}{m^4}(m^2-2)D_n+\frac{(D_n)^2}{m^4}}},\\ \tilde{k}_n&=\frac{\tau_0\cdot m}{\sqrt{2}}\sqrt{1+\frac{D_n}{m^2}-\sqrt{1+\frac{2}{m^4}(m^2-2)D_n+\frac{(D_n)^2}{m^4}}},\end{aligned}\right\} \tag{38}$$

where

$$D_n=\frac{M(m^2-1)\cdot n\cdot(n+1)}{2}. \tag{39}$$

Finally

$$A_n^h=\frac{1}{2}\frac{\beta(1-\beta)^2(m^2-1)W_n^h}{\beta^2(\beta^2-m^2)+D_n(1-\beta^2)}. \tag{40}$$

E.N. Blinova adopted the following numerical values of the physical parameters: m = 1.75; τ_0 = 12.6; β = 0.2; q/f = 1.15; in evaluating

$$\frac{M(m^2-1)}{2}=\frac{K'}{K}\frac{1}{(a\rho_w a_0)^2}$$

she takes: $K'/K = 10^6$; for a and ρ_w she adopts their mean values at the earth's surface: $a = 7.27\ \mathrm{cm^2/g}$, $\rho_w = 6.2\cdot10^{-6}\ \mathrm{g/cm^3}$. Then $M(m^2-1)/2 = 0.00128$.

Besides, we may write to a high degree of accuracy:

$$k_n\approx\tau_0\cdot m;\qquad \tilde{k}_n\approx\frac{\tau_0\sqrt{D_n}}{m}.$$

For the determination of the arbitrary constants of all E^h_n and E'^h_n, except E^0_0, the following condition was adopted: $E^h_n = E'^h_n = 0$ at x = 0. This corresponds to the boundary condition (26).

Accordingly, the arbitrary constants assume the following values:

$$
\left.
\begin{aligned}
(C_1)_n^h &\approx A_n^h \left(Q+\frac{m}{\beta}\right)\cdot e^{-\tilde{k}_n-\tau_0\beta};\\
(C_2)_n^h &\approx \frac{A_n^h}{2\Lambda_n}\left[\left(\frac{1}{m-1}+\frac{1}{1-\beta}\right)e^{-\tilde{k}_n}+\right.\\
&\qquad \left.+Q\left(\frac{\sqrt{D_n}}{m}+\frac{m}{m-1}\right)\frac{\sqrt{D_n}}{m^2}e^{-\tau_0\beta}\right];\\
(C_3)_n^h &\approx \frac{A_n^h}{\Lambda_n}\left[\frac{\sqrt{D_n}}{m}\cdot e^{\tilde{k}_n}-\right.\\
&\qquad \left.-\left(\frac{\beta}{1-\beta}+\frac{\sqrt{D_n}}{m}\right)\operatorname{ch}\tilde{k}_n-\frac{D_n}{m^3}Qe^{-\tau_0\beta}\right];\\
(C_4)_n^h &\approx \frac{A_n^h}{\Lambda_n}\left[\left(\frac{1}{m-1}+\frac{1}{1-\beta}\right)e^{\tilde{k}_n}+\right.\\
&\qquad \left.+Q\left(\frac{\sqrt{D_n}}{m}-\frac{m}{m-1}\right)\frac{\sqrt{D_n}}{m^2}e^{-\tau_0\beta}\right],
\end{aligned}
\right\} \qquad (41)
$$

where

$$
Q=\frac{\frac{1}{1+\beta}-q}{\frac{1}{m-1}+q}-\frac{m}{\beta};\quad \Lambda_n=\left(\frac{\sqrt{D_n}}{m}-\frac{m}{m-1}\right)\operatorname{ch}\tilde{k}_n-\frac{\sqrt{D_n}}{m}e^{\tilde{k}_n}. \qquad (42)
$$

For determination of E_0^0 the boundary condition (27) was used. Besides, it follows from (36), that E_0^0 satisfies the equation

$$
\frac{2}{m^2-1}\frac{d^4E_0^0}{dx^4}-\frac{2m^2\tau_0^2}{m^2-1}\frac{d^2E_0^0}{dx^2}=\beta(1-\beta)\tau_0^4e^{-\tau_0\beta x}W_0^0. \qquad (43)
$$

Equation (43) was already solved by Kibel', and hence his solution could be employed to determine E_0^0.

After E_n^h and E'^h_n were determined by formulas (37), the solution for E could be constructed in the form of a series (35), which for different x, θ, ψ, yields the altitude, longitude and latitude distribution of E. Knowing the distribution $E=f\sigma T^4$, it is easy to pass to the temperature distribution (see Appendix 15).

Blinova carried out the calculation of the zonal distribution of annual mean atmospheric temperature in altitude and along the meridian. The albedo of the earth was assumed to be constant and equal to 0.7. The function W, representing the effective insolation of the atmosphere, is now a function of θ only. It may be developed in a series of Legendre polynomials:

$$
W(\theta)=\sum_{n=0}^{\infty}W_n^0P_n(\cos\theta). \qquad (44)
$$

Then the solution of the problem also appears in the form of a series of Legendre polynomials

$$
E(x,\theta)=\sum_{n=0}^{\infty}E_n^0(x)\cdot P_n(\cos\theta). \qquad (45)
$$

We know that the annual mean insolation of the earth is symmetrically distributed with respect to the equator. Therefore W (θ) should be an even function of θ, i.e.,

$$W(\theta) = W(-\theta).$$

Accordingly the coefficients of all odd Legendre polynomials should identically vanish, i.e.,

$$W_1^0 = W_3^0 = W_5^0 = \ldots = 0.$$

Blinova applied the values of W(0) calculated by Milankovich and cited in his book "Mathematical Climatology and Astronomical Theory of Climate Variation". Using the data of Milankovich, Blinova obtained the following values for the coefficients of the series (44): $W_0^0 = 0.1498$; $W_2^0 = -0.0709$; $W_4^0 = 0.0057$. The remaining coefficients W_n^0 with $n \geqslant 0$ proved to be negligibly small.

Calculating k_n, $\tilde{k}_n$, A_n^0, $(C_1)_n^0$, $(C_2)_n^0$, $(C_3)_n^0$, $(C_4)_n^0$ for n = 2 and 4 by formulas (38)–(42), Blinova obtained the following expressions for E_2^0 and E_4^0, according to formula (37):

$$\left.\begin{aligned} E_2^0(x) &= 10^{-2}[-0.12\,e^{22(x-1)} - 2.97\cdot e^{0.61x} + 1.11e^{-22x} - \\ &\quad - 10.48e^{-0.61x} + 12.29e^{-2.52x}], \\ E_4^0 &= 10^{-3}[-0.11e^{22(x-1)} - 1.06e^{1.12x} + 0.79e^{-22x} - \\ &\quad - 11.26e^{-1.12x} + 11.47e^{-2.52x}], \end{aligned}\right\} \qquad (46)$$

and for E_0^0:

$$E_0^0 = 0.051\,[6.73 - 4.80e^{-2.52x} + 0.55e^{-22x} - 0.26e^{22(x-1)}]. \qquad (47)$$

The final expression for the function E(x, θ) has the form

$$E(x, \theta) = E_0^0(x) + E_2^0(x)\cdot P_2(\cos\theta) + E_4^0(x)\cdot P_4(\cos\theta). \qquad (48)$$

Using formulas (46)–(48) Blinova calculated the annual mean temperature distribution in altitude and along the meridian. In order to pass from the optical thickness x to the altitude z she used the following relationship:

$$(1 + \tau_0 x)^{5/4} - 1 = [(1 + \tau_0)^{5/4} - 1]\cdot 10^{-\frac{z}{l}} \quad (l = 8\cdot 10^5\ cm). \qquad (49)$$

Table 57 gives the results of these calculations.

The theoretical distribution of zonal temperature obtained by Blinova is in very good agreement with empirical results. In Figure 158 the solid line represents the temperature distribution along the meridian at sea level, according to empirical data, after Hann and Süring. The broken line represents the theoretical temperature at the sea level given in the first row of Table 57. It can be seen from the figure that in the Southern Hemisphere both curves practically coincide, and in the Northern they diverge by no more than 2°C. It is remarkable that the theoretical and empirical temperatures at the North Pole (−20°C) and on the equator (26°C) do coincide.

This shows that the shape of the curve is really accurate.

Table 57

Annual mean temperature distribution in altitude and along the meridian

Altitude z km	Latitude φ°									
	90	80	70	60	50	40	30	20	10	0
0	253	255	261	269	277	284	291	295	298	299
0.5	57	59	64	71	79	86	92	96	99	99.5
1	56	58	63	70	77	84	90	94	97	98
2	52	54	59	65	73	80	85	90	92	93
3	47	49	53	60	67	74	79	83	83	86
4	41	44	48	54	61	67	73	77	79	80
5	36	38	42	48	54	60	65	69	72	72
6	33	35	38	43	49	55	59	63	64	65
7	26	28	31	36	41	46	51	55	57	58
8	21	22	26	30	35	40	44	47	49	50
9	18	20	22	27	31	35	39	42	43	44
10	16	17	19	23	27	30	33	35	37	38
11	15	16	18	21	24	27	30	32	33	34
12	15	16	17	20	22	25	27	29	30	30
13	15	16	17	19	21	24	25	27	28	28
14	16	17	18	19	21	23	24	25	26	27
15	17	17	18	19	21	22	23	24	24	25
16	18	18	18	19	21	21	22	22	23	23

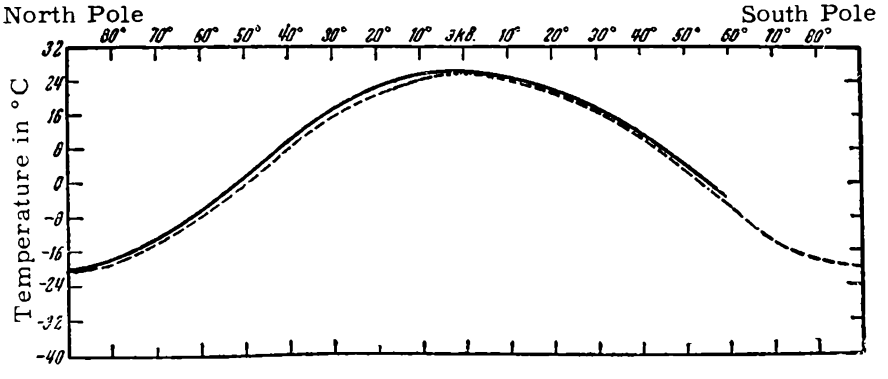

Figure 158. Zonal temperature distribution along the meridian: solid line — observed, broken line — calculated by Blinova.

Figures 159a, b, c, d, represent the distribution of temperature in altitude for different latitudes. Figure 159 a refers to the latitude 78°C. The solid line shows results obtained by Byrd's Antarctic expedition in Little America (φ =78°34'S, ψ = 163°55'W). Figure 159 b gives data for the Soviet Union from 55 to 60° N and from 30 to 39° E. In Figure 159 c are given observarions conducted at the Madrid station. Figure 159 d gives the results of radiosonde measurements conducted at the station of San Diego (32°42'N, 117°9'W). It is seen from all these figures, that within the entire atmospheric layer extending from the earth's surface up to 9—10 km, the calculated curves practically coincide with the empirical. Only at great altitudes (particularly for low latitudes) do the theoretical results significantly depart from observation, whereby the lower atmosphere proves to be actually colder than in theory.

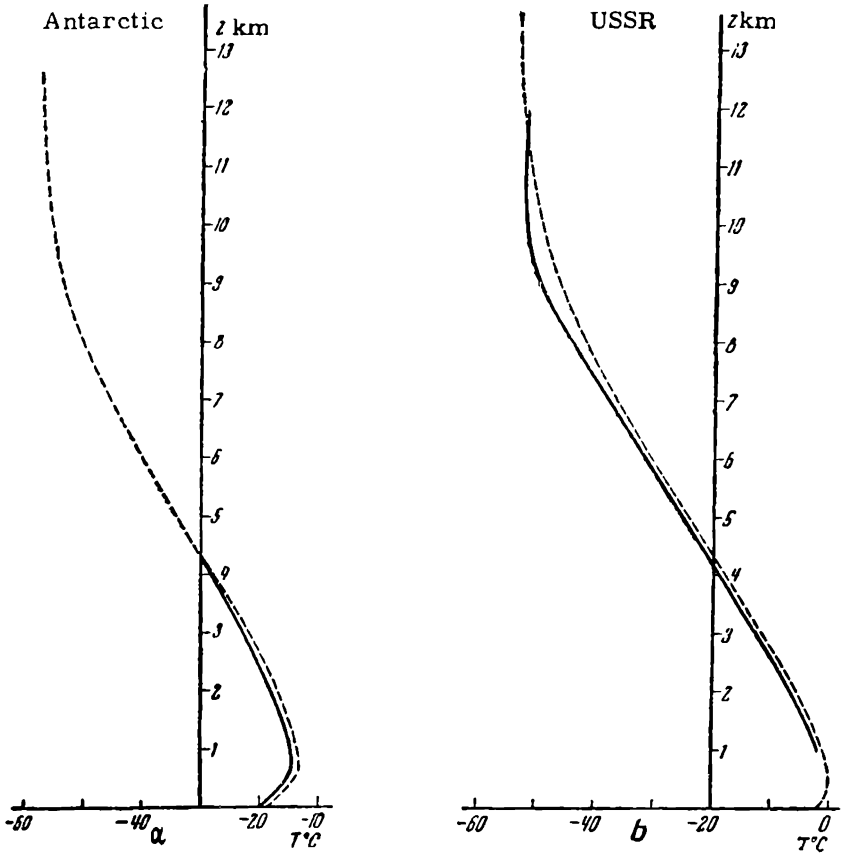

Figure 159 a and b. Temperature distribution in altitude. Solid line — observed. Broken line — calculated by Blinova

§ 4. Diurnal Temperature Cycle in the Mixing Layer According to Dorodnitsyn

A. Dorodnitsyn considered the question of the 24 hour temperature variations in the mixing layer, regarding the influx of heat by radiative exchange to be given as a function of the time.

Air temperature is determined by the eddy flux of heat and by radiation processes, and also by the advective heat transfer. Writing separately the heat influx due to eddy transport of mass, Dorodnitsyn sets up the equation of the problem in the following form:

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(K \frac{\partial T}{\partial z}\right) + \rho c_p \Phi(z, t) \tag{1}$$

or

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + \Phi(z, t), \tag{2}$$

where $\lambda = K/\rho c_p$ and $\Phi(z, t)$ is a function accounting for the radiant heat influx and heat advection.

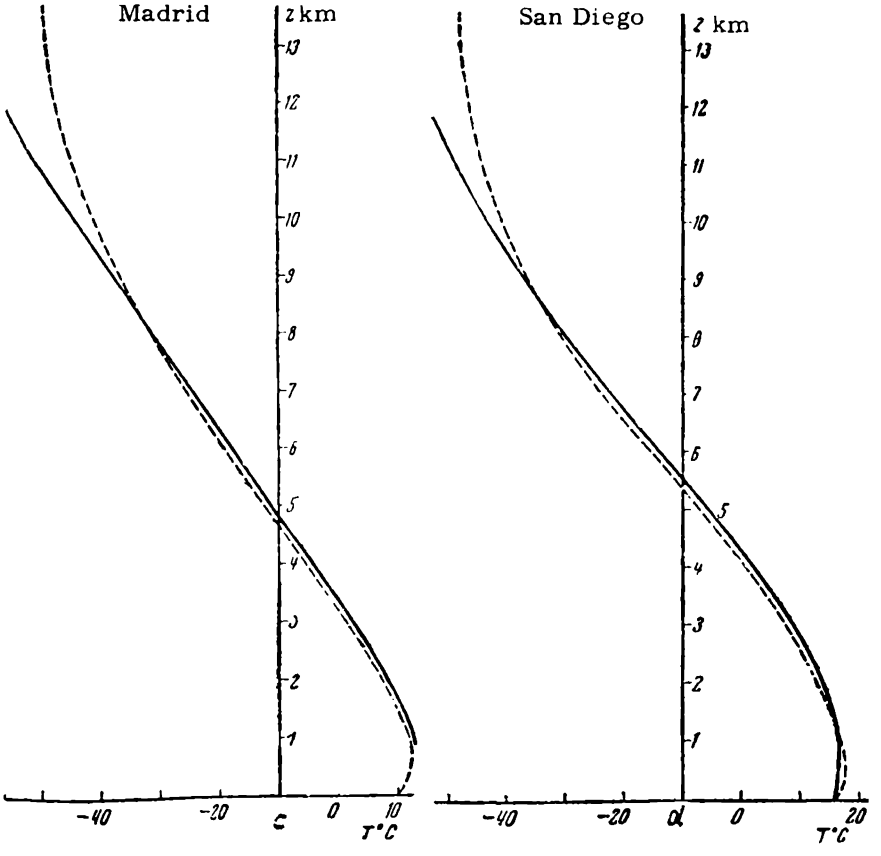

Figure 159 c and d. Temperature distribution in altitude. Solid line — observed. Broken line — calculated by Blinova

Advection obviously depends on altitude and time. The rate of advective temperature variation is, however, assumed by Dorodnitsyn to be so slow that in the consideration of the problem of the diurnal temperature variation this change may be ignored. Dorodnitsyn further assumes that the radiant heat influx in the lower layer (1–2 km) is small in comparison with the eddy flux of heat. This assumption is supported by the fact that the diurnal temperature cycle is observed only in the lowest 1 kilometer layer. If radiational processes were as significant as the turbulent heat transfer, the diurnal temperature cycle would be observed at all altitudes, which is not the case at all. Dorodnitsyn evaluates the various factors influencing heat exchange, and finds that in the lower 1 km layer, radiant heat influx amounts to no more than a few per cent, and eventually less than one per cent on the eddy flux of heat. Thus Dorodnitsyn believes that Φ depends only on z and not on t, since radiant energy plays but a minor role, and advection has a long period.

The solution of the heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + \Phi(z) \tag{3}$$

is sought by Dorodnitsyn in the form

$$T(z, t) = T_0(z) + \tau(z, t), \tag{4}$$

where T_0 is the 24 hour mean temperature. Thence

$$\frac{\partial T}{\partial t} = \frac{\partial \tau}{\partial t}; \quad \frac{\partial T}{\partial z} = \frac{dT_0}{dz} + \frac{\partial \tau}{\partial z}. \tag{5}$$

Consequently, equation (3) takes the form

$$\frac{\partial \tau}{\partial t} = \frac{d}{dz}\left(\lambda \frac{dT_0}{dz}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{d\tau}{\partial z}\right) + \Phi(z); \tag{6}$$

since $\frac{d}{dz}\left(\lambda \frac{dT_0}{dz}\right)$ and $\Phi(z)$ are functions of the altitude only, equation (6) splits into independent equations

$$\frac{d}{dz}\left(\lambda \frac{dT_0}{dz}\right) + \Phi(z) = 0, \tag{7}$$

$$\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial z}\left(\lambda \frac{\partial \tau}{\partial z}\right). \tag{8}$$

Dorodnitsyn is not concerned with equation (7), believing that the diurnal mean temperature T_0 may be determined, for instance, by Kibel's methods (see Chapter XX). Dorodnitsyn considers only equation (8).

He assumes the following dependence of λ on altitude:

$$\lambda = \lambda_0 (1 + \varepsilon - e^{-mz}). \tag{9}$$

The quantity ε is introduced to obtain a λ value of the order of the molecular heat conductivity at $z = 0$. Let us note that this corresponds to the assumption that the earth's surface is smooth and devoid of vegetation.

Dorodnitsyn considers, in addition to equation (8), the process of heat conduction in the soil, assuming the soil coefficient of thermal conductivity λ^* to be constant. Thus the problem reduces to the integration of the following system of equations:

$$\left.\begin{aligned} \frac{\partial \tau}{\partial t} &= \frac{\partial}{\partial z}\left(\lambda \frac{\partial \tau}{\partial z}\right) \quad \text{for } z > 0, \\ \frac{\partial \tau^*}{\partial t} &= \frac{\partial}{\partial z}\left(\lambda^* \frac{\partial \tau}{\partial z}\right) \quad \text{for } z < 0, \end{aligned}\right\} \tag{10}$$

where τ is the deviation of the soil temperature from its 24 hour mean value.

No initial conditions are given for the problem since we deal here with periodic temperature variations.

The boundary conditions are formulated as follows: at infinity all periodic temperature variations are levelled down; therefore

$$\tau(z, t) \to 0 \quad \text{for} \quad z \to +\infty, \tag{11}$$

$$\tau^*(z, t) \to 0 \quad \text{for} \quad z \to -\infty. \tag{12}$$

Further Dorodnitsyn stipulates a continuous temperature variation at the earth's surface, i. e.,

$$\tau(0, t) = \tau^*(0, t). \tag{13}$$

Finally, he sets up the condition of heat balance at the surface of the earth. If radiative exchange did not exist, the boundary condition would have had the following form

$$K \frac{\partial T}{\partial z} = K^* \frac{\partial T^*}{\partial z} \quad \text{or} \quad K \frac{\partial \tau}{\partial z} = K^* \frac{\partial \tau^*}{\partial z} \quad \text{at} \quad z = 0.$$

However, in the presence of both incoming and outgoing radiant energy, the boundary condition expressing the energy balance should be written in the form

$$-K^* \frac{\partial T^*}{\partial z} = -K \frac{\partial T}{\partial z} - \sigma T^4 - (1 - \Gamma) W(t), \tag{14}$$

where Γ is the albedo of the earth's surface.

However, since the equations to be integrated are given in terms of τ, and not of T, it is necessary to rewrite the boundary condition (14) for τ too.

$$\left.\begin{aligned} T^*(z,t) &= T_0^*(z) + \tau^*(z,t), \\ T(z,t) &= T_0(z) + \tau(z,t), \\ W(t) &= W_0 + W_1(\tau), \end{aligned}\right\} \tag{15}$$

where W_0 is the 24 hour mean insolational and atmospheric heat influx. The boundary condition (14) will be rewritten as follows:

$$-K^*\frac{\partial T_0^*}{\partial z} - K^*\frac{\partial \tau^*}{\partial z} = -K\frac{\partial T_0}{\partial z} - K\frac{\partial \tau}{\partial z} + \sigma[T_0(0)+\tau(0,t)]^4 - \tag{16}$$
$$-(1-\Gamma)\cdot W_0 - (1-\Gamma)\cdot W_1(t) \quad (\text{at } z=0).$$

When, in addition, the mean diurnal balance is considered, we have

$$-K^*\frac{\partial T_0^*}{\partial z} = -K^*\frac{\partial T_0}{\partial z} + \sigma T_0^4(0) - (1-\Gamma)\,W_0 \quad \text{at} \quad z=0. \tag{17}$$

Combining (16) and (17) we obtain:

$$(1-\Gamma)\cdot W_1(t) = \sigma[T_0(\tau)+\tau(0,t)]^4 - \sigma T_0^4(0) - K\frac{\partial \tau}{\partial z} + \tag{18}$$
$$+K^*\frac{\partial \tau^*}{\partial z} \quad \text{at} \quad z=0.$$

This is the fourth boundary condition.

But this equation introduces a new function $W_1(t)$; Dorodnitsyn considers it as given. The boundary condition (18) is not linear in τ. This circumstance renders the solution of the problem more difficult; Dorodnitsyn avoids this difficulty by linearizing the boundary condition in the following manner. Evidently

$$\sigma[T_0(0)+\tau(0,t)]^4 - \sigma T_0^4(0) = 4\sigma T_0^3\tau + \ldots \tag{19}$$

Assuming

$$4\sigma T_0^3 = \mu = \text{const}. \tag{20}$$

and disregarding all powers of τ higher than the first, Dorodnitsyn believes this simplification causes an error of no more than 10%, if instead of the right side of (19) $\mu\tau$ is substituted. But since the term expressing radiant energy flux is small in comparison with the term expressing heat conduction, Dorodnitsyn estimates the resulting error in the boundary condition at no more than 3%. Consequently, the boundary condition (18) is replaced by the following:

$$(1-\Gamma)\,W_1(t) = \mu\tau(0,t) - K\frac{\partial \tau}{\partial z} + K^*\frac{\partial \tau^*}{\partial z} \quad \text{at} \quad z=0. \tag{21}$$

Thus the determination of the diurnal temperature cycle is reduced to the integration of the system (10) for the boundary conditions (11), (12), (13) and (21).

A solution of these equations is sought in the form of an infinite series:

$$\left.\begin{aligned} \tau(z,t) &= \sum_{\nu=1}^{\infty}[\tau_{1,\nu}(z)\cos\nu\omega t + \tau_{2,\nu}(z)\sin\nu\omega t], \\ \tau^*(z,t) &= \sum_{\nu=1}^{\infty}[\tau_{1,\nu}^*(z)\cos\nu\omega t + \tau_{2,\nu}^*(z)\sin\nu\omega t]. \end{aligned}\right\} \tag{22}$$

The given function $W_1(t)$ is also expanded in an infinite series:

$$W_1(t) = \sum_{\nu=1}^{\infty}[a_\nu\cos\nu\omega t + b_\nu\sin\nu\omega t]. \tag{23}$$

Substituting the expression (22) and (23) in the equation of the problem

and equating the corresponding coefficients, we obtain a complete system of equations for the determination of $\tau_{1,\nu}$, $\tau_{2,\nu}$, $\tau^*_{1,\nu}$, $\tau^*_{2,\nu}$.

Dorodnitsyn does not examine the convergence of this series solution, but only seeks a particular solution. Substituting (22) in (10), we obtain:

$$-\nu\omega\tau_{1,\nu}(z)\sin\nu\omega t+\nu\omega\tau_{2,\nu}(z)\cos\nu\omega t=$$
$$=\lambda_0\frac{d}{dz}\left[(1+\varepsilon-e^{-mz})\frac{d\tau_{1,\nu}}{dz}\right]\cos\nu\omega t+$$
$$+\lambda_0\frac{d}{dz}\left[(1+\varepsilon-e^{-mz})\frac{d\tau_{2,\nu}}{dz}\right]\sin\omega t. \tag{24}$$

Since the equality (24) holds for any t, we get

$$\left.\begin{aligned}\lambda_0\frac{d}{dz}\left[(1+\varepsilon-e^{-mz})\frac{d\tau_{1,\nu}}{dz}\right]&=\nu\omega\tau_{2,\nu}(z),\\ \lambda_0\frac{d}{dz}\left[(1+\varepsilon-e^{-mz})\frac{d\tau_{2,\nu}}{dz}\right]&=-\nu\omega\tau_{1,\nu}(z);\end{aligned}\right\} \tag{25}$$

as well as

$$\left.\begin{aligned}\lambda^*\frac{d^2\tau^*_{1,\nu}}{dz^2}&=\nu\omega\tau^*_{2,\nu}(z),\\ \lambda^*\frac{d^2\tau^*_{2,\nu}}{dz^2}&=-\nu\omega\tau^*_{1,\nu}(z).\end{aligned}\right\} \tag{26}$$

For the boundary conditions we obtain

$$\left.\begin{aligned}&\tau_{1,\nu}(\infty)=\tau_{2,\nu}(\infty)=0;\\ &\tau^*_{1,\nu}(-\infty)=\tau^*_{2,\nu}(-\infty)=0;\\ &\tau_{1,\nu}(0)=\tau^*_{1,\nu}(0);\\ &\tau_{2,\nu}(0)=\tau^*_{2,\nu}(0);\\ &(1-\Gamma)a_\nu=\mu\tau_{1,\nu}(0)-K\frac{d\tau_{1,\nu}}{dz}+K^*\frac{d\tau^*_{1,\nu}}{dz}\quad\text{at}\quad z=0;\\ &(1-\Gamma)b_\nu=\mu\tau_{2,\nu}(0)-K\frac{d\tau_{2,\nu}}{dz}+K^*\frac{d\tau^*_{2,\nu}}{dz}\quad\text{at}\quad z=0.\end{aligned}\right\} \tag{27}$$

Introducing the complex functions

$$\tau_\nu=\tau_{1,\nu}+i\tau_{2,\nu};\quad \tau^*_\nu=\tau^*_{1,\nu}+i\tau^*_{2,\nu}, \tag{28}$$

multiplying the second of equations (25) by i and adding it to the first, we obtain

$$\lambda_0\frac{d}{dz}\left[(1+\varepsilon-e^{-mz})\frac{d\tau_\nu}{dz}\right]+i\nu\omega\tau_\nu=0. \tag{29}$$

In the same way, equation (26) will be rewritten:

$$\lambda^*\frac{d^2\tau^*_\nu}{dz^2}+i\nu\omega\tau^*_\nu=0. \tag{30}$$

We first solve equation (30), whose auxiliary equation has the following form

$$r^2+\frac{i\nu\omega}{\lambda^*}=0,$$

whence

$$r=\pm\sqrt{-\frac{i\nu\omega}{\lambda^*}}=\pm\frac{1-i}{\sqrt{2}}\sqrt{\frac{\nu\cdot\omega}{\lambda^*}}.$$

Consequently, two particular solutions of equation (30) have the form

$$e^{\frac{1-i}{\sqrt{2}}\sqrt{\frac{\nu\omega}{\lambda_0}}\cdot z}; \quad e^{-\frac{1-i}{\sqrt{2}}\sqrt{\frac{\nu\cdot\omega}{\lambda_0}}\cdot z}.$$

The second of these particular solutions must be rejected, since it tends to infinity for $z = -\infty$. Consequently

$$\tau_\nu^* = (C_\nu + iD_\nu)\cdot e^{\frac{1-i}{\sqrt{2}}\sqrt{\frac{\nu\omega}{\lambda_0}}\cdot z}. \tag{31}$$

Separating the real and imaginary parts of (31) we find $\tau_{1,\nu}$ and $\tau_{2,\nu}$.

Equation (29) is much more complicated since its coefficients are variable. We will first determine the type of this equation. Differentiating equation (29) we obtain:

$$\lambda_0[1+\varepsilon-e^{-mz}]\cdot\frac{d^2\tau_\nu}{dz^2}+\lambda_0\cdot m\cdot e^{-mz}\frac{d\tau_\nu}{dz}+i\nu\omega\tau_\nu=0.$$

Dorodnitsyn further changes the independent variable as follows:

$$x = 1-(1+\varepsilon)e^{mz}. \tag{32}$$

Since

$$\frac{d\tau_\nu}{dz}=\frac{d\tau_\nu}{dx}\cdot\frac{dx}{dz};\quad \frac{d^2\tau_\nu}{dz^2}=\frac{d^2\tau_\nu}{dx^2}\left(\frac{dx}{dz}\right)^2+\frac{d\tau_\nu}{dx}\cdot\frac{d^2x}{dz^2}$$

and

$$\frac{d\tau_\nu}{dz}=(1+\varepsilon)\cdot me^{mz}\frac{d\tau_\nu}{dx},$$
$$\frac{d^2\tau_\nu}{dz^2}=(1+\varepsilon)^2\cdot m^2e^{2mz}\frac{d^2\tau_\nu}{dx}-(1+\varepsilon)m^2e^{mz}\cdot\frac{d\tau_\nu}{dx},$$

we get

$$\frac{dx}{dz}=-(1+\varepsilon)me^{mz};\quad \frac{d^2x}{dz^2}=-(1+\varepsilon)m^2e^{mz},$$

and equation (29) is brought to the form

$$\lambda_0(1+\varepsilon-e^{-mz})(1+\varepsilon)^2m^2e^{2mz}\frac{d^2\tau_\nu}{dx^2}-[\lambda_0(1+\varepsilon)m^2+$$
$$+\lambda_0(1+\varepsilon-e^{-mz})(1+\varepsilon)m^2e^{mz})]\frac{d\tau_\nu}{dx}+i\nu\omega\tau_\nu=0.$$

Let us express the coefficient of the first term of equation (33) in terms of x. From (32) we have:

$$(1+\varepsilon)e^{mz}=1-x;$$

consequently,

$$\lambda_0(1+\varepsilon-e^{-mz})(1+\varepsilon)^2\cdot m^2e^{2mz}=$$
$$=\lambda_0(1+\varepsilon)\cdot m^2(1+\varepsilon)\cdot e^{mz}[(1+\varepsilon)e^{mz}-1]=\lambda_0(1+\varepsilon)m^2(1-x)\cdot x.$$

From (33) we obtain

$$-\lambda_0(1+\varepsilon)m^2(1-x)x\frac{d^2\tau_\nu}{dx^2}-[\lambda_0(1+\varepsilon)m^2+(1-x)]\frac{d\tau_\nu}{dx}+i\nu\omega\tau_\nu=0$$

or

$$x(1-x)\frac{d^2\tau_\nu}{dx^2}+(1-x)\frac{d\tau_\nu}{dx}-\frac{i\nu\omega}{\lambda_0(1+\varepsilon)m^2}\tau_\nu=0. \tag{34}$$

Writing

$$\delta^2=\frac{\omega}{\lambda_0(1+\varepsilon)m^2},$$

we have

$$x(1-x)\frac{d^2\tau_\nu}{dx^2}+(1-x)\frac{d\tau_\nu}{dx}-i\nu\delta^2\tau_\nu=0, \tag{35}$$

and putting

$$i\nu\delta^2=-\alpha^2,$$

we have

$$\alpha=\sqrt{-i\nu\delta^2}=\frac{(1-i)}{\sqrt{2}}\cdot\delta\sqrt{\nu}=\frac{1-i}{2}\cdot\sqrt{\frac{\omega}{\lambda_0(1+\varepsilon)m^2}}\cdot\sqrt{\nu}, \tag{36}$$

finally we obtain:

$$x(1-x)\frac{d^2\tau_\nu}{dx^2}+(1-x)\frac{d\tau_\nu}{dx}+\alpha^2\tau_\nu=0. \tag{37}$$

This equation is a particular case of the thoroughly studied hypergeometric equation

$$x(1-x)\frac{d^2y}{dx^2}+[\gamma-(\alpha+\beta+1)x]\frac{dy}{dx}-\alpha\beta y=0 \tag{38}$$

for

$$\beta=-\alpha;\quad \gamma=1.$$

Equation (38) is not integrable in terms of the elementary functions. The integral of this equation is the infinite hypergeometric series having the form

$$y_1=F(\alpha,\beta,\gamma,x)=1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma\cdot(\gamma+1)}x^2+\ldots \tag{39}$$

However, the hypergeometric series (39) cannot be admitted as a solution of our problem, since $z\to\infty$ as $x\to-\infty$ and consequently this solution does not satisfy the boundary condition at infinity. Another form of solution must be sought. For this purpose Dorodnitsyn uses the same series, but with different parameters:

$$y_2=(-x)^\beta\cdot F\left(\alpha,\alpha-\gamma+1,\alpha-\beta+1,\frac{1}{x}\right). \tag{40}$$

In our case $\beta=-\alpha$, $\gamma=1$. As a result

$$\tau_\nu=(A_\nu+iB_\nu)\cdot\left(-\frac{1}{x}\right)^\alpha\cdot F\left(\alpha,\alpha,2\alpha+1,\frac{1}{x}\right), \tag{41}$$

where A_ν, B_ν are arbitrary constants, to be determined from the boundary conditions, and F is the hypergeometric function.

Consequently the sought solution is determined by the parameter α. However, this series converges only for $|x|>1$, and the boundary condition was formulated at $z=0$, i.e., for $x=-\varepsilon$, where ε is a very small magnitude. Thus the obtained solution will not converge for $x=-\varepsilon$.

Dorodnitsyn constructed an analytic continuation of the hypergeometric series $F\left(\alpha,\alpha,2\alpha+1,\frac{1}{x}\right)$ into the region $|x|<1$, and found the solution of equation (37) in the form

$$\tau_\nu = (A_\nu + iB_\nu)(\theta_{1,\nu} + i\theta_{2,\nu}). \tag{42}$$

The previously obtained solution (31) may be taken in the form

$$\tau^*_\nu = (C_\nu + iD_\nu)(\theta^*_{1,\nu} + i\theta^*_{2,\nu}). \tag{43}$$

Separating in (42) and (43) the real and the imaginary parts we obtain

$$\left.\begin{aligned} \tau_{1,\nu} &= A_\nu \theta_{1,\nu} - B_\nu \theta_{2,\nu}, \\ \tau_{2,\nu} &= A_\nu \theta_{2,\nu} + B_\nu \theta_{1,\nu}, \\ \tau^*_{1,\nu} &= C_\nu \theta^*_{1,\nu} - D_\nu \theta^*_{2,\nu}, \\ \tau^*_{2,\nu} &= C_\nu \theta^*_{2,\nu} + D_\nu \theta^*_{1,\nu}. \end{aligned}\right\} \tag{44}$$

Substituting (44) in the boundary conditions we find after some transformations the equations

$$\left.\begin{aligned} (1-\Gamma)\, a_\nu &= A_\nu \gamma_{1,\nu} - B_\nu \gamma_{2,\nu}, \\ (1-\Gamma)\, b_\nu &= A_\nu \gamma_{2,\nu} + B_\nu \gamma_{1,\nu}, \end{aligned}\right\} \tag{45}$$

where

$$\left.\begin{aligned} \gamma_{1,\nu} &= \mu\theta_{1,\nu} - K\theta'_{1,\nu} + K^*(\theta_{1,\nu} + \theta_{2,\nu}) \cdot \sqrt{\frac{\omega\nu}{2\lambda^*}}, \\ \gamma_{2,\nu} &= \mu\theta_{2,\nu} - K\theta'_{2,\nu} - K^*(\theta_{1,\nu} - \theta_{2,\nu}) \cdot \sqrt{\frac{\omega\nu}{2\lambda^*}} \end{aligned}\right\} \tag{46}$$

Hence

$$\left.\begin{aligned} A_\nu &= (1-\Gamma)\frac{a_\nu\gamma_{1,\nu} + b_\nu\gamma_{2,\nu}}{\gamma^2_{1,\nu} + \gamma^2_{2,\nu}}, \\ B_\nu &= (1-\Gamma)\frac{b_\nu\gamma_{1,\nu} - a_\nu\gamma_{2,\nu}}{\gamma^2_{1,\nu} + \gamma^2_{2,\nu}}. \end{aligned}\right\} \tag{47}$$

For the illustration of the obtained solution Dorodnitsyn calculated the theoretical 24 hour temperature cycle in Slutsk for a clear, cloudless weather on 13 July 1932. The values of W were obtained by the actinograph. The following parameter values were used:

$\lambda_0 = 8.1\ m^2/sec$, $K = 0.5 \cdot 10^{-5}$ kcal/m sec deg,
$m = 0.02\ m^{-1}$ $\lambda^* = 0.9 \cdot 10^{-6}\ m^2/sec$,
$K^* = 3 \cdot 10^{-4}$ kcal/m sec deg,
$\Gamma = 0.1$

The results of the calculations agreed very well with the observations, as may be seen from Figure 160.

§5. Theoretical Method of Precalculation of the Diurnal Temperature Variation

M. E. Shvets has investigated the problem of the 24 hour variation of the temperature both in the air and in the underlying surface, taking into consideration the radiative heat exchange in the atmosphere. By successive approximations he jointly solved the equation of heat balance and the equation of radiative energy transfer, on the basis of very general assumptions.

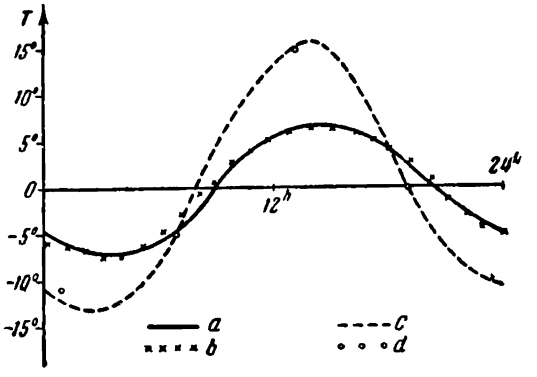

Figure 160. 24 hour temperature variation according to Dorodnitsyn:

a - calculated temperature at 2 m altitude, b - observed temperature at 2 m altitude, c - calculated soil temperature, d - observed soil temperature.

Shvets starts his study of periodic temperature variations from the equation of heat balance

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\lambda\frac{\partial T}{\partial z} + \frac{a\rho_w}{c_p \cdot \rho}(A + B + \beta I - 2fE), \qquad (1)$$

combining it with the equations of radiant energy transfer

$$\frac{\partial A}{\partial z} = a\rho_w(A - fE), \qquad (2)$$

$$\frac{\partial B}{\partial z} = a\rho_w(fE - B), \qquad (3)$$

$$\frac{\partial I}{\partial z} = -a\beta\rho_w \sec\theta, \qquad (4)$$

where I is the solar short wave radiation, and θ - the zenith distance of the sun.

Since heat exchange takes place constantly between the air and the underlying surface, in addition to the effects described by the equations, the temperature distribution in the soil should be considered as well

$$\frac{\partial T^*}{\partial t} = \frac{\partial}{\partial z}\lambda^*\frac{\partial T}{\partial z} \quad \text{for} \quad z < 0, \qquad (5)$$

where the quantities referring to the underlying surface (soil or water) are distinguished by an asterisk.

Moisture is also exchanged between the underlying surface and the air, as manifested in the evaporation of water from the surface, or in vapor condensation on it. Therefore it is necessary to add the equation of moisture transport to the foregoing equations.

Denoting the specific humidity by Q and assuming that the horizontal eddy exchange of moisture may be ignored, we obtain the equation of eddy transport of moisture in the following form:

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial Q}{\partial z}\right). \qquad (6)$$

Let us formulate the relevant boundary conditions. For the upper boundary

of the atmosphere we adopt:

$$\left.\begin{aligned} A &= 0 \quad \text{for} \quad z = \infty, \\ I &= W(t) \quad \text{for} \quad z = \infty. \end{aligned}\right\} \tag{7}$$

At the underlying surface the temperature is assumed to have a discontinuity:

$$T(0, t) = T^*(0, t), \tag{8}$$

and also

$$B(0, t) = \delta E(0, t), \tag{9}$$

where δ is the "grayness factor", accounting for the difference between the emissivity of the surface and that of a black body.

At the underlying surface the condition of heat balance should be fulfilled too. It is expressed by

$$-k\frac{\partial T}{\partial z} + k^*\frac{\partial T^*}{\partial z} + Lm = A - B + (1 - \Gamma)I, \tag{10}$$

where m is the amount of water evaporating from a surface unit, equal to the rate of evaporation $m = -D\rho\frac{\partial Q}{\partial z}$, L is the latent heat of evaporation, Γ – the albedo of the underlying surface.

Finally, it is still necessary to formulate the boundary condition for Q at z = 0, or for the rate of evaporation. The rate of evaporation from the soil surface depends on many factors, most of them highly variable, which cannot be taken into consideration in the general case. However, when the evaporation occurs from a water surface, the boundary condition for the moisture has a simple form. It is natural to assume that close to the water surface the air is permanently saturated:

$$Q(0, t) = Q_{max}, \tag{11}$$

In order to evaluate the relative effect of evaporation in tne heat balance, M. E. Shvets adopts the condition of complete saturation for the soil surface too, noting, however, that the rate of evaporation from dry soil is considerably smaller than from the surface of very damp soil.

All elements of heat balance of the atmosphere come well into play during a 24 hour course. The classical example is the 24 hour variation of direct insolation. In order to find the most general expression for this diurnal variation, Shvets proceeds in the following manner.

Neglecting for the sake of simplicity the changes of the distance of the earth from the sun the amount of energy incident on a unit horizontal surface per unit time in the absence of atmosphere is given by the formula

$$I = I_0 \sin\theta,$$

where I_0 is the solar constant. Using the relationship known from spherical trigonometry

$$\sin\theta = \sin\varphi \cdot \sin\delta + \cos\varphi \cdot \cos\delta \cdot \cos\omega t,$$

we obtain

$$I = I_0 (\sin\varphi \cdot \sin\delta + \cos\varphi \cdot \cos\delta \cdot \cos\omega t), \tag{12}$$

where φ is the latitude of the locality, δ the declination of the sun, ω the angular velocity of the earth.

Taking δ as invariable over 24 hours and assuming

$$\left.\begin{aligned} M &= I_0 \sin\varphi \cdot \sin\delta, \\ N &= I_0 \cos\varphi \cdot \cos\delta, \end{aligned}\right\} \tag{13}$$

we obtain

$$I = M + N \cdot \cos\omega t. \tag{14}$$

Formula (14) is applied only in the case of those values ωt for which $I \geqslant 0$, i. e., between the moments of sunrise and sunset, when $M + N \cos \omega t \neq 0$, whereby the negative argument, $-\omega t_0$ refers to the sunrise, and the positive $+\omega t_0$ to sunset. For the remaining time I = 0.

M. E. Shvets expresses both parts of the curve of the 24 hour variation of direct solar radiation by the same continuous function, in the form of a Fourier expansion:

$$I(t) = \sum_{-\infty}^{+\infty} j_n e^{-in\omega t}; \tag{15}$$

whereby he believes the first four terms to suffice for practical purposes. The first four coefficients of the series (15) are the following:

$$\left.\begin{aligned} j_0 &= \frac{N}{\pi} [\sin\omega t_0 - \omega t_0 \cos\omega t_0], \\ J_1 &= \frac{N}{\pi} \left[\omega t_0 - \frac{\sin 2\omega t_0}{2}\right], \\ j_2 &= \frac{N}{3\pi} \sin^3 \omega t_0, \\ j_3 &= \frac{N}{3\pi} \sin^3 \omega t_0 \cos\omega t_0. \end{aligned}\right\} \tag{16}$$

The attenuation of solar radiation by the atmosphere may also easily be taken into account. Integrating equation (4) for the boundary condition (7) we obtain:

$$I(z, t) = W(t) \cdot e^{-\sec\theta \int_z^\infty a\rho_\omega \, dz} \tag{17}$$

Consequently, in the presence of atmosphere the following amount of solar radiation falls on the earth's surface:

$$I(0, t) = W(t) \cdot e^{-\sec\theta \int_0^\infty a\rho_{t0} \, dz} \tag{18}$$

or, when a coefficient of transmittance $p = e^{-\int_0^\infty a\rho_{t0} \, dz}$ is introduced

$$I(0, t) = W(t) \cdot P^{\sec\theta} = \sum_{-\infty}^{+\infty} j_k e^{-ik\omega t}. \tag{19}$$

Shvets computed detailed tables of the values of three coefficients of the series (19) for various φ, δ and P.

Thus the diurnal variation of short-wave insolational heat influx can be represented by Fourier series

$$I(z, t) = \bar{I}(z) + \sum j_n e^{-i_n \omega t} = \bar{I}(z) + J(z, t), \qquad (20)$$

where $\bar{I}(z)$ is the 24 hour mean heat influx.

It is natural to look for a solution of the system (1)–(6) in the form

$$\left.\begin{aligned} T = \bar{T}(z) + \tau(z, t); \quad T^* = \bar{T}^*_{(z)} + \tau^*(z, t); \\ E = \bar{E}(z) + e(z, t), \\ A = \bar{A}(z) + a(z, t); \quad B = \bar{B}(z) + b(z, t); \\ Q = \bar{Q}(z) + q(z, t). \end{aligned}\right\} \qquad (21)$$

Substituting (21) in equations (1)–(6), we obtain two systems of equations. The first system will involve the 24 hour average of the sought functions. Shvets is not interested in this system, assuming that the 24 hour average values can be determined by other methods, such as Kibel's method (Chapter XX). The second system involves the unknown deviations of the sought functions from their 24 hour mean values and has the form

$$\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial z} \lambda \frac{\partial \tau}{\partial z} + \frac{\alpha \rho_w}{c_p \rho} (a + b + \beta j - 2fe), \qquad (1')$$

$$\frac{\partial a}{\partial z} = \alpha \rho_w (a - fe), \qquad (2')$$

$$\frac{\partial b}{\partial z} = \alpha \rho_w (fe - b), \qquad (3')$$

$$\frac{\partial \tau^*}{\partial t} = \frac{\partial}{\partial z} \lambda^* \frac{\partial \tau^*}{\partial z}, \qquad (5')$$

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial z} D \frac{\partial q}{\partial z}. \qquad (6')$$

Shvets changes the boundary conditions, too, in a suitable manner. Since complete saturation was assumed to prevail over the earth's surface and the density of saturated water vapor depends only on the temperature through Magnus' formula, the boundary condition (11) may be taken in the form

$$Q(0, t) = \bar{Q}(0) + q(0, t) = \frac{0.622 \times 6.10}{p_0} \cdot 10^{\frac{7.45 (T - 273)}{T - 38}},$$

where p_0 is the air pressure at the earth's surface. Remembering that $T = \bar{T} + \tau$ we obtain:

$$\bar{Q}(0) + q(0, t) \approx \frac{0.622 \times 6.10}{p_0} \cdot 10^{\frac{7.45 (\bar{T} - 273)}{\bar{T} - 38}} \cdot 10^{\frac{7.45}{\bar{T} - 38} \tau} =$$

$$= \bar{Q}_{\max}(0) \left[1 + \frac{7.45}{\bar{T} - 38} \ln 10 \cdot \tau(0, t)\right].$$

Thus the boundary condition for q(0, t) may be taken in the following form to a sufficient degree of accuracy:

$$q(0, t) = \nu \cdot \tau(0, t), \qquad (11')$$

where

$$\nu = \frac{7.45 \cdot \ln 10}{\bar{T} - 38} \cdot \bar{Q}_{max}(0).$$

The remaining boundary conditions are obvious

$$\left.\begin{aligned} a &= 0 \quad \text{at} \quad z = \infty, \\ j &= w\ (t) \quad \text{at} \quad z = \infty, \end{aligned}\right\} \tag{7'}$$

$$\tau = \tau^* \quad \text{at} \quad z = 0, \tag{8'}$$

$$b = \delta e \quad \text{at} \quad z = 0, \tag{9'}$$

$$-K\frac{\partial \tau}{\partial z} + K^*\frac{\partial \tau^*}{\partial z} - LD\rho\frac{\partial q}{\partial z} = a - b + (1 - \Gamma) j. \tag{10'}$$

Finally, a natural boundary condition is the effacement of all temperature fluctuations at infinity

$$\left.\begin{aligned} \tau &= 0 \quad \text{at} \quad z = \infty, \\ \tau^* &= 0 \quad \text{at} \quad z = -\infty. \end{aligned}\right\} \tag{22}$$

Shvets regards $\frac{a\rho_w}{c_p\rho} = \varepsilon$ as a very small constant magnitude. Therefore he looks for the solution of the system (1') - (6') in the form of an infinite series in powers of the small parameter ε

$$\left.\begin{aligned} \tau &= \tau_0(z, t) + \varepsilon \cdot \tau_1(z, t) + \ldots \\ \tau^* &= \tau_0^*(z, t) + \varepsilon \cdot \tau_1^*(z, t) + \ldots \\ a &= a_0(z, t) + \varepsilon \cdot a_1(z, t) + \ldots \\ b &= b_0(z, t) + \varepsilon \cdot b_1(z, t) + \ldots \\ q &= q_0(z, t) + \varepsilon \cdot q_1(z, t) + \ldots \end{aligned}\right\} \tag{23}$$

Substituting the series (23) in the system of equations (1') - (6') and in the boundary conditions and equating the coefficients of equal powers of ε, we obtain for τ_0, τ_0^* etc, the following system of equations

$$\frac{\partial \tau_0}{\partial t} = \frac{\partial}{\partial z}\lambda\frac{\partial \tau_0}{\partial z}, \tag{24}$$

$$\frac{\partial a_0}{\partial z} = 0, \tag{25}$$

$$\frac{\partial b_0}{\partial z} = 0, \tag{26}$$

$$\frac{\partial \tau_0^*}{\partial t} = \frac{\partial}{\partial z}\lambda^*\frac{\partial \tau_0^*}{\partial z}, \tag{27}$$

$$\frac{\partial q_0}{\partial t} = \frac{\partial}{\partial z} D \frac{\partial q}{\partial z}, \tag{28}$$

and the following boundary conditions:

$$-K\frac{\partial T_0}{\partial z} + K^*\frac{\partial T_0^*}{\partial z} - L\rho D\frac{\partial q_0}{\partial z} = a_0 - b_0 + (1 - \Gamma)\, j \quad \text{at} \quad z = 0, \tag{29}$$

$$\tau_0 = \tau_0^* \qquad \text{at} \quad z=0, \tag{30}$$

$$b_0 = \delta e_0 \qquad \text{at} \quad z=0, \tag{31}$$

$$q_0 = \nu\tau_0 \qquad \text{at} \quad z=0, \tag{32}$$

$$\tau_0 = a_0 = q_0 = 0 \quad \text{at} \quad z=\infty, \tag{33}$$

$$j = w(t) \qquad \text{at} \quad z=\infty, \tag{34}$$

$$\tau_0^* = 0 \qquad \text{at} \quad z=-\infty. \tag{35}$$

Exactly the same procedure for τ_1, τ_1^*, , a_1, etc yields:

$$\frac{\partial \tau_1}{\partial t} = \frac{\partial}{\partial z}\lambda\frac{\partial \tau_1}{\partial z} - (a_0 + b_0 - \beta j_1 - 2fe_0), \tag{36}$$

$$\frac{\partial \tau_1^*}{\partial t} = \frac{\partial}{\partial z}\lambda^*\frac{\partial \tau^*}{\partial z}, \tag{37}$$

$$\frac{\partial a_1}{\partial z} = a_0 - fe_0, \tag{38}$$

$$\frac{\partial b_1}{\partial z} = fe_0 - b_0, \tag{39}$$

$$\frac{\partial q_1}{\partial z} = \frac{\partial}{\partial z} D\frac{\partial q_1}{\partial z}, \tag{40}$$

$$-K\frac{\partial \tau_1}{\partial z} + K^*\frac{\partial \tau_1^*}{\partial z} - L\rho D\frac{\partial q_1}{\partial z} = a_1 - b_1 \quad \text{at} \quad z=0, \tag{41}$$

$$\tau_1 = \tau_1^* \qquad \text{at} \quad z=0, \tag{42}$$

$$b_1 = \delta\mu\tau_1 \qquad \text{at} \quad z=0, \tag{43}$$

$$q_1 = \nu\tau_1 \qquad \text{at} \quad z=0, \tag{44}$$

$$\tau_1 = a_1 = q_1 = 0 \quad \text{at} \quad z=\infty, \tag{45}$$

$$\tau_1^* = 0 \qquad \text{at} \quad z=-\infty. \tag{46}$$

The solution of equations (24), (27) and (28) is sought in the form of infinite series

$$\left.\begin{aligned} \tau_0 &= \sum_{-\infty}^{+\infty} \tau_{0n}(z)\cdot e^{-in\omega t}, \\ \tau_0^* &= \sum_{-\infty}^{+\infty} \tau_{0n}^*(z)\cdot e^{-in\omega t}, \\ q_0 &= \sum_{-\infty}^{+\infty} q_{0n}(z)\cdot e^{-in\omega t}. \end{aligned}\right\} \tag{47}$$

The substitution of (47) in equations (24), (27) and (28) gives the following equations for determination of τ_{0n}, τ_{0n}^*, q_{0n}:

$$\frac{d}{dz}\lambda\frac{d\tau_{0n}}{dz} + in\omega\tau_{0n} = 0, \tag{48}$$

$$\frac{d}{dz}\lambda^*\frac{d\tau_{0n}^*}{dz} + in\omega\tau_{0n}^* = 0, \tag{49}$$

$$\frac{d}{dz}D\frac{dq_{0n}}{dz} + in\omega q_{0n} = 0. \tag{50}$$

Integrating (25) for the boundary condition (33) we obtain

$$a_0 = 0. \tag{51}$$

Noting that

$$e_0 \approx 4\sigma T^3 \cdot \tau_0 = \mu \cdot \tau_0, \tag{52}$$

we obtain, after integration of (26) for the boundary condition (31):

$$b_0 = \delta\mu \sum_{-\infty}^{+\infty} \tau_{0n}(0)\, e^{-in\omega t}. \tag{53}$$

The boundary conditions for τ_{0n}, τ^*_{0n} and q_{0n} assume the following form

$$\delta\mu\tau_{0n} + K^* \frac{\partial \tau^*_{0n}}{\partial z} - K \frac{\partial \tau_{0n}}{\partial z} - zD\rho \frac{\partial q_n}{\partial z} = (1-\Gamma) j_n$$

$$\text{at} \quad z = 0, \tag{54}$$

$$\tau_{0n} = \tau^*_{0n} \qquad \text{at} \quad z = 0, \tag{55}$$

$$q_{0n} = \nu\tau_{0n} \qquad \text{at} \quad z = 0, \tag{56}$$

$$\tau_{0n} = q_{0n} = 0 \quad \text{at} \quad z = \infty, \tag{57}$$

$$\tau^*_{0n} = 0 \qquad \text{at} \quad z = -\infty. \tag{58}$$

Shvets believes that the coefficient of heat conductivity of the soil λ^* is independent of the soil depth, and the coefficient of eddy heat conductivity λ equals the coefficient of eddy diffusion and changes with altitude according to the law

$$\lambda = \lambda_0 + cz \quad \text{at} \quad z \leqslant h,$$

$$\lambda = \lambda_0 + ch \approx ch = \text{const.} \quad \text{at} \quad z > h,$$

where λ_0 is the coefficient of molecular heat conductivity.

The solution of equation (49) for the boundary condition (58) will be the following:

$$\tau^*_{0n} = B_n \cdot e^{\sqrt{-\frac{in\omega}{\lambda^*}} \cdot z}, \tag{59}$$

where B_n is an integration constant.

By making the substitution

$$\frac{z}{c}\sqrt{in\omega\lambda} = x; \qquad \frac{dx}{dz} = \frac{z}{c} \cdot in\omega x^{-1} \tag{60}$$

equations (48) and (50) are brought to the form

$$\frac{d}{dx} x \frac{d\tau_{0n}}{dx} + x\tau_{0n} = 0, \tag{61}$$

$$\frac{d}{dx} x \frac{dq_{0n}}{dx} + xq_{0n} = 0. \tag{62}$$

The integrals of these equations are expressed by the Bessel and Neumann functions:

$$\left.\begin{aligned}\tau_{0n} &= R_n \cdot I_0(x) + S_n \cdot N_0(x) = Z_0(x),\\ q_{0n} &= \tilde{R}_n \cdot I_\epsilon(x) + \tilde{S}_n \cdot N_0(x) = \tilde{Z}_0(x)\end{aligned}\right\} \text{ - at } \quad z \geqslant h, \tag{63}$$

where R_n, S_n, $\tilde{R}_n$, $\tilde{S}_n$ are constants of integration; I_0 — the Bessel function of zero order, N_0 — the Neumann function of zero order, Z_0 — a cylinder function.

For $z \geqslant h$

$$\left.\begin{aligned}\tau_{0n} &= A_n \cdot e^{-\sqrt{-\frac{in\omega}{\lambda_0 + ch}} \cdot z},\\ q_{0n} &= \tilde{A}_n \cdot e^{-\sqrt{-\frac{in\omega}{\lambda_0 + ch}} \cdot z},\end{aligned}\right\} \tag{64}$$

where A_n and $\tilde{A}_n$ — are constants of integration.

Besides the boundary conditions (41) - (46) the obtained solutions should satisfy also the conditions of conjugation:

$$\left.\begin{aligned}&\tau_{0,n}(h-0) = \tau_{0,n}(h+0),\\ &q_{0,n}(h-0) = q_{0,n}(h+0),\\ &\frac{\partial \tau_{0,n}(h-0)}{\partial z} = \frac{\partial \tau_{0,n}(h+0)}{\partial z},\\ &\frac{\partial q_{0,n}(h-0)}{\partial z} = \frac{\partial q_{0,n}(h+0)}{\partial z}.\end{aligned}\right\} \tag{65}$$

Fulfilling the boundary condition (55) we obtain:

$$B_n = R_n \cdot I_n(x_0) + S_n \cdot N_0(x_0). \tag{66}$$

We now eliminate from the boundary condition (54) the temperature of the underlying surface τ^*. According to (59),

$$K^* \frac{\partial \tau_{0n}^*}{\partial z} = c^* \rho^* \sqrt{-in\omega\lambda^*} \cdot B_n = c^* \rho^* \sqrt{-in\omega\lambda^*} \cdot \tau_{0,n}(0). \tag{67}$$

We further notice that, according to (32),

$$\tilde{R}_n = \nu R_n; \quad \tilde{S}_n = \nu S_n; \tag{68}$$

therefore

$$q_{0n} = \nu \tau_{0n} \tag{69}$$

at every point of space. As a result, the equation of heat balance takes the form

$$-\lambda \rho (c_p + L\nu) \frac{d\tau_{0n}}{dz} + (\rho^* c^* \sqrt{-in\omega\lambda^*} + \delta\mu)\tau_{0n} = (1-\Gamma) J_n. \tag{70}$$

Using the equation of heat balance in the form (70), Shvets evaluates the effects of various factors influencing the incoming and outgoing flux of heat. For example:

$\delta \cdot \mu = 1.2 \cdot 10^{-4}$ cal/deg cm^{2}sec; $\rho^* = 1.5$ g/cm^3
$c^* = 0.33$ cal/g deg ; $\lambda^* = 7 \cdot 10^{-3}$ cm^2/sec;
$c_p = 0.24$ cal/g deg ; $n = 1$; $\nu z = 0.3$ cal/g deg.

(this corresponds to a vapor pressure of 10 mm Hg).

For these values of the parameters the heat absorption by the soil is roughly three times greater than the heat loss by radiation. The heat spent on vaporization is smaller or of the same order as the portion of the eddy flux of heat which is absorbed in the air.

Further, for calm water $\lambda^* = 1.8 \cdot 10^{-2}$ cm^2/sec; consequently for water $\rho^* c^* \sqrt{\lambda^*} = 0.135$; and for dry soil, 0.19. Thus the 24 hour temperature variation over ideally calm water will be roughly the same as over dry soil.

However, the turbulent mixing which takes place in flowing water substantially changes the relationship between the elements of the heat balance. The coefficient of eddy heat conductivity of water is several thousand times greater than the molecular conductivity. Therefore, for the sea the expression $\rho^* c^* \sqrt{-in\omega\lambda^*} \cdot \tau_{0n}$ in equation (70) is hundreds of times greater than the remaining terms on the left side of this equation. This means that the solar radiation is spent mainly on the heating of the sea and only a negligible part of it on the heating of the atmosphere.

Omitting the very small terms in the equation of heat balance for a sea surface, we can write this equation in the form:

$$\tau_{0n} = \frac{(1-\Gamma) j_n}{\sqrt{-in\omega\lambda^*}}. \tag{71}$$

We now insert the values of τ_{0n} from (63) in the equation of heat balance (70), using thereby the following relationship, borrowed from the theory of cylinder functions:

$$\frac{dZ_0}{dx} = -Z_1.$$

Then after obvious transformations, the equation of heat balance will assume the following form

$$(1-\Gamma) j_n = R_n X_n + S_n Y_n, \tag{72}$$

where

$$\left.\begin{aligned} X_n &= [\delta\mu + c^*\rho^* \sqrt{\tfrac{n\omega\lambda^*}{2}} (1-i)] I_0(x_0) + \\ &\quad + \rho (c_p + \nu L) \cdot \sqrt{\tfrac{n\omega\lambda_0}{2}} (1+i) I_1(x_0), \\ Y_n &= [\delta\mu + c^*\rho^* \sqrt{\tfrac{n\omega\lambda^*}{2}} (1-i)] N_0(x_0) + \\ &\quad + \rho (c_p + \nu L) \cdot \sqrt{\tfrac{n\omega\lambda_0}{2}} (1+i) N_1(x_0). \end{aligned}\right\} \tag{73}$$

The conditions of conjugation (65) give the following two equations

$$\left.\begin{aligned} &A_n e^{-\sqrt{-\frac{in\omega}{\lambda_0 + ch}} \cdot h} = R_n I_0(x_h) + S_n N_0(x_h), \\ &-\sqrt{-\frac{in\omega}{\lambda_0 + ch}} \cdot A_n \cdot e^{-\sqrt{-\frac{in\omega}{\lambda_0 + ch}} \cdot h} = \\ &= -\sqrt{-\frac{in\omega}{\lambda_0 + ch}} [R_n I_1(x_h) + S_n N_1(x_h)]. \end{aligned}\right\} \tag{74}$$

Eliminating A_n from (74), we obtain

$$R_n [I_0(x_h) - iI_1(x_h)] + S_n [N_0(x_h) - iN_1(x_h)] = 0. \tag{75}$$

Equations (72) and (75) allow to determine the constants of integration

$$\left.\begin{aligned}\frac{R_n}{(1-\Gamma)f_n} &= -\frac{N_0(x_h)-iN_1(x_h)}{[I_0(x_h)-iI_1(x_h)]\cdot Y_n-[N_0(x_h)-iN_1(x_h)]X_n},\\ \frac{S_n}{(1-\Gamma)f_n} &= -\frac{I_0(x_h)-iI_1(x_h)}{[I_0(x_h)-iI_1(x_h)]Y_n-[N_0(x_h)-iN_1(x_h)]X_n},\end{aligned}\right\} \tag{76}$$

and for z > h

$$\tau_n=\tau_n(h)\,e^{-\sqrt{-\frac{in\omega}{\lambda_0+ch}}(z-h)}. \tag{77}$$

We now turn to the determination of the functions τ, a, b, which depend on the radiative exchange in the atmosphere. Substituting the obtained values τ_0, a_0, b_0, in equations (36) – (40) and in the boundary conditions (41) – (46), we obtain:

$$\frac{\partial\tau_1}{\partial t}=\frac{\partial}{\partial z}\lambda\frac{\partial\tau_1}{\partial z}+\sum_{-\infty}^{+\infty}e^{-in\omega t}[\delta\mu\tau_{0n}(0)+\beta J_n(0)-2\mu f\tau_{0n}(z)], \tag{78}$$

$$\frac{\partial a_1}{\delta z}=-c_p\rho f\mu\sum_{-\infty}^{+\infty}\tau_{0n}(z)\,e^{-in\omega t}, \tag{79}$$

$$\frac{\partial b_1}{\partial z}=c_p\rho\mu\delta\sum_{-\infty}^{+\infty}[\tau_{0n}(z)-\tau_{0n}(0)]\,e^{-in\omega t}, \tag{80}$$

$$\frac{\partial\tau_1^*}{\partial t}=\frac{\partial}{\partial z}\lambda^*\frac{\partial\tau_1^*}{\partial z}, \tag{81}$$

$$\frac{\partial q_1}{\partial t}=\frac{\partial}{\partial z}\lambda\frac{\partial q_1}{\partial z}, \tag{82}$$

$$\left.\begin{aligned}-K\frac{\partial\tau_1}{\partial z}+K^*\frac{\partial\tau_1^*}{\partial z}-L\rho\lambda\frac{\partial q_1}{\partial z}&=a_1-b_1 &(83)\\ b_1&=\delta\mu\tau_1, &(84)\\ \tau_1&=\tau_1^*, &(85)\\ q_1&=\nu\tau_1, &(86)\end{aligned}\right\}\ \text{at}\quad z=0,$$

$$\tau_1=a_1=q_1=0\quad\text{at}\quad z=\infty, \tag{87}$$

$$\tau^*=0\quad\text{at}\quad z=-\infty. \tag{88}$$

Integrating (79) and taking into consideration the boundary condition (87), we obtain:

$$a_1=-c_p\rho f\mu\sum_{-\infty}^{+\infty}e^{-in\omega t}\int_{\infty}^{z}\tau_{0n}dz.$$

But according to (48) we have

$$\int_{-\infty}^{z}\tau_{0n}dz=-\frac{1}{in\omega}\int_{\infty}^{z}\frac{\partial}{\partial z}\lambda\frac{\partial\tau_{0n}}{\partial z}dz=-\frac{\lambda}{in\omega}\frac{\partial\tau_{0n}}{\partial z}.$$

consequently

$$a_1=\frac{Kf\mu}{i\omega}\sum_{-\infty}^{+\infty}\frac{1}{n}\frac{\partial\tau_{0n}}{\partial z}\cdot e^{-in\omega t}. \tag{89}$$

We now rewrite the boundary condition (83) in the following manner:

$$-K\frac{\partial\tau_1}{\partial z}+K^*\frac{\partial\tau^*}{\partial z}-L\rho\lambda\frac{\partial q_1}{\partial z}+\delta\mu\tau_1=\frac{Kf\mu}{i\omega}\sum\frac{1}{n}\frac{\partial\tau_{0n}}{\partial z}e^{-in\omega t}, \tag{90}$$

and we seek a solution of equations (78), (81) and (82) in tne form of infinite series

$$\left.\begin{aligned}\tau_1 &= \sum_{-\infty}^{+\infty} \tau_{1n}(z)\cdot e^{-in\omega t},\\ \tau_1^* &= \sum_{-\infty}^{+\infty} \tau_{1n}^*(z)\cdot e^{-in\omega t},\\ q_1 &= \sum_{-\infty}^{+\infty} q_{1n}(z)\, e^{-in\omega t}.\end{aligned}\right\} \tag{91}$$

For determination of τ_{1n}, τ^*_{1n}, and q_{1n} we obtain the following equations:

$$\frac{d}{dz}\lambda\frac{d\tau_{1n}}{dz} - in\omega\tau_{1n} = \delta\mu\tau_{0n}(0) + \beta J_n(0) - 2\mu f\tau_{0n}(z), \tag{92}$$

$$\frac{d}{dz}\lambda^*\frac{d\tau^*_{1n}}{dz} + in\omega\tau^*_{1n} = 0, \tag{93}$$

$$\frac{d}{dz}\lambda\frac{dq_{1n}}{dz} + in\omega q_{1n} = 0, \tag{94}$$

$$-K\frac{\partial\tau_{1n}}{\partial z} + K^*\frac{\partial\tau^*_{1n}}{\partial z} + \delta\mu\tau_{1n} - L\rho\lambda\frac{\partial q_{1n}}{\partial z} = \frac{Kf\mu}{in\omega}\frac{\partial\tau_{0n}}{\partial z} \quad \text{at} \quad z > 0, \tag{95}$$

$$\tau_{1n} = \tau^*_{1n}, \tag{96}$$

$$q_{1n} = \nu\tau_{1n}, \tag{97}$$

$$\tau_{1n} = q_{1n} = 0 \quad \text{at} \quad z = \infty, \tag{98}$$

$$\tau^*_{1n} = 0 \quad \text{at} \quad z = -\infty. \tag{99}$$

Again we change the variable in equation (92):

$$\frac{z}{c}\sqrt{in\omega\lambda} = x.$$

This equation then assumes the form

$$\frac{d}{dx}x\frac{d\tau_{1n}}{dx} + x\tau_{1n} = x\left[\delta\mu\tau_{0n}(x_0) + \beta J_n(x_0) - 2\mu f\tau_{1n}(x_0)\right]\frac{1}{in\omega}. \tag{100}$$

A particular solution of the inhomogeneous equation (100) is

$$\tau_{1n} = \frac{1}{in\omega}\left[\delta\mu\tau_{0n}(x_0 + \beta J_n(x_0) + \mu f x\frac{\partial\tau_{0n}}{\partial x}\right], \tag{101}$$

this may be directly verified. The general solution of equation (100) for z < h will have the form

$$\tau_{1n} = R_{1n}I_0(x) + S_{1n}N_0(x) + \frac{1}{in\omega}\left[\delta\mu\tau_{0n}(x_0) + \beta J_n(x_0) + \mu f x\frac{d\tau_{0n}}{dx}\right], \tag{102}$$

and for z > h

$$\tau_{1n} = A_{1n}e^{-\sqrt{-\frac{in\omega}{\lambda_0+ch}}\cdot z} + \frac{1}{in\omega}\left[\delta\mu\tau_{0n}(x_0) + \beta J_n(0) + \mu f z\frac{\partial\tau_{0n}}{\partial z}\right], \tag{103}$$

where A_{1n}, R_{1n}, S_{1n} are arbitrary constants. Equations (102) and (103) can be simplified by omitting from the right sides the terms $\mu f x\frac{\partial\tau_n}{\partial x}$ and $\mu f z\frac{\partial\tau_{1n}}{\partial z}$, which are hundreds of times smaller than the remaining terms. For determination of the arbitrary constants, besides the boundary conditions, the condition of conjugation of the solutions (102) and (103) at z = h should be applied. This condition, as was

shown above, is given by the relationship

$$R_{1n}(I_0 - iI_1) + S_{1n}[N_0 - iN_1] = 0.$$

Shvets showed that the coefficients R_{1n} and S_{1n} are small in comparison with R_n and S_n. Therefore, the terms containing R_{1n} and S_{1n} may be disregarded. The solution for $z < h$ will then take the form

$$\tau_n = R_n\left[I_0(x) - i\varepsilon\frac{\delta\mu}{\omega n}I_0(x_0)\right] + S_n\left[N_0(x) - i\varepsilon\frac{\delta\mu}{\omega n}N_0(x_0)\right] - \\ - \frac{i\varepsilon\beta}{\omega n}J_n(0), \qquad (104)$$

and for $z > h$

$$\tau_n = \tau_n(h)e^{-\sqrt{-\frac{in\omega}{\lambda_0 + ch}}} - \frac{i\varepsilon}{\omega n}[\delta\mu\tau_{0n}(0) + \beta J_n(0)]. \qquad (105)$$

The expressions (104) and (105) show that the thermal radiation of air has a greater effect on the 24 hour temperature variation in summer, when the humidity (ε) and the insolation are greater.

The solution of the system (105) obtained by Shvets, also allows to solve the problem of the diurnal variation of the effective radiation.

Finally, Shvets succeeded theoretically in explaining the peculiarity of the diurnal temperature variation over the sea, which consists in the growth of the amplitude of temperature fluctuations with increasing altitude. Over the land this amplitude decreases. This peculiarity is explained by the effect of radiative heat exchange. It was shown above that in the diurnal temperature cycle over land the role of radiative heat exchange is considerably smaller than that of the turbulent heat exchange. Above the sea the effects of these processes become comparable.

Satisfying the conditions of conjugation at $z = h$ and the condition of equality of the water and air temperatures at $z = 0$, we obtain condition (57) and

$$\left(1 - \varepsilon i\frac{\delta\mu}{\omega n}\right)R_n I_0(x_0) + S_n N_0(x_0) - \varepsilon\frac{i\beta J_n}{\omega n} = \frac{(1-\Gamma)j_n}{\sqrt{2\lambda^* n\omega}}(1+i).$$

Whence:

$$\frac{R_n}{j_n} = \frac{i_n}{1 - i\varepsilon\frac{\delta\mu}{\omega n}}\left\{\frac{1-\Gamma}{\sqrt{2\lambda^* n\omega}}(1+i) + \frac{\varepsilon\cdot i\beta}{\omega n}\right\},$$

$$\frac{S_n}{j_n} = \frac{m_n}{1 - i\varepsilon\frac{\delta\mu}{\omega n}}\left\{\frac{1-\Gamma}{\sqrt{2\lambda^* n\omega}}(1+i) + \frac{\varepsilon i\beta}{\omega n}\right\},$$

where

$$j_n = \frac{N_0(x_h) - iN_1(x_h)}{[I_0(x_h) - iI_1(x_h)]N_0(x_0) - [N_0(x_h) - iN_1(x_h)]I_0(x_0)}$$

$$m_n = \frac{I_0(x_h) - iI_1(x_h)}{[I_0(x_h) - iI_1(x_h)]\cdot N_0(x_0) - [N_0(x_h) - iN_1(x_h)]\cdot I_0(x_0)}$$

Shvets gives the following example: $j_1 = 7.3 \cdot 10^{-3}$; $j_2 = 3.7 \cdot 10^{-3}$ (which corresponds to latitude 50°, the month of March and a coefficient of transmittance 0.7); $\lambda^* = 200$; $\delta\mu = 1.2 \cdot 10^{-4}$; $\Gamma = 0.1$; $\beta = 0.04$; $a = 2$; $c = 14$; $h = 4{,}000$ cm. Then

$$\tau_0 = 0.04\cos\omega t + 0.04\sin\omega t + 0.014\cos 2\omega t + 0.014\sin 2\omega t,$$

and at an altitude of 20 m:

$$\tau_{20} = -0.32 \sin \omega t - 0.08 \sin 2\omega t.$$

Summing up the results of his very detailed investigations of the 24 hour variation of temperature, Shvets draws the following significant conclusions.

The variation of air temperature depends upon a great number of physical parameters, first and foremost upon these characterizing the physical properties of the underlying surface. Shvets gives a series of tables which show convincingly the effect of these parameters on the separate purely periodic terms into which the diurnal temperature variation is resolved.

The heat spent on vaporization of moisture from the soil surface has an appreciable effect on the heat balance. However, the effect of this factor does not readily lend itself to quantitative treatment, since it depends upon the humidity and structure of the soil, varying very irregularly in time and space. The calculations of Shvets also showed, that the effect of this factor can often be ignored. Some corroboration of this conclusion is found by Shvets in the known reciprocal compensation between the heat spent on vaporization from the soil surface (which may contain different amounts of moisture) and the albedo. Thus, for instance, a very moist but bare soil loses by evaporation a considerably greater amount of heat than dry soil. On the other hand, the albedo of damp soil is considerably smaller than that of dry soil. Therefore a great loss of heat by vaporization is associated with a large influx of radiant heat, and vice versa.

The theory elaborated by Shvets also reveals the effect of heat radiation of the air on the diurnal temperature variation. Shvets gives detailed tables for the separate periodic components which make up the correction for atmospheric radiation of the diurnal temperature variation. The effect of radiation proves to be greater, the higher the air humidity and the insolational heat influx.

The theoretical results obtained by Shvets allow to calculate beforehand the temperature variation at various altitudes. Figure 161 shows side by side some observed and calculated temperature variations. It should be strongly emphasized that this remarkable coincidence of theory with observation was attained by the adjustment of only two parameters, $\frac{K^*}{\sqrt{\lambda^*}}$ and the albedo. The coefficient of exchange at 40 m altitute was assumed to be equal to 60 g/cm sec.

However, the precalculation of the diurnal temperature variation is complicated by the erratic variability of cloudiness, which distorts the temperature variation curve. Therefore the prognosis of the 24 hour variation presupposes a prognosis of cloudiness.

§ 6. Transformation of Air Masses by the Influence of the Underlying Surface

One of the most difficult and simultaneously the most important problems of contemporary meteorology is the transformation of air masses. The essence of the problem consists in the following.

According to modern meteorological conceptions, the air persisting for a long time over the same ground surface gradually acquires peculiarities characteristic of that surface, of the latitude and the season.

Among these characteristic properties the vertical temperature distribution is most important. When the air mass is present for a long time above a certain underlying surface, making but little headway, the transformation process or, in other words, the process of acquisition of new properties by the air mass admits of quantitative evaluation.

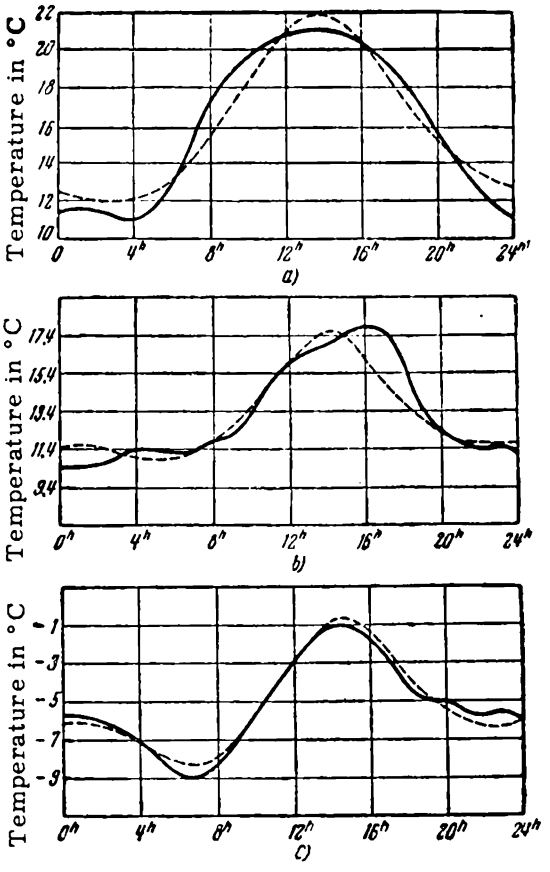

Figure 161. Comparison of diurnal temperature variations calculated by Shvets (broken line) and as recorded at the observatory "Vysokaya Dubrava" (56° N lat.):
a--9 August 1941, b-- 15 September 1941, c--16 October 1941.

The first important contributions to the transformation theory of air masses were the investigations of I. A. Kibel' and N. R. Malkin. We shall report here on the work of I. A. Kibel'.

Kibel' starts from the equation of heat balance, as he did in the study of the temperature distribution in altitude

$$c_p\rho\frac{dT}{dt}-\frac{dp}{dt}=\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)+\alpha\rho_w(A+B+\beta S-2fE). \quad (1)$$

In the latter problem he assumes the left side of equation (1) to be equal to zero. In the problem of air mass transformation it is obviously indinspensable to retain the term $c_p\rho\frac{\partial T}{\partial t}$ in the left side of this equation.

The equations for the problem are the following

$$c_p\rho\frac{\partial T}{\partial t}=\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)+\alpha\rho_w(A+B+\beta S-2fE), \quad (2)$$

$$E=\sigma T^4,$$

$$\frac{\partial A}{\partial z}=\alpha\rho_w(A-fE), \quad (3)$$

$$\frac{\partial B}{\partial z}=\alpha\rho_w(fE-B), \quad (4)$$

$$\frac{\partial S}{\partial z}=-\alpha\beta\rho_w S, \quad (5)$$

and the boundary conditions

$$-K\frac{\partial T}{\partial z}+K^*\frac{\partial T^*}{\partial z}=A+S-B;$$

$$B=qfE \quad \text{at} \quad z=0,$$

$$A=0,$$

$$B=W, \tag{6}$$

$$S=W \quad \text{or} \quad \frac{\partial T}{\partial t}=0 \quad \text{at} \quad z\to\infty. \tag{7}$$

Besides, for the soil temperature T* (a new undetermined function) the equation of heat conduction should hold

$$c^*\rho^*\frac{\partial T^*}{\partial t}=K^*\frac{\partial^2 T^*}{\partial z^2}, \tag{8}$$

where c* is the specific heat of the soil, , ρ^* – the soil density, K* – the coefficient of thermal conductivity of the soil. The boundary conditions for T* are the following:

$$\left.\begin{aligned} T^*&=T \quad \text{at} \quad z=0,\\ T^* &\text{ is bounded at } \quad z\to-\infty. \end{aligned}\right\} \tag{9}$$

Following Kibel', let us introduce instead of the altitude a a new independent variable – the normalized optical thickness x:

$$x=\frac{1}{\tau_0}\int_z^\infty \alpha\rho_w dz, \tag{10}$$

where

$$\tau_0=\int_0^\infty \alpha\rho_w dz, \tag{11}$$

and instead of the time t – the dimensionless time s:

$$s=\omega t, \tag{12}$$

where $1/\omega$ is the characteristic time.

Equation (2) then takes the form

$$\frac{c_p\rho_\omega}{\alpha\rho_w}\frac{\partial T}{\partial s}=\frac{1}{\tau_0^2}\frac{\partial}{\partial x}\left(K\alpha\rho_w\frac{\partial T}{\partial x}\right)+(A+B+\beta S-2fE). \tag{13}$$

Noting that

$$\frac{\partial E}{\partial x}=4\sigma T^3\frac{\partial T}{\partial x},$$

and passing from T to E we obtain:

$$\frac{c_p\rho_\omega}{4\alpha\rho_w f\sigma T^3}\frac{\partial E}{\partial s}=\frac{1}{\tau_0^2}\frac{\partial}{\partial x}\left(\frac{K\alpha\rho_w}{4f\sigma T^3}\frac{\partial E}{\partial x}\right)+A+B+\beta S-2fE. \tag{14}$$

Kibel' regards the coefficients of equation (14) as constant and uses the notation

$$\frac{4f\sigma T^3}{K\alpha\rho_w}=\frac{m^2-1}{2}; \quad \frac{c_p\rho_\omega}{4\alpha\rho_w f\sigma T^3}=N. \tag{15}$$

Equation (14) then assumes the form

$$N\frac{\partial E}{\partial s}=\frac{1}{\tau_0^2}\frac{\partial}{\partial x}\left(\frac{2}{m^2-1}\frac{\partial E}{\partial x}\right)+A+B+\beta S-2fE. \tag{16}$$

Let us introduce new variables a and y, defined by the equalities

$$a = A + B; \quad y = B - A. \tag{17}$$

Solving (16) with respect to a we obtain:

$$a = -\frac{1}{\tau_0^2}\frac{\partial}{\partial x}\left(\frac{2}{m^2-1}\frac{\partial E}{\partial x}\right) + N\frac{\partial E}{\partial s} - \beta S + 2fE. \tag{18}$$

Recalling that according to (4) $\frac{1}{\tau_0}\frac{da}{dx} = y$, differentiating (18) with respect to x and applying (15) we obtain:

$$y = -\frac{1}{\tau_0^3}\frac{\partial^2}{\partial x^2}\left(\frac{2}{m^2-1}\frac{\partial E}{\partial x}\right) + \frac{2}{\tau_0}\frac{\partial E}{\partial x} + \frac{1}{\tau_0}\frac{\partial}{\partial x}\left(N\frac{\partial E}{\partial s}\right) + \beta^2 S. \tag{19}$$

Noting further that

$$\frac{1}{\tau_0}\frac{dy}{dx} = a - 2fE, \tag{20}$$

we differentiate (19) too with respect to x and apply (18). We then obtain

$$\frac{1}{\tau_0^4}\frac{\partial^3}{\partial x^3}\left(\frac{2}{m^2-1}\frac{\partial E}{\partial x}\right) - \frac{1}{\tau_0^2}\frac{\partial}{\partial x}\left(\frac{2}{m^2-1}\frac{\partial E}{\partial x}\right) - \frac{2}{\tau_0^2}\frac{\partial^2 E}{\partial x^2} - \\ - \frac{1}{\tau_0^2}\frac{\partial^2}{\partial x^2}\left(N\frac{\partial E}{\partial s}\right) + N\frac{\partial E}{\partial s} = \beta(1-\beta^2)s. \tag{21}$$

The boundary conditions will have the form

$$\left.\begin{aligned} y + \frac{2}{m^2-1}\frac{1}{\tau_0}\frac{\partial E}{\partial x} + K^*\frac{\partial T^*}{\partial z} = S \quad \text{at} \quad x = 1, \\ a + y = 2qfE, \\ a - y = 0, \end{aligned}\right\} \tag{22}$$

$$a + y = 2W \quad \text{or} \quad \frac{\partial E}{\partial s} = 0 \quad \text{at} \quad x = 0. \tag{23}$$

Substituting the numerical values, Kibel' takes: T = 280°, α = 7.25 cm^2/g, ρ_w = = 6.2 · 10^{-6} g/cm^3, ρ = 1.3 · 10^{-3} g/cm^3. Whence

$$N = 10^5\,\omega.$$

When the characteristic time equals 24 hours ($1/\omega \approx 10^5$ sec), N is of the order of unity. Noting that $m^2 \approx 3$, $\tau \approx 10$, Kibel' evaluates the orders of the terms of equation (21), and reaches the following very important result: far from the boundaries (x = 0, x = 1) equation (21) may be replaced by a simpler approximate equation

$$N\frac{\partial E}{\partial s} \approx \beta(1-\beta^2)s, \tag{24}$$

and only near the boundaries is the full equation (21) really essential.

However, this conclusion is justified only in case the characteristic time is of the order of 24 hours. When the characteristic time is longer, the coefficients of the previous terms of equation (21) will have the same order as N. Equation (21) can then not be replaced by equation (24).

Kibel' indicates that in this case the boundary layer "swells", so to speak. This is confirmed by the fact that annual temperature fluctuations are appreciable much

higher than the diurnal fluctuations. The temperature changes extend to high altitudes and in the considered case express the transformation of air masses.

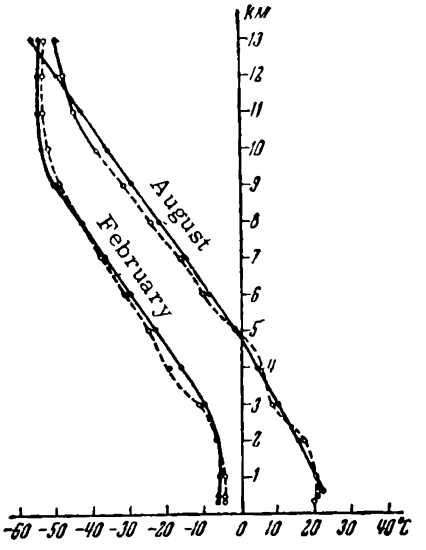

Figure 162. Actual (solid line) and calculated (broken line) change of temperature with altitude, after Kibel'

Kibel' dealt only with annual temperature variations.

Since equation (21) is linear, as are the boundary conditions, the solution of (21) may be sought in the form of an infinite trigonometric series

$$E = E_0(x) + \sum_1^\infty [E_{1n}(x)\cos ns + E_{2n}(x)\sin ns], \qquad (25)$$

where E_0 corresponds to the annual mean temperature distribution in altitude.

The direct insolation at the upper boundary of the atmosphere can also be expressed by such an infinite series

$$W = W_0 + \sum_1^\infty (W_{1n}\cos ns + W_{2n}\sin ns). \qquad (26)$$

Substituting (25) and (26) in (21) and equating the coefficients of corresponding terms, Kibel' obtained the following approximate solution of equation (21), fulfilling the boundary conditions (22) and (23):

$$E_n(x) \approx \left\{\frac{m-\beta}{1-\beta}\frac{1}{m}\frac{\operatorname{ch}[\lambda_2(1-x)+\zeta]}{\operatorname{ch}(\lambda_2+\zeta)} - \frac{m-1}{m(1-\beta)}\beta\cdot e^{-\lambda_1 x} - e^{-\beta\tau_0 x}\right\}\frac{1-\beta^2}{2\beta}\frac{m^2-1}{m^2-\beta^2} - \frac{W_n}{1-\frac{1-\beta^2}{m^2-\beta^2}\frac{m^2-1}{2\beta^2}nNi}, \qquad (27)$$

where

$$E_n = E_{1n} - iE_{2n};$$

$$W_n = W_{1n} - iW_{2n};$$

$$\lambda_1 \approx m\tau_0;$$

$$\lambda_2 \approx \sqrt{\frac{m^2-1}{2}\frac{Nni}{m^2}}\tau;$$

$$\zeta = \frac{1}{2}\ln\frac{1+M}{1-M};$$

$$M = \sqrt{\frac{m^2-1}{2m^2N}}\frac{\sqrt{\lambda^*\omega}}{4\zeta_0 T^{*4}}.$$

As a numerical example Kibel' took the following values of the parameters: $m = 1.75$, $\tau_0 = 12.6$; $\beta = 0.2$. Then $M \approx 0.45$; $N \approx 0.024$; $\lambda_1 \approx 22$; $\lambda_2 \approx 0.8\,(1 + i)$; $\zeta = 0.3$. W_n was taken by Kibel' from Milankovich's work for latitude 40°, assuming the albedo to equal 0.6.

Kibel' obtained $W_1 = 0.07\,i$; $W_2 = W_3 = 0$. He calculated the temperature distribution for February and August. Figure 162 shows the results of Kibel's calculations side by side with the observed values. The agreement is seen to be very good indeed.

APPENDIXES

Appendix 1

Essential magnitudes in the CGS and MTS systems

Designation	Dimension	Unit		Coefficient of conversion from MTS to CGS
		MTS	CGS	
Fundamental Magnitudes				
Length	[L]	1 m	1 cm	10^2
Mass	[M]	1 t	1 g	10^6
Time	[T]	1 sec	1 sec	1
Derived Magnitudes				
Area	$[L^2]$	1 m^2	1 cm^2	10^4
Volume	$[L^3]$	1 m^3	1 cm^3	10^6
Gradient (operator) Curl (operator) Divergence (operator)	$[L^{-1}]$	1 m^{-1}	1 cm^{-1}	10^{-2}
Individual, local, convective derivatives	$[T^{-1}]$	1 sec^{-1}	1 sec^{-1}	1
Density	$[ML^{-3}]$	1 t m^{-3}	1 g cm^{-3}	1
Specific volume	$[M^{-1}L^3]$	1 $t^{-1}m^3$	1 $g^{-1}cm^3$	1
Velocity	$[LT^{-1}]$	1 m sec^{-1}	1 cm sec^{-1}	10^2
Acceleration	$[LT^{-2}]$	1 m sec^{-2}	1 cm sec^{-2}	10^2
Force	$[MLT^{-2}]$	1 t m sec^{-2}	1 dyne= = 1 g cm sec^{-2}	10^8
Pressure, stress	$[ML^{-1}T^{-2}]$	1 centibar= =1 t $m^{-1}sec^{-2}$	1 bar= =1 g $cm^{-1}sec^{-2}$	10^4
Momentum Impulse	$[MLT^{-1}]$	1 t m sec^{-1}	1 g cm sec^{-1}	10^6
Work, energy enthalpy quantity of heat	$[ML^2T^{-2}]$	1 kilojoule= = 1 t m^2sec^{-2}	1 erg= =1 g cm^2sec^{-2}	10^{10}
Specific work Specific energy Specific enthalpy Specific heat quantity Geopotential	$[L^2T^{-2}]$	1 m^2sec^{-2}	1 cm^2sec^{-2}	10^4
Dissipation of mechanical energy	$[ML^{-1}T^{-3}]$	1 t $m^{-1}sec^{-3}$	1 g $cm^{-1}sec^{-3}$	10^4
Pressure gradient	$[ML^{-2}T^{-2}]$	1 t $m^{-2}sec^{-2}$	1 g $cm^{-2}sec^{-2}$	10^2
Coefficient of viscosity Exchange coefficient	$[ML^{-1}T^{-1}]$	1 t $m^{-1}sec^{-1}$	1 g $cm^{-1}sec^{-1}$	10^4
Coefficient of kinematic viscosity Coefficient of thermal conductivity Coefficient of diffusion Coefficient of turbulence	$[L^2T^{-1}]$	1 m^2sec^{-1}	1 cm^2sec^{-1}	10^4

Designation	Dimension	Unit		Coefficient of conversion from MTS to CGS
		MTS	CGS	
Angular velocity Vorticity Velocity divergence Velocity of deformation	$[T^{-1}]$	1 sec^{-1}	1 sec^{-1}	1
Stream function Velocity potential	$[L^2T^{-1}]$	1 m^2sec^{-1}	1 cm^2sec^{-1}	10^4
Temperature	$[\theta]$	1 degree	1 degree	1
Temperature gradient	$[L^{-1}\theta]$	1 m^{-1}degree	1 cm^{-1}degree	10^2
Coefficient of thermal conductivity Coefficient of eddy heat conductivity	$[LT^{-3}\theta^{-1}]$	1 m $sec^{-3}\times$ $\times degree^{-1}$	1 cm $sec^{-3}\times$ $\times degree^{-1}$	10^2
Heat capacity Entropy	$[ML^2T^{-2}\theta^{-1}]$	1 t $m^2sec^{-2}\times$ $\times degree^{-1}$ = 1 t m^2sec^{-2} $\times degree^{-1}$	1 g $cm^2sec^{-2}\times$ $\times degree^{-1}$	10^{10}
Specific heat Specific entropy Gas constant	$[L^2T^{-2}\theta^{-1}]$		1 $cm^2sec^{-2}\times$ $\times degree^{-1}$	10^4
Coefficient of absorption Coefficient of scattering	$[M^{-1}L^2]$	1 $t^{-1}m^2$	1 $g^{-1}cm^2$	10^2

Some important geodetic and astronomical constants

Dimensions of the terrestrial spheroid, accepted by the international convention, 1924	
Semi-major axis (equatorial radius)	$a = 6,378,388$ m
Semi-minor axis (polar radius)	$b = 6,356,912$ m
Oblateness	$\frac{a-b}{a} = 1/297$
Eccentricity	$\sqrt{\frac{a^2-b^2}{a^2}} = 0.0819919$
Meridional circumference	40,009,153 m
Equatorial circumference	40,076,594 m
Surface area	510,100,900 km^2
Volume	1,083,320,000,000 km^2
Radius of the sphere of equal volume	6,371,221 m
1° latitude at the equator	111,307 km
Mass of the earth	$5.98 \cdot 10^{27}$ g
Mean density of the earth	5.52 g$\cdot$cm^{-3}
Mean duration of the solar year	365 days 5 hours 48.8 minutes
Duration of mean solar day	24 h 0 min 0sec=1440 min = 86,400 sec
Duration of sidereal day	24 h–3min 55.9 sec = 23 h 56 min 4.1 sec= =86,164 sec
Angular velocity of the earth	$\omega = \frac{2\pi}{86,164} = 7.292 \cdot 10^{-5}$ sec^{-1}
Distance from the earth to the sun	
maximal	$1.5200\ 10^{11}$ m
minimal	$1.4701\ 10^{11}$ m
mean	$1.491\ 10^{11}$ m
Eccentricity of the terrestrial orbit	0.01675
Solar radius	695,300 km

Appendix 3

Values of acceleration of gravity in $m \cdot sec^{-2}$

($g_{45°} = 9.8062\ m \cdot sec^{-2}$)

h km \ $\varphi°$	0	10	20	30	40	50	60	70	80
0	9.780	9.782	9.786	9.793	9.802	9.811	9.819	9.826	9.831
2	9.774	9.776	9.780	9.787	9.796	9.804	9.813	9.820	9.824
4	9.768	9.770	9.774	9,781	9.789	9.798	9.807	9.814	9.818
6	9.762	9.763	9.768	9.775	9,783	9.792	9.801	9.808	9.812
8	9.756	9,757	9.762	9.768	9.777	9.786	9.794	9.801	9.806
10	9.749	9.751	9.755	9.762	9.771	9.780	9.788	9.795	9.800
12	9.743	9.745	9.749	9.756	9.765	9.774	9.782	9.789	9,794
14	9.737	9.739	9.743	9.750	9.758	9.768	9.776	9.783	9.787
16	9,731	9.732	9.737	9.744	9.752	9.761	9.770	9.777	9,781
18	9.725	9.726	9.731	9.738	9.746	9.755	9.764	9.771	9.775
20	9.719	9.720	9.725	9.732	9.740	9,749	9.757	9.764	9.769
22	9.712	9.714	9,718	9.725	9.734	9.743	9.751	9.758	9.763
24	9.706	9.708	9,712	9,719	9.728	9.737	9.745	9.752	9.756
26	9.700	9.702	9,706	9,713	9.722	9.730	9,739	9.746	9.750
28	9.694	9.696	9.700	9,707	9.715	9.724	9,733	9.740	9.744

Appendix 4

Major gaseous constituents of dry pure air

Gases	Nitrogen	Oxygen	Argon	Carbon dioxide	Hydrogen	Neon	Helium	Dry air	Units
Chemical formula	N_2	O_2	Ar	CO_2	H_2	Ne	He		
Molecular weight	28.016	32.000	39.944	44.000	2.016	20.183	4.002	28.97	
Density at 760 mm	0.001251	1 429	1 783	1 977	90	900	178	0.001293	g cm^{-3}
Density at 1000 mb	0.001235	1 411	1 760	1 951	89	888	176	0.001276	g cm^{-3}
Density relative to air	0.967	1.105	1.379	1.529	0.0695	0.695	0.138	1.00	
Mean molecular velocity (at 0°C and 760 mm Hg)	$4.9 \cdot 10^4$	4.6	4.1	3.9	18.4	7.4	13.1	$4.8 \cdot 10^4$	cm/sec^{-1}
Composition by volume at earth surface	78.09	20.95	0.93	0.03	0.001	0.0015	0.0004	100	%
Composition by weight	75.52	23.15	1.28	0.046	—	—	—	100	
Partial pressure at the earth's surface	593,6	159.3	7.06	0.228	0.0076	0.0114	0.00304	760	mm Hg
Gas constant	$2.97 \cdot 10^6$	2.59	2.08	1.88	41.30	4.16	20.94	$2.87 \cdot 10^6$	cm^2 sec^{-2} deg^{-1}
Altitude of homogeneous atmosphere (at 0°C)	8 260	7 231	5 795	5 226	114 980	11 481	58 051	7 991	m
	8 125	7 113	5 700	5 140	113 100	11 290	57 100	7 863	dyne m
The constant in the barometric formula	19 021	16 654	13 344	12 035	264 649	26 437	133 670	18 401	m
	18 710	16 380	13 230	11 940	260 300	26 010	131 500	18 100	dyne m
Specific heat C_p at 18°C	0.249	0.218	0.127	0,202	3.41	—	1.25	0.241	cal g^{-1} deg^{-1}
	1.40	1.40	1.65	1.30	1.41	1.64	1.66	1.40	
Boiling point at 760 mm Hg	—196	—183	—186	—78,5	—253	—246	—269	—	°C
Critical temperature at 760 mm Hg	—143	—119	—121	—31	—240	—229	—268	—	°C
Heat conductivity at 0°C	0.0000580	586	388	342	418	111	340	0.0000578	cal cm^{-1} sec^{-1} deg^{-1}
Coefficient of viscosity at 0°C	0.000167	192	210	140	84	298	189	0.000172	g cm^{-1} sec^{-1}

Appendix 5

The state of air in the upper layers of atmosphere

Altitude km	Temperature °K	Pressure mb	Density g. m^{-3}	Number of molecules per 1 cm^3	Mean free path	
					273° K cm	220° K cm
0	280	1000	1260	$2.7 \cdot 10^{19}$	$8.0 \cdot 10^{-6}$	
10	220	260	414	$8.6 \cdot 10^{18}$	$3.0 \cdot 10^{-5}$	$2.4 \cdot 10^{-5}$
20	220	55	87	$1.8 \cdot 10^{18}$	$1.4 \cdot 10^{-4}$	$1.1 \cdot 10^{-4}$
30	220	12	19	$4.0 \cdot 10^{17}$	$6.6 \cdot 10^{-4}$	$5.3 \cdot 10^{-4}$
40	220	3	4.8	$1.0 \cdot 10^{17}$	$2.6 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$
50	220	1.3	2.1	$4.3 \cdot 10^{16}$	$6.0 \cdot 10^{-3}$	$4.8 \cdot 10^{-3}$
60	220	0.3	0.5	$1.0 \cdot 10^{16}$	$2.6 \cdot 10^{-2}$	$2.1 \cdot 10^{-2}$
80	220	0.03	0.05	$1.0 \cdot 10^{15}$	$2.6 \cdot 10^{-1}$	$2.1 \cdot 10^{-1}$
100	220	0.004	0.006	$1.3 \cdot 10^{14}$	2.0	1.6

Note: The air composition is considered as equal at all altitudes

Appendix 6

Gas constant R for dry air

System of units		$T_0 = 273.0$	$T_0 = 273.16$
p In kg m^{-2}	ρ In kg m^{-3}	287.04	286.86 $m^2 sec^{-2} deg^{-1}$
p In dyne cm^{-2}	ρ In g cm^{-3}	$2.8704 \cdot 10^6$	$2.8686 \cdot 10^6$ $cm^2 sec^{-2} deg^{-1}$ or $erg \cdot g^{-1} deg^{-1}$
p In mb	ρ In $kg \cdot m^{-3}$	2.8704	2.8686
p In mb	ρ In $g \cdot cm^{-3}$	$2.8704 \cdot 10^3$	$2.8686 \cdot 10^3$
p In mm Hg	ρ In $kg \cdot m^{-3}$	2.153	2.152
p In mm Hg	ρ In $g \cdot cm^{-3}$	2.153	2.152

Specific heat of dry and of moist air in cal· g^{-1} ·deg^{-1} at p = 760 mm Hg

Temperature °C	Specific humidity in g/kg				
	0	5	10	15	20
			c_p		
40	0.2409	0.2419	0.2429	0.2446	0.2450
30	07	17	27	38	
20	06	16	26		
10	05	15			
0	04				
—20	05				
—40	0.241				
—60	0.242				
			c_v		
40	0.1718	0.1726	0.1734	0.1743	0.1751
30	16	24	32	41	
20	15	23	31		
10	14	22			
0	12				
—20	15				
—40	0.172				
—60	0.172				
			$\varkappa = c_p : c_v$		
40	1.401	1.400	1.400	1.400	1.398
20	402	401	401		
0	404				
—20	402				
—40	402				
—60	403				
			$\lambda = R : c_p$		
40	0.2843	0.2841	0.2838	0.2834	0.2832
20	2845	2843	2840		
0	2848				
—20	2847				
—40	284				
—60	283				
			$c_p - c_v$		
40	0.0691	0.0693	0.0695	0.0697	0.0699
			R		
	0.0685	0.0687	0.0689	0.0692	0.0694

Saturation pressure of water vapor in mm Hg
over a plane surface

t	0	1	2	3	4	5	6	7	8	9
					over water (observed)					
+ 40	55.324	58.34	61.50	64.80	68.26	71.88	75.65	79.60	84.71	88.02
30	31.824	33.695	35.663	37.729	39.898	42.175	44.563	47.067	49.692	52.442
20	17.535	18.650	19.827	21.068	22.377	23.756	25.209	26.739	28.349	30.043
10	9.209	9.844	10.518	11.231	11.987	12.788	13.634	14.530	15.477	16.477
0	4.579	4.926	5.294	5.685	6.101	6.543	7.013	7.513	8.045	8.609
— 0	4.579	4.256	3.952	3.669	3.404	3.158	2.928	2.712	2.509	2.320
—10	2.143	1.979	1.826	1.684	1.551	1.429	1.315			
					over water (calculated)					
— 0	4.579	4.257	3.954	3.670	3.405	3.157	2.926	2.710	2.507	2.319
—10	2.142	1.978	1.826	1.684	1.551	1.429	1.314	1.208	1.109	1.018
—20	0.934	0.856	0.783	0.716	0.654	0.596	0.543	0.494	0.449	1.408
—30	0.371	0.337	0.307	0.279	0.253	0.230	0.208	0.188	0.170	0.154
—40	0.139	0.125	0.113	0.102	0.092	0.083	0.075	0.067	0.060	0.054
—50	0.049	0.044	0.039	0.034	0.031	0.027	0.024	0.021	0.018	0.016
—60	0.014	0.012	0.010	0.009	0.007	0.006	0.005	0.004	0.002	
					over ice.					
— 0	4.579	4.216	3.879	3.566	3.277	3.010	2.763	2.535	2.323	2.128
—10	1.947	1.731	1.628	1.487	1.357	1.238	1.128	1.027	0.934	0.849
—20	0.771	0.700	0.635	0.575	0.520	0.470	0.424	0.382	0.344	0.310
—30	0.279	0.251	0.226	0.203	0.183	0.164	0.148	0.132	0.118	0.106
—40	0.095	0.085	0.076	0.068	0.060	0.054	0.048	0.043	0.038	0.034
—50	0.030	0.027	0.024	0.021	0.018	0.016	0.014	0.012	0.011	0.009
—60	0.008	0.007	0.006	0.005	0.004	0.003	0.003	0.002	0.001	
					The ratio $E_i : E_w$ at saturation, relative to ice					
— 0	1.000	0.990	0.981	0.972	0.962	0.953	0.944	0.935	0.927	0.918
—10	0.909	0.900	0.892	0.833	0.875	0.866	0.858	0.850	0.842	0.834
—20	0.826	0.818	0.811	0.803	0.795	0.788	0.780	0.773	0.765	0.758
—30	0.751	0.744	0.737	0.730	0.723	0.716	0.709	0.702	0.696	0.689
—40	0.682	0.676	0.669	0.663	0.658	0.651	0.645	0.639	0.633	0.627
—50	0.621	0.615	0.609	0.603	0.597	0.592	0.586	0.580	0.575	0.570
—60	0.564	0.559	0.554	0.549	0.543	0.538	0.533	0.528	0.523	0.518
					$E_i - E_w$					
— 0	0.000	0.041	0.075	0.104	0.128	0.147	0.163	0.175	0.184	0.191
—10	0.195	0.197	0.198	0.197	0.194	0.191	0.186	0.181	0.175	0.169
—20	0.162	0.156	0.148	0.141	0.134	0.127	0.119	0.112	0.105	0.099
—30	0.092	0.086	0.031	0.075	0.070	0.065	0.061	0.056	0.052	0.048
—40	0.044	0.041	0.037	0.034	0.032	0.029	0.027	0.024	0.022	0.020
—50	0.019	0.017	0.015	0.014	0.012	0.011	0.010	0.009	0.008	0.007
—60	0.006	0.005	0.005	0.004	0.003	0.003	0.002	0.002	0.001	0.001

Appendix 9

Physical constants of water vapor

Molecular weight	18.02
Gas constant	$460\ m^2 \cdot sec^{-2}\ deg^{-1} = 0.07\ cal^2 \cdot g^{-1}\ deg^{-1}$
Specific weight relative to air	$0.6221 \approx 5/8$
Density at $0^\circ C$ and 760 mm Hg	$0.804\ kg \cdot m^{-3}$
Density at $0^\circ C$ and 1000 mb	$0.794\ kg \cdot m^{-3}$

Specific heat

$t = 0$	10	$20 - 40^\circ C$
$c_{pw} = 0.445$	0.445	$0.440\ cal \cdot g^{-1}\ deg^{-1}$
$c_{vw} = 0.335$	0.335	$0.330\ cal \cdot g^{-1}\ deg^{-1}$

$c_p : c_v = 1.330$

Coefficient of diffusion of water vapor in air

at $0^\circ C$	$0.198\ cm^2 \cdot sec^{-1}$
at $19.5^\circ C$	0.283 " "

Appendix 10

Physical constants of water and ice

Specific weight of water at $4^\circ C$	1.00000
at $0^\circ C$	0.99987
Specific weight of ice	0.917
Specific volume of ice	1.091
Specific heat of water at $15^\circ C$	$1\ cal \cdot g^{-1}\ deg^{-1}$
Specific heat of ice	$0.505\ cal.g^{-1}\ deg^{-1}$
Boiling point of water at 760 mm Hg	$100.00^\circ C$
at 1000 mb	$99.66^\circ C$
Melting point of ice	$0.00^\circ C$
Latent heat of fusion of ice	$79.7\ cal \cdot g^{-1}$

Latent heat of vaporization of water L at various temperatures:

t =	-80	-40	-30	-20	-10	0	10	20	30	40°C
L =	630.0	624.0	617.0	608.7	603.0	597.3	591.6	585.9	580.2	574.5 cal·g^{-1}

Latent heat of vaporization of ice L_1 at various temperatures:

t =	-80	-70	-60	-50	-40	-30	-20	-10	0°C
L =	675.1	675.8	676.6	677.1	677.4	677.5	677.6	677.3	677.1

Coefficient of heat conductivity	
of water	$0.001325 + 4 \cdot 10^{-6}\ t\ cal \cdot cm^{-1} \cdot sec^{-1} \cdot deg^{-1}$
of ice	$0.002\ cal \cdot cm^{-1} \cdot sec^{-1} \cdot deg^{-1}$
of snow	0.0002 (at density 0.05 $g \cdot cm^{-3}$)
	0.005 (at " 0.90 $g \cdot cm^{-3}$)

Coefficient of viscosity of water at $20^\circ C$	$0.0100\ g \cdot cm^{-1} \cdot sec^{-1}$
Capillary constant for water (at $18^\circ C$)	$72.8\ dyne \cdot cm^{-1}$

Appendix 11

Pressure of water vapor and supersaturation necessary for equilibrium of a drop

Radius of drop in cm	$t = 30°$ C		$t = 0°$ C		$t = -20°$ C	
	$\frac{\Delta E}{E}$	$100\left(1+\frac{\Delta E}{E}\right)\%$	$\frac{\Delta E}{E}$	$100\left(1+\frac{\Delta E}{E}\right)\%$	$\frac{\Delta E}{E}$	$100\left(1+\frac{\Delta E}{E}\right)\%$
$3\cdot10^{-7}$	0.92	192	2.01	301	4.35	535
$5\cdot10^{-7}$	0.343	134.3	0.46	146	0.58	158
10^{-6}	0.128	112.8	0.157	115.7	0.185	118.5
10^{-5}	0.0104	101.04	0.0124	101.24	0.0139	101.39
10^{-4}	0.00102	100.10	0.0012	100.12	0.00134	100.13
10^{-3}	0.000102	100.01	0.00012	100.01	0.00013	100.01
10^{-2}	0.0000102	100.001	0.000012	100.001	0.000013	100.001

Appendix 12

Virtual difference $T_v - T$ for air saturated with water vapor

Temperature °C	Pressure in mb			
	1000	900	700	500
−40	0.0	0.0	0.0	0.0
−30	0.0	0.0	0.1	0.1
−20	0.1	0.1	0.1	0.2
−10	0.2	0.2	0.2	0.3
− 5	0.3	0.3	0.4	0.5
0	0.6	0.6	0.8	1.2
5	0.8	0.9	1.2	
10	1.3	1.5	1.9	
15	1.8	2.0	2.6	
20	2.6	2.9	3.7	
25	3.6	4.0		
30	4.8	5.4		
35	6.6	7.3		
40	8.8	9.9		

Condensation-adiabatic lapse rate in °C/100 dyne·m

Temperature °C	Pressure					
	1000	900	700	500	300	100
40	0.315	0.304				
35	337	326				
30	366	352				
25	400	384				
20	440	422	0.382			
15	487	468	422			
10	539	520	471			
5	596	578	527			
0	659	640	587	0.517		
— 5	723	705	653	579		
—10	780	762	717	650		
—15	833	819	780	720		
—20	879	867	835	785	0.692	
—25	913	904	880	842	764	
—30	941	936	920	890	829	0.629
—35	964	961	948	927	883	710
—40	978	976	968	954	922	800
—45	989	988	983	973	952	870
—50	995	994	991	984	971	909
—55	999	998	995	991	983	941
—60	1.001	1.001	1.000	998	996	966
—65	1.003	1.002	1.002	1.001	999	985

Appendix 14

Sublimation-adiabatic lapse rate in °C/100 dyne·m

Temperature °C	Pressure					
	1000	900	700	500	300	100
0	0.600	0.578	0.525	0.457		
— 5	680	658	606	535		
—10	751	732	683	617		
—15	818	802	760	679		
—20	872	862	827	773	0.679	
—25	914	905	880	839	761	
—30	945	940	922	894	835	0.640
—35	970	965	953	935	896	743
—40	983	980	974	962	935	825
—45	992	991	986	979	964	891
—50	997	996	994	990	980	934
—55	1.000	999	998	996	991	961
—60	1.003	1.002	1.002	1.000	998	981
—65	1.004	1.004	1.003	1.003	1.001	994

Appendix 15

Radiation of surface unit of a black body into a hemisphere

$$E = \sigma T^4 \text{ cal} \cdot \text{cm}^{-2} \text{ min}^{-1}$$

where $\sigma = 0.826 \times 10^{-10}$

Temperature °C	0	1	2	3	4	5	6	7	8	9
—40	0.244	0.239	0.235	0.231	0.227	0.223	0.219	0.216	0.212	0.208
—30	288	283	279	274	270	265	261	256	252	248
—20	339	333	328	323	318	312	308	303	298	293
—10	395	389	384	378	372	366	360	355	349	344
— 0	459	452	446	439	432	426	420	414	407	401
0	459	466	472	479	486	493	500	508	515	522
10	530	537	542	553	560	568	576	584	592	601
20	609	617	626	634	643	651	660	669	678	687
30	696	706	715	724	734	743	753	762	772	782
	$t = -50$		—55	—60	—65	—70° C				
	$E = 0.204$		0.187	0.170	0.155	0.140				
	$t = 40$		45	50	55	60	65	70° C		
	$E = 0.792$		0.844	0.899	0.956	1.015	1.078	1.143		

Appendix 16

Intensity of radiation $\pi E_{\lambda T} \Delta\lambda$ of unit black surface into a hemisphere, according to Planck's law

$\Delta\lambda = 1\ \mu$ (in 10^{-3} cal·cm^{-2}·min^{-1})

λ \ T	200	210	220	230	240	250	260	270	273	280	290	300
3.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.3
3.5	0.0	0.0	0.0	0.0	0.0	0.1	0.2	0.3	0.3	0.5	0.8	1.2
4.0	0.0	0.0	0.0	0.1	0.2	0.3	0.6	1.0	1.0	1.4	2.3	3.5
4.5	0.0	0.1	0.1	0.3	0.5	0.9	1.4	2.3	2.6	3.3	4.9	7.1
5.0	0.1	0.2	0.4	0.6	1.1	1.8	2.8	4.2	4.8	6.2	8.8	12.4
5.5	0.2	0.4	0.8	1.3	2.1	3.2	4.8	6.9	7.7	9.7	13.5	18.1
6.0	0.5	0.8	1.3	2.1	3.3	4.9	7.1	9.9	11.1	13.6	18.4	24.1
6.5	0.8	1.3	2.1	3.2	4.8	6.9	9.6	13.2	14.5	17.6	23.1	29.7

λ \ T	200	210	220	230	240	250	260	270	273	280	290	300
7.0	1.2	1.9	2.9	4.4	6.2	8.9	12.1	16.3	17.8	21.4	27.4	34.8
7.5	1.6	2.5	3.8	5.6	7.9	10.9	14.6	19.2	20.7	24.6	31.2	38.6
8.0	2.1	3.3	4.8	6.8	9.3	12.6	16.6	21.5	23.2	27.3	33.9	41.8
8.5	2.7	4.0	5.7	8.0	10.8	14.3	18.5	23.4	25.1	29.3	36.1	43.8
9.0	3.2	4.6	6.5	9.0	11.9	15.6	19.9	25.0	26.6	30.8	37.5	45.0
9.5	3.7	5.3	7.3	9.8	12.9	16.6	21.0	26.0	27.6	31.7	38.2	45.5
10.0	4.2	5.8	8.0	10.6	13.7	17.4	21.7	26.5	28.2	32.2	38.4	45.3
11	4.9	6.7	8.9	11.5	14.6	18.2	22.2	26.8	28.3	31.9	37.5	43.6
12	5.5	7.3	9.5	12.0	14.9	18.2	21.9	26.0	27.3	30.5	35.5	40.7
13	5.8	7.6	9.7	12.0	14.7	17.7	21.0	24.6	25.8	28.5	32.7	37.3
14	6.0	7.6	9.5	11.7	14.1	16.8	19.7	22.9	23.9	26.3	29.9	33.7
15	6.0	7.5	9.2	11.2	13.3	15.7	18.2	21.0	21.9	23.8	27.0	30.2
16	5.8	7.2	8.8	10.6	12.5	14.5	16.7	19.1	19.8	21.6	24.3	26.9
17	5.6	6.9	8.3	9.9	11.5	13.3	15.2	17.3	17.9	19.4	21.6	24.1
18	5.4	6.5	7.8	9.1	10.6	12.1	13.8	15.5	16.1	17.4	19.3	21.4
19	5.1	6.1	7.2	8.4	9.7	11.1	12.5	14.0	14.4	15.6	17.2	19.0
20	4.8	5.7	6.7	7.7	8.8	10.0	11.3	12.5	13.0	13.9	15.3	16.8
22	4.1	4.9	5.6	6.4	7.3	8.2	9.2	10.2	10.5	11.1	12.2	13,3
24	3.5	4.1	4.7	5.4	6.0	6.7	7.5	8.2	8.4	9.0	9.7	10.6
26	3.0	3.5	4.0	4.5	5.0	5.6	6.1	6.7	6.9	7.2	7.9	8.5
28	2.6	3.0	3.3	3.7	4.1	4.6	5.0	5.4	5.6	5.9	6.4	6.9
30	2,2	2.5	2.8	3.1	3.5	3.8	4.1	4.5	4.6	4.9	5.2	5:6
35	1.5	1.7	1.9	2.0	2.3	2.4	2.6	2.8	2.9	3.0	3.2	3.5
40	1.0	1.1	1.3	1.4	1.5	1.6	1.7	1.9	1.9	2.0	2.1	2.3
45	0.7	0.8	0.9	1.0	1.0	1.1	1.2	1.3	1.3	1.4	1.4	1.5
50	0.5	0.6	0.6	0.7	0.7	0.8	0.8	0.9	0.9	1.0	1.0	1:1
60	0.3	0.3	0.3	0.4	0.4	0.4	0.5	0.5	0.5	0.5	0.5	0.6
70	0.2	0.2	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.3	0.3	0.3
80	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2	0.2	0.2
90	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
100	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

Appendix 17

Solar energy spectrum beyond the earth's atmosphere (in 10^{-3} cal·cm^{-2}·min^{-1}) for a mean solar constant equal to 1.880 cal·cm^{-2}·min^{-1}

λ in μ	$I_{0\lambda}$	λ in μ	$I_{0\lambda}$	λ in μ	$I_{0\lambda}$	λ in μ	$I_{0\lambda}$
0.28—0.30	2.5	0.52—0.54	55.6	0.76—0.78	34.3	1.0—1.1	92.4
0.30—0.32	11.2	0.54—0.56	53.8	0.78—0.80	32.2	1.1—1.2	71.1
0.32—0.34	22.1	0.56—0.58	52.9	0.80—0.82	30.7	1.2—1.3	54.2
0.34—0.36	23.0	0.58—0.60	52.6	0.82—0.84	29.2	1.3—1.4	45.4
0.36—0.38	29.6	0.60—0.62	51.2	0.84—0.86	27.7	1.4—1.5	36.7
0.38—0.40	38.8	0.62—0.64	48.7	0.86—0.88	26.3	1.5—1.6	30.9
		0.64—0.66	46.9	0.88—0.90	25.1	1.6—1.7	25.1
0.40—0.42	53.3	0.66—0.68	45.4	0.90—0.92	23.6	1.7—1.8	21.2
0.42—0.44	55.2	0.68—0.70	44.0	0.92—0.94	2.28	1.8—1.9	18.3
0.44—0.46	59.1	0.70—0.72	41.6	0.94—0.96	2.19	1.9—2.0	15.4
0.46—0.48	61.0	0.72—0.74	39.0	0.96—0.98	21.2	2.0—2.5	54.3
0.48—0.50	60.6	0.74—0.76	36.6	0.98—1.00	20.3	2.5—3.0	23.0
0.50—0.52	57.8					> 3.0	30.0

Appendix 18

Values of $E = f \cdot \sigma \cdot T^4$ (in cal·cm^{-2}·min^{-1})

T °K	0	1	2	3	4	5	6	7	8	9
210	0.106	0.108	0.110	0.111	0.113	0.115	0.117	0.119	0.121	0.123
220	124	126	129	130	132	135	136	138	140	143
230	145	147	149	152	154	156	158	160	163	165
240	167	170	172	174	177	180	182	185	187	189
250	192	194	197	200	203	205	208	210	213	216
260	219	222	225	227	231	234	237	240	243	246
270	249	253	256	259	263	266	270	273	277	280
280	284	287	291	295	299	303	307	310	314	318
290	322	326	331	335	339	343	347	352	355	360
300	365	369	373	378	383	387	392	396	400	405
310	410	415	420	425	430	434	439	445	451	456
320	0.462									

Appendix 19

Deflective force of earth rotation, centrifugal force and angular momentum

φ	Coriolis parameter		Centrifugal force			Angular momentum
	horizontal $2\omega \sin\varphi$	vertical $2\omega \cos\varphi$	total $R\omega^2 \cos\varphi$ $m\,sec^{-2}$	horizontal $R\omega^2 \sin\varphi \cos\varphi$ $m\,sec^{-2}$	vertical $R\omega^2 \cos^2\varphi$ $m^2\,sec^{-1}$	$R^2\omega \cos^2\varphi$ $m\,sec^{-2}$
0	$0.000 \cdot 10^{-4}$	$1.458 \cdot 10^{-4}$	0.0338	0.0000	0.0338	$296 \cdot 10^7$
10	0.253	1.436	0.0333	0.0057	0.0328	287
20	0.499	1.370	0.0318	0.0108	0.0298	261
30	0.729	1.263	0.0293	0.0146	0.0254	222
40	0.937	1.117	0.0259	0.0166	0.0199	173
50	1.117	0.937	0.0218	0.0166	0.0140	122
60	1.263	0.729	0.0169	0.0146	0.0085	74
70	1.370	0.499	0.0116	0.0108	0.0040	35
80	1.436	0.253	0.0057	0.0057	0.0010	8.9
90	1.458	0.000	0.0000	0.0000	0.0000	0.0

Appendix 20

Radius of curvature of inertial circle and time of run along inertial circle for v = 1 m/sec

φ	5	10	20	30	40	50	60	70	80	90
r	78.6	39.5	20.0	13.7	10.7	9.0	7.9	7.3	7.0	6.9 km
τ	137.1	68.9	35.0	23.9	18.6	15.6	13.8	12.7	12.0	12.0 hours

Appendix 21

Slope of isobaric surfaces

v_g / φ	1	5	10	20	30	50	m/sec
10°	0.26	1.29	2.58	5.16	7.74	12.50	$\cdot 10^{-5}$
	1″	3″	5″	11″	16″	27″	
20°	0.51	2.53	5.06	10.12	15.18	25.30	$\cdot 10^{-5}$
	1″	5″	10″	21″	31″	52″	
30°	0.74	3.70	7.40	14.80	22.20	37.00	$\cdot 10^{-5}$
	2″	8″	15″	31″	46″	1′16″	
40°	0.95	4.76	9.52	19.03	28.55	47.58	$\cdot 10^{-5}$
	2″	10″	20″	39″	59″	1′38″	
50°	1.13	5.67	11.34	22.67	34.00	56.68	$\cdot 10^{-5}$
	2″	12″	23″	46″	1′10″	1′57″	
60°	1.28	6.41	12.82	25.63	38.45	64.08	$\cdot 10^{-5}$
	3″	13″	26″	53″	1′19″	2′12″	
70°	1.39	7.00	13.91	27.82	41.74	69.55	$\cdot 10^{-5}$
	3″	14″	29″	57″	1′26″	2′23″	
80°	1.46	7.28	14.58	29.16	43.74	72.90	$\cdot 10^{-5}$
	3″	14″	30″	1′00″	1′30″	2′30″	
90°	1.48	7.40	14.80	29.60	44.40	74.00	$\cdot 10^{-5}$
	3″	15″	30″	1′01″	1′31″	2′32″	

Appendix 22

Coefficient of thermal conductivity and coefficient of viscosity for dry air

t	—70	—50	—30	—20	—10	—0	10	20	30	40° C	
λ	0.0000422	453	485	501	517	533	549	565	581	0.0000597	cal $cm^{-1} sec^{-1} deg^{-1}$
μ	0.000137	147	157	162	167	172	177	182	187	0.000192	g $cm^{-1} sec^{-1}$

Physical constants

Gravitation constant	$6.665 \cdot 10^{-8}\ cm^3 \cdot g^{-1} \cdot sec^{-2}$
Universal gas constant	$8.314 \cdot 10^{7}\ erg \cdot deg^{-1} \cdot mole^{-1} =$ $= 1.9867\ cal \cdot deg^{-1} \cdot mole^{-1}$
Avogadro's number (number of molecules per 1 cm^3)	
at 0°C 760 mm Hg	$2.70 \cdot 10^{19}$
at 0°C 1000 mb	$2.66 \cdot 10^{19}$
Loschmidt's number (number of gas molecules per mole)	$6.03 \cdot 10^{23}$
Absolute zero, 0°K	−273.16°C
Volume of 1 mole of ideal gas at 0°C and p = 760 mm Hg	22.4160 l
Mean "diameter" of a diatomic gas molecule	$2.7 \cdot 10^{-8}$ cm
Mass of single molecule for molecular weight M	$1.65 \cdot 10^{-24} \cdot M$
Heat equivalent of work	$A = \frac{1}{J} = 0.2389 \cdot 10^{-7}\ cal \cdot erg^{-1}$
1 erg = 1 $g \cdot cm^2 \cdot sec^{-2}$	$0.2389 \cdot 10^{-7}$ cal = 10^{-7} joule
1 calorie	$4.1855 \cdot 10^{-7}$ erg
Planck's constant	$h = 6.61 \cdot 10^{-27}\ erg \cdot sec$
Boltzmann's constant	$k = 1.379 \cdot 10^{-16}\ erg \cdot deg^{-1}$
	$\frac{h}{k} = 4.795 \cdot 10^{-11}\ deg \cdot sec$
Velocity of light in vacuo	$c = 2.9977 \cdot 10^{10}\ cm \cdot sec^{-1}$
Constants occurring in Planck's radiation law: $c_1 = 2c^2h$	$11.88 \cdot 10^{-6}\ erg \cdot cm^2 \cdot sec^{-1} =$ $= 17.02 \cdot 10^{-12}\ cal \cdot cm^2 \cdot min^{-1}$
$c_2 = \frac{ch}{k}$	$1.432\ cm \cdot deg = 1.432 \cdot 10^{-4}\ \mu \cdot deg$
Radiation constant in Stefan-Boltzmann's law	$\sigma = 5.75 \cdot 10^{-5}\ erg \cdot cm^{-2} \cdot sec^{-1}\ deg^{-4} =$ $= 1.374 \cdot 10^{-12}\ cal \cdot cm^{-2} \cdot sec^{-1}\ deg^{-4}$
Constant of Wien's displacement law	$0.2884\ \mu \cdot deg$
Mean solar constant:	
according to American scale	$I_0 = 1.94\ cal \cdot cm^{-2} \cdot min^{-1}$
" " European scale	$I_0 = 1.88\ cal \cdot cm^{-2} \cdot min^{-1}$

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